Turk J Math
30 (2006) , 177 – 185.

Quasi-Dual Modules

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Abstract

Let R be a ring, M be a right R-module and $S = End_R(M)$. M is called a quasi-dual module if, for every R-submodule N of M, N is a direct summand of $r_M(X)$ where $X \subseteq S$. In this article, we study and provide several characterizations of this module classes. We show that if M is quasi-dual module, then, for all $m \in M$, $r_M \ell_S(m) = mR \oplus K$ for some submodule K of M. We also show that every quasi-dual module is a Kasch module and $Z(_S M) \subseteq Rad(M_R)$.

Key Words: Quasi-dual module, Kasch module, Ikeda-Nakayama module.

1. Introduction

Throughout this paper, R is an associative ring with identity, modules are right and unitary over it and $S = End_R(M)$ is the ring of R-endomorphisms of M. Submodules of M will be right R-modules unless specified otherwise. Clearly, the module M is a left S and right R-bimodule.

A ring R is called a *right dual ring* if every right ideal of R is an annihilator and R is called *right quasi-dual ring* if every right ideal of R is a direct summand of a right annihilator. Right dual and, as a generalization of right dual rings, right quasi-dual rings were discussed in detail in [4] and [9]. Some of the known results on right quasi-dual rings can be recalled as follows: R is a right quasi-dual ring if and only if $r\ell(I) = I$ for every essential ideal I of R; if R is a right quasi-dual ring then, R is a right Kasch ring and $r\ell(Soc(R_R)) = Soc(R_R)$ and $r\ell(J(R)) = J(R)$.

In this paper, the notion of a quasi-dual module is introduced as a generalization of quasi-dual rings to modules.

2. Preliminaries

Let R and S be rings and ${}_SM_R$ be a bimodule. For any $X \leq M$ and $T \subseteq S$, denote $\ell_S(X) = \{s \in S : sX = 0\}$ and $r_M(T) = \{m \in M : Tm = 0\}$.

Lemma 2.1 *For a right R-module M, let $S = End_R(M)$, $N \leq M$, $I \leq R_R$, $J \leq {}_SS$ and $0 \in S$; we then have*

$$r_M(0) = M$$

$$\ell_S(0) = S$$

$$r_M(S) = \ell_S(S) = \ell_S(M) = 0$$

$$\ell_M(r_R(\ell_M(I))) = \ell_M(I)$$

$$\ell_S(r_M(\ell_S(N))) = \ell_S(N)$$

$$r_R(\ell_M(r_R(N))) = r_R(N)$$

$$r_M(\ell_S(r_M(J))) = r_M(J)$$

$$\ell_S(\oplus_{i\in I} N_i) = \cap_{i\in I} \ell_S(N_i).$$

Proof. See [2, 12]. □

Definition 2.2 A ring R is said to be a *right dual* if every right ideal of R is an annihilator ([4]).
Definition 2.3 A ring R is called a *right quasi-dual* if every right ideal of R is a direct summand of a right annihilator ([9]).

Definition 2.4 A module M is called *Ikeda-Nakayama module* if

$$\ell_S(A \cap B) = \ell_S(A) + \ell_S(B)$$

for any submodules A, B of M_R (see [10]).

Definition 2.5 A module M is called *Kasch module* if $\hat{M}$ is an (injective) cogenerator in $\sigma[M]$, where $\hat{M}$ is injective hull of M in $\sigma[M]$ ([1]).

The notations, "$\leq$" will denote a submodule, "$\leq_e$" an essential submodule, and "$<<$" a small submodule.

We will refer to [2, 3, 4, 8, 9, 11] for all undefined notions used in the text, and also for basic facts concerning (quasi-)dual rings and annihilators.

3. Quasi-Dual Modules

In this paper, we shall introduce the notion of quasi-dual modules and try to give a module theoretic characterizations of quasi-dual ring.

Definition 3.1 (See [5]) Let R be a ring, M be a right R-module and $S = End_R(M)$. M is called *a dual module* if

1. $r_M\ell_S(N) = N$ for every submodule N of M;
2. $\ell_S r_M(I) = I$ for every right ideal I of S.

Definition 3.2 Let R be a ring, M be a right R-module and $S = End_R(M)$. We shall call M *a quasi-dual module* if, for every R-submodule N of M, N is a direct summand of $r_M(X)$, where $X \subseteq S$(compare with [6] and [7]). Trivially,

1. A right quasi-dual ring is a quasi-dual module as right module.
2. Every dual module is a quasi-dual module.
3. Every semisimple module is a quasi-dual module.

