Turk J Math
30 (2006) , 277 – 284.

The Radius of Starlikeness p-Valently Analytic Functions in the Unit Disc

Yaşar Polatoğlu, Metin Bolcal, Arzu Şen, and H. Esra Özkan

Abstract

In the present paper we shall give the radius of starlikeness for the classes of p-valent analytic functions in the unit disc $D = \{z \mid |z| < 1\}$.

Key Words: p-valent analytic functions, Radius of starlikeness, Radius of convexity.

1. Introduction

Let A_p the class of $f(z)$ normalized by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad p \in N = \{1, 2, 3, ...\} \tag{1.1}$$

which are analytic and p-valent in D. Further, let Ω be the family of functions $\omega(z)$ which are regular in D and satisfying the conditions $\omega(0) = 0,\ |\omega(z)| < 1$ for $z \in D$. Next, for arbitrary fixed numbers $A,\ B,\ -1 \leq B < A \leq 1$, denote by $P(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \tag{1.2}$$

which are regular in D such that $p(z) \in P(A, B)$ if and only if

2000 *AMS Mathematics Subject Classification:* 30C45.

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \tag{1.3}$$

for some function $\omega(z) \in \Omega$ and every $z \in D$. This class was introduced by W. Janowski [4].

Morever, let $S^*(A, B, b, p, q)$ denote the family of functions $f(z) \in A_p$ and such that $f(z)$ is in $S^*(A, B, b, p, q)$ if and only if

$$1 + \frac{1}{b}(z.\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) = p(z) \tag{1.4}$$

for some functions $p(z) \in P(A, B)$ and all $z \in D$, and $q \in N_0 = N \cup \{0\}$, whereas, as usual, $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order q, and

$$f^{(0)}(z) = f(z).$$

We note that by giving specific values to A, B, b, p and q, we obtain the subclasses of the class $S^*(A, B, b, p, q)$ which were considered earlier by various authors [1], [2], [5], [6], [9], and [10].

We shall need the following definition and lemma.

Definition 1.1 *The radius for the property $\Im$ in the class F is denoted by $R_{\Im}(F)$ and is the largest R such that every function in the class F has the property $\Im$ in each disc D_r for every $r < R$.*

2. New Results

In this section of this paper, we shall give the radius of starlikeness and the radius of convexity for the class $S^*(A, B, b, p, q)$.

Lemma 2.1 *Let $\omega(z)$ be regular in the unit disc with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, we can write $z_1\omega'(z_1) = k\omega(z_1)$, where k is real and $k \geq 1$.*

This lemma was proved by I. S. Jack [3].

Lemma 2.2 *The function*

$$w = \begin{cases} \frac{1+Az}{1+Bz} & , \ B \neq 0 \\ \\ 1 + Az & , \ B = 0 \end{cases}$$

maps $|z| = r$ *onto a disc centred at* $C(r)$*, and having the radius* $\rho(r)$*, viz.*

$$\begin{cases} C(r) = (\frac{(1-ABr^2)}{1-B^2r^2}, 0) & , \rho(r) = \frac{(A-B).r}{1-B^2r^2} \ , \ B \neq 0 \\ \\ C(r) = (1, 0) & , \rho(r) = |A|.r \ , \ B = 0. \end{cases}$$

Proof.

$$\begin{cases} \left. \begin{array}{l} w = \frac{1+Az}{1+Bz} \Leftrightarrow z = \frac{w-1}{A-Bw} \Leftrightarrow |z|^2 = r^2 = \frac{|w-1|^2}{|A-Bw|^2} \\ \Rightarrow u^2 + v^2 + \frac{(2ABr^2-2)}{1-B^2r^2}u + \frac{(1-A^2r^2)}{1-B^2r^2} = 0 \end{array} \right\} , \ B \neq 0 \\ \\ \left. \begin{array}{l} w = 1 + Az \Leftrightarrow z = \frac{w-1}{A} \Leftrightarrow |z|^2 = r^2 = \frac{|w-1|^2}{|A|^2} \\ \Rightarrow u^2 + v^2 - 2u + (1 - A^2r^2) = 0 \end{array} \right\} , \ B = 0. \end{cases} \tag{2.1}$$

Lemma follows from (2.1). □

Lemma 2.3 *The function*

$$w = \begin{cases} \frac{(A-B)z}{1+Bz} & , \ B \neq 0 \\ \\ Az & , \ B = 0 \end{cases}$$

maps $|z| = r$ *onto the disc centred at* $C(r)$*, and having radius* $\rho(r)$

$$\begin{cases} C(r) = (-\frac{B(A-B)r^2}{1-B^2r^2}, 0) & , \rho(r) = \frac{(A-B).r^2}{1-B^2r^2} \ , \ B \neq 0 \\ \\ C(r) = (0, 0) & , \rho(r) = |A|.r \ , \ B = 0. \end{cases}$$