Lemma 3.3 *The following conditions are equivalent for a right R-module M.*

1. *M is a quasi-dual module.*
2. *For every essential submodule K of M, $r_M\ell_S(K) = K$*
3. *For every submodule L of M, L is a direct summand of $r_M\ell_S(L)$.*

Proof. (1) $\Rightarrow$ (2) Let M be a quasi-dual module and K be an essential submodule of M. Then K is a direct summand of $r_M(Y)$ for some $Y \subseteq S$. Let $r_M(Y) = K \oplus K'$ for some K'. Then $K = r_M(Y)$. Note that $\ell_S(K) = \ell_S r_M(Y)$ implies $r_M\ell_S(K) = r_M\ell_S r_M(Y) = r_M(Y) = K$.
(2) $\Rightarrow$ (3) Let L be a submodule of M. If L is essential in M, $r_M\ell_S(L) = L$ by (2). Hence L is a direct summand of $r_M\ell_S(L)$. Assume that L is not essential in M. Then

$L \oplus L'$ is an essential for some submodule L' of M. By (2), $r_M\ell_S(L \oplus L') = L \oplus L'$. Since $L \subseteq r_M\ell_S(L) \subseteq r_M\ell_S(L \oplus L')$, L is a direct summand of $r_M\ell_S(L)$ by modularity.
(3) $\Rightarrow$ (1) clear. □

Following [10], M is called *almost principally injective* (AP-injective for short) if, for any $m \in M$, there exists an S-submodule K of M such that $r_M\ell_S(m) = mR \oplus K$.

Theorem 3.4 *Every quasi-dual right R-module is an AP-injective module.*

Proof. Clear. □

Let N be any module. N is said to be $M-$*cyclic module* if N is isomorphic to M/X for some $X \leq M$, and in case $N \leq M$ and N is M-cyclic module then it is called $M-$*cyclic submodule* of M and N is called M*-singular* if $N \cong M/K$ with $K \leq_e M$.

Proposition 3.5 *Let M be an R-module. Then*

1. *If, for every essential submodule K of M, $r_M\ell_S(K) = K$ then, every M-cyclic singular R-module is cogenerated by M.*
2. *If every singular factor submodule (i.e. M-cyclic submodule) of M is cogenerated by M, then $r_M\ell_S(K) = K$ for every essential submodule K of M.*

Proof. (1) Let N be a singular R-module with $N \cong M/K$ and $K \leq_e M$. Since K is essential in M, $r_M\ell_S(K) = K$ by assumption. Let $I = \ell_S(K)$. We define $\phi : M/K \longrightarrow \Pi_{\alpha\in I}M_\alpha$ by $m + K \rightarrow \phi(m + K) = (\alpha m)_{\alpha\in I}$. Let $(\alpha m)_{\alpha\in I} = 0$. Then $\alpha m = 0$ for all $\alpha \in I$. Hence $\alpha \in \ell_S(K)$ and so $m \in r_M\ell_S(K) = K$. Therefore ϕ is a monomorphism.
(2) Let M/K be a singular module for some $K \leq_e M$. By hypothesis, there exists a monomorphism $\sigma : M/K \longrightarrow \Pi_{\alpha\in I}M_\alpha$ for some index set I with $M_\alpha = M$ for all $\alpha \in I$. We consider the natural epimorphism $\pi : M \longrightarrow M/K$ and canonical projection $p_\alpha : \Pi_{\alpha\in I}M_\alpha \longrightarrow M_\alpha$. Then $p_\alpha\sigma\pi \in \ell_S(K)$. Let $m \in r_M\ell_S(K)$. Then $p_\alpha\sigma\pi(m) = 0$ for all $\alpha \in I$. Therefore $\sigma\pi(m) \in Ker(p_\alpha)$ for all $\alpha \in I$ and so $\sigma\pi(m) \in \cap_{\alpha\in I}Ker(p_\alpha)$. Since $\cap_{\alpha\in I}Ker(p_\alpha) = 0$, $\sigma\pi(m) = 0$. But σ is a monomorphism, so $\pi(m) = 0$. Therefore $m \in K$. Other side is obvious. Hence $r_M\ell_S(K) = K$. □

$\sigma[M]$ will denote the full subcategory of left R-modules whose objects are the submodules of $M-$generated modules. Hence

$$\sigma[M] = \{N \in R - Mod : N \cong K/L \leq M^{(\Lambda)}/L \ for \ \ some \ \Lambda\}.$$

Following [1], a module M is called *Kasch module* if $\hat{M}$ is an (injective) cogenerator in $\sigma[M]$, where $\hat{M}$ is injective hull of M in $\sigma[M]$.

Proposition 3.6 *For a module M, the following are equivalent;*

1. *M is a Kasch module;*
2. *Any simple module in $\sigma[M]$ can be embedded in M;*
3. *Any simple module in $\sigma[M]$ is cogenerated by M;*
4. *$Hom(C, M) \neq 0$ for any nonzero (cyclic) R-module C from $\sigma[M]$;*
5. *$\ell_S(N) \neq 0$ for every proper submodule N of M;*
6. *$r_M \ell_S(N) = N$ for every maximal submodule N of M.*

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ by [1, Proposition 2.6], the other equivalences follows from Lemma 3.3 and Proposotion 3.5. □

Theorem 3.7 *Let M be a quasi dual module.*

1. *$r_M \ell_S(Soc(M)) = Soc(M)$.*
2. *For every maximal submodule N of M, $r_M \ell_S(N) = N$. Therefore, M is a Kasch module and $r_M \ell_S(Rad(M)) = Rad(M)$.*
3. *If L is a submodule of M, then $r_M \ell_S(L) = L \oplus L'$ for a submodule L' with $\ell_S(L) \leq \ell_S(L')$.*

Proof. (1) Let M be a quasi dual module. Then, for each essential submodule K of M, $r_M \ell_S(K) = K$ by Lemma 3.3. By Proposition 3.5, M/K is cogenerated by M. Since $Soc(M)$ is the intersection of all essential submodules, $M/Soc(M)$ is cogenerated by M. Since $Soc(M)$ is an essential submodule of M and $M/Soc(M)$ is singular factor module, so $r_M \ell_S(Soc(M)) = Soc(M)$ by Lemma 3.3.
(2) Let N be a maximal submodule of M. Assume that $r_M \ell_S(N) \neq N$. By maximality

of N, $r_M\ell_S(N) = M$. Note that, for $x \in \ell_S(N)$, $xN = 0$ implies $xM = 0$. Since M is a quasi-dual module, N is a direct summand of $r_M\ell_S(N)$ by Lemma 3.3, and so of M. Let $M = N \oplus N'$ for some submodule N' of M. We consider the canonical projection π on N'. Since $\pi(N) = 0$ implies $\pi(M) = 0$, we have $M = N$. It is a contradiction by maximality of N. Hence $r_M\ell_S(N) = N$. So, M is a Kasch module by Proposition 3.6. Let $x \in r_M\ell_S(Rad(M))$. Then $\ell_S(Rad(M))x = 0$. Note that $M/Rad(M) = M/\cap_{N\leq_{max}M} N$. We consider

$$M \xrightarrow{\pi} M/Rad(M) = M/\cap_{N\leq_{max}M} N \xrightarrow{\sigma} \Pi_{N\leq_{max}M} M/N \xrightarrow{\beta} \Pi_{\alpha\in I} M_\alpha \xrightarrow{p_\alpha} M_\alpha = M.$$

We know that σ and β are one to one. Since $p_\alpha\beta\sigma\pi \in \ell_S(Rad(M))$, we have $(p_\alpha\beta\sigma\pi)(x) = 0$ for all $\alpha \in I$. Then $\beta\sigma\pi(x) = 0$ and so $\pi(x) = 0$. This implies that $x \in Rad(M)$. Other side is obvious.
(3) Let L be a submodule of M. Then $r_M\ell_S(L) = L \oplus L'$ for a submodule L' by Lemma 3.3. Note that $\ell_S(r_M\ell_S(L)) = \ell_S(L \oplus L') = \ell_S(L) \cap \ell_S(L')$ by Lemma 2.1. Hence $\ell_S(L) \leq \ell_S(L')$, as required. □

Recall that;
$(C1)$ Every complement submodule is a direct summand.
$(C2)$ If every submodule isomorphic to a direct summand of M is itself a direct summand.
$(C3)$ If N and K are direct summands of M and $N \cap K = 0$, then $N \oplus K$ is a direct summand of M.

M is called a *continous* (or a *quasi-continous*) module if M has $C1$ and $C2$ (or $C1$ and $C3$).

Theorem 3.8 *Let M be a finitely generated Kasch module such that, any complement submodule N of M, $r_M\ell_S(N) = N$. Then M is quasi-continuous.*

Proof. Let N_1 and N_2 be submodules of M such that they are complements of each other in M. Then $N_1 \cap N_2 = 0$. So $0 = N_1 \cap N_2 = r_M\ell_S(N_1) \cap r_M\ell_S(N_2) = r_M(\ell_S(N_1) + \ell_S(N_2))$. Since M is a Kasch module, by Proposition 3.6, $\ell_S(N_1) + \ell_S(N_2) = M$. Hence M is a quasi-continuous by [11, Theorem 8]. □

Question : When M is a semiperfect module with essential socle in $\sigma[M]$ under the conditions of Theorem 3.8 ?