Proof.

$$\begin{cases} \left. \begin{array}{l} w = \frac{(A-B)z}{1+Bz} \Leftrightarrow z = \frac{w}{(A-B)-Bw} \Leftrightarrow |z|^2 = r^2 = \frac{|w|^2}{|(A-B)-Bw|^2} \\ \Rightarrow u^2 + v^2 + \frac{(2B(A-B)r^2)}{1-B^2r^2} u + \frac{(A-B)^2 r^2}{1-B^2 r^2} = 0 \end{array} \right\} , \ B \neq 0 \\ \\ \left. \begin{array}{l} w = Az \Leftrightarrow z = \frac{w}{A} \Leftrightarrow |z|^2 = r^2 = \frac{|w|^2}{|A|^2} \\ \Rightarrow u^2 + v^2 - r^2 A^2 = 0 \end{array} \right\} , \ B = 0. \end{cases} \tag{2.2}$$

□

Lemma follows from (2.2).

Theorem 2.1 *Let $f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + ...$ be an analytic function in the unit disc D. If $f(z)$ satisfies*

$$\frac{1}{b}(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) \prec \begin{cases} \frac{(A-B)z}{1+Bz} = F_1(z) \ , \ B \neq 0 \\ \\ A.z = F_2(z) \quad , \ B = 0, \end{cases} \tag{2.3}$$

then $f(z) \in S^*(A, B, b, p, q)$, and this result is as sharp as the function $(\frac{1+Az}{1+Bz})$.

Proof. We define the function $w(z)$ by

$$\frac{f^{(q)}(z)}{z^{p-q}} = \begin{cases} (1 + Bw(z))^{\frac{b(A-B)}{B}} \ , \ B \neq 0 \\ \\ e^{Abw(z)} \quad , \ B = 0, \end{cases} \tag{2.4}$$

where $(1 + Bw(z))^{\frac{b(A-B)}{B}}$ and $e^{bAw(z)}$ have the values 1 at the origin.

Then $w(z)$ is analytic in D and $w(0) = 0$. If we take the logarithmic derivative from the equality (2.4) and after the brief calculations we get

$$\frac{1}{b}(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) \prec \begin{cases} \frac{(A-B)zw'(z)}{1+Bw(z)} \quad , \ B \neq 0 \\ \\ A.z.w'(z) \quad , \ B = 0. \end{cases} \tag{2.5}$$

Now it is easy to realize that subordination (2.3) is equivalent to $|w(z)| < 1$ for all $z \in D$. Indeed, assume the contrary: there exists a $z_1 \in D$ such that $|w(z_1)| = 1$. Then by the Lemma of I. S. Jack, $z_1 w'(z_1) = kw(z_1)$ and $k \geq 1$ for such $z_1 \in D$ (using Lemma 2.3), and we have

$$\frac{1}{b}(z_1 \frac{f^{(q+1)}(z_1)}{f^{(q)}(z_1)} - p + q) \prec \begin{cases} \frac{(A-B)kw(z_1)}{1+Bw(z_1)} = F_1(w(z_1)) \notin F_1(D) & , B \neq 0 \\ \\ A.k.w(z_1) = F_2(w(z_1)) \notin F_2(D) & , B = 0. \end{cases} \tag{2.6}$$

But this is a contradiction of (2.3) of this theorem; so our assumption is wrong, i.e., $|w(z)| < 1$ for all $z \in D$. By using condition (2.5), we get

$$1 + \frac{1}{b}(z \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) = \begin{cases} \frac{1+Aw(z)}{1+Bw(z)} & , B \neq 0 \\ \\ 1 + Aw(z) & , B = 0. \end{cases} \tag{2.7}$$

Then we obtain from equality (2.7)

$$1 + \frac{1}{b}(z \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) \prec \begin{cases} \frac{1+A.z}{1+B.z} & , B \neq 0 \\ \\ 1 + A.z & , B = 0. \end{cases} \tag{2.8}$$

From equality (2.8), we get $f(z) \in S^*(A, B, b, p, q)$. □

Corollary 2.1 *Let $f(z) \in S^*(A, B, b, p, q)$. Then $f(z)$ can be written in the form*

$$f_*^{(q)}(z) = \begin{cases} z^{p-q}(1 + Bw(z))^{\frac{b(A-B)}{B}} & , B \neq 0 \\ \\ z^{p-q}.e^{Abw(z)} & , B = 0, \end{cases}$$

Theorem 2.2 *The radius of starlikeness and the radius of convexity of the class $S^*(A,B,b,p,q)$ is*

$$R_{sc} = \frac{2(p-q)}{|b|(A-B)+\sqrt{|b|^2(A-B)^2-4(p-q)\left[(B^2-AB)Reb+(q-p)B^2\right]}}. \tag{2.9}$$

This radius is sharp because the extremal function is

$$f_*^{(q)}(z) = \begin{cases} z^{p-q}(1+Bw(z))^{\frac{b(A-B)}{B}} & , \ B \neq 0 \\ z^{p-q}.e^{Abw(z)} & , \ B = 0 \end{cases}.$$

Proof. By using Lemma 2.2. set of values $(z \cdot \frac{f^{(q+1)}(z)}{f^{(q)}(z)})$ is obtained which comprises the closed disc with centre $C(r)$ and the radius $\rho(r)$, where

$$C(r) = \frac{(p-q)-\left[(AB-B^2)b+(p-q)B^2\right].r^2}{1-B^2r^2},$$

$$\rho(r) = \frac{|b|(A-B)r}{1-B^2r^2}.$$

Therefore, by using the definition of the class $S^*(A,B,b,p,q)$, we have

$$\left| z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - C(r) \right| \leq \rho(r).$$

This gives

$$\mathrm{Re}(z.\frac{f^{(q+1)}(z)}{f^{(q)}(z)}) \geq \frac{(p-q)-|b|(A-B)r+\left[(B^2-AB)Reb+(q-p)B^2\right].r^2}{1-B^2.r^2}. \tag{2.10}$$

Hence for $r < R_{sc}$ the first hand side of the preceeding inequality is positive, implying that

$$R_{sc} = \frac{2(p-q)}{|b|(A-B)+\sqrt{|b|^2(A-B)^2-4(p-q)\left[(B^2-AB)Reb+(q-p)B^2\right]}} \tag{2.11}$$

Also note that inequality (2.9) becomes an equality for the function $f_*^{(q)}(z)$; it follows that

$$R_{sc} = \frac{2(p-q)}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 - 4(p-q)\left[(B^2-AB)Reb + (q-p)B^2\right]}}.$$

□

Remark 2.3 (i) By taking $q = 0,\ p = 1,\ A = 1,$ *and* $B = -1$ in (2.9), we obtain

$$R_s = \frac{1}{|b| + \sqrt{|b|^2 - 2Reb + 1}}.$$

This is the radius of starlikeness for the class of starlike functions of complex order which was obtained by M. A. Nasr and M. K. Aouf [6].

(ii) By setting , $q = 0$ in (2.9), then we obtain the radius of starlikeness for the class $S^*(A, B, b, p, 0)$

$$R_s = \frac{2p}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 - 4p\left[(B^2-AB)Reb + -pB^2\right]}}.$$

(iii) *By letting* $q = 1$ in (2.9), we also obtain the radius of convexity for the class $S^*(A, B, b, p, 1)$

$$R_c = \frac{2(p-1)}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 - 4(p-1)\left[(B^2-AB)Reb + (1-p)B^2\right]}}.$$

References

[1] Altintaş, O., Srivastava, H. M.: Some majorization problems associated with p-valently starlike and convex functions of complex order. East Asian Math. J. 17(2), 173-183(2001).

[2] Goodman, A. W.: Univalent Functions. Volume I and II. Mariner Publishing Company, Tampa Florida (1984).

[3] Jack, I. S.: Functions starlike and convex of order α. London. J. Math. Soc. 3(2), 469-471(1971).

[4] Janowski, W.: Some extremal problems for certain families of analytic functions. Annales Polon. Math XXIII., 297-326(1973).

[5] Libera, R. J.: Univalent α-spiral functions. Canad. J. Math. 19, 449-456(1969).

[6] Nasr, M. A., Aouf, M. K.: Starlike functions of complex order. Journal of Natural Sciences and Mathematics. 25(1), 1-12(1985).

[7] Robertson, M. S.: On the theory of univalent functions. Ann of Math 37, 374-408(1936).

[8] Spacek, L.: Prispevek k teorii funkci prostych. Casopis Pest. Math. Fys 62, 12-19(1933).

Yaşar POLATOĞLU, Metin BOLCAL,
Arzu ŞEN, H. Esra ÖZKAN
Department of Mathematics and Computer Sciences,
Faculty of Science and Arts
İstanbul Kültür University, İstanbul, TURKEY
e-mail: y.polatoglu@iku.edu.tr - e-mail: m.bolcal@iku.edu.tr
e-mail: a.sen@iku.edu.tr - e.ozkan@iku.edu.tr

Received 02.02.2005