Proposition 3.9 *The following conditions are equivalent for a right R-module M.*

1. *M is a quasi-dual module and, for every right ideal I of S, I is a direct summand of $\ell_S(K)$ where $K \leq M$.*
2. (a) *For every essential submodule K of M, $r_M \ell_S(K) = K$*

   (b) *For every essential right ideal I of S, $\ell_S r_M(I) = I$*
3. (a) *For every submodule L of M, L is a direct summand of $r_M \ell_S(L)$*

   (b) *For every essential right ideal I of S, I is a direct summand of $\ell_S r_M(I)$.*

Proof. Similar to Lemma 3.3. □

Definition 3.10 We shall call M *a strongly quasi-dual module* if, for every R-submodule N of M and for every right ideal I of S, N is a direct summand of $r_M(X)$ and I is a direct summand of $\ell_S(K)$ where $X \subseteq S$ and $K \leq M$.

Let R and S be any rings and M be an $S-R$-bimodule. Following [6,7], if M is strongly quasi-dual module, then M is called *quasi-dual bimodule*

Proposition 3.11

1. *Let M be a quasi-dual module and A be a submodule of M. Then we have:*

   (i) *If $\ell_S(A) = 0$, then $A = M$.*

   (ii) *If M is an IN-module and $\ell_S(A) << S$, then $A \leq_e M$.*
2. *Let M be a strongly quasi dual module and I be a right ideal of S. Then we have:*

   (i) *If $r_M(I) = 0$, then $I = S$.*

   (ii) *If $\ell_S(A) \leq_e S$, then $A << M$.*

   (iii) *If M is indecomposable and $A \leq_e M$, then $\ell_S(A) << S$.*

   (iv) *If $r_M(I) \leq_e M$, then $I << S$.*

Proof. **1.(i)** Assume that A is an essential submodule of M. By Lemma 3.3, $r_M \ell_S(A) = A$. But $\ell_S(A) = 0$ and M is a quasi-dual module, we have $M = A$. If A is not essential submodule of M, then there exists a submodule B of M such that $A \oplus B$ is essential. So $M = A \oplus B$. Let π_B projection on B. Then $\pi_B(A) = 0$, and so $\pi_B \in \ell_S(A)$. Therefore $B = 0$.

(ii) Assume that A is not essential in M. Then there exist a non-zero submodule K of M such that $A \cap K = 0$. Hence $\ell_S(A \cap K) = S$. Since M is an IN-module, $\ell_S(A \cap K) = \ell_S(A) + \ell_S(K) = S$. Then $\ell_S(K) = S$. Therefore, $K = 0$.

2. **(i)** Similar to 1.(i).

(ii) Let $A+B = M$ for some submodule B of M. Then $\ell_S(A+B) = \ell_S(A) \cap \ell_S(B) = 0$. By assumption, $\ell_S(B) = 0$. By 1.(i), we have $B = M$.

(iii) Let $\ell_S(A) + X = S$ for $X \subseteq S$. Then $r_M(\ell_S(A) + X) = r_M(S) = 0$. But $r_M(\ell_S(A)+X) = r_M\ell_S(A) \cap r_M(X) = A \cap r_M(X) = 0$. Since A is an essential submodule of M, $r_M(X) = 0$. Then $X = S$ by 2.(i).

(iv) Let $I + J = S$ for some $J \subseteq S$. Then $0 = r_M(I + J) = r_M(I) \cap r_M(J)$. Since $r_M(I)$ is essential in M, $r_M(J) = 0$ and so $J = S$ by 2.(i). □

In Theorem 3.4, shown that every quasi-dual module is AP-injective. Following [9], we have $Z(R_R) = J(R)$, where $J(R)$ and $Z(M_R)$ denote Jacobson radical of R and the singular submodule of an R-module M, respectively. Therefore,

Theorem 3.12 *Let M be a quasi-dual module. Then $Z({}_SM) \subseteq Rad(M_R)$.*

Proof. If $x \in Z({}_SM)$, then xR is small in M by Proposition 3.12 and hence $x \in Rad(M_R)$. □

Question : Let M be a quasi-dual module. When $Z({}_SM) \subseteq Rad(M_R)$?

Acknowledgments

The author wishes to express his sincere gratitude to his Ph. D. advisor Prof. Abdullah Harmanci (Ankara) for his encouragement and direction. The author also thank the referee for valuable comments.

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Received 04.10.2004