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On Lifts of Paracomplex Structures

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Abstract

In this paper, we obtain vertical, complete and horizontal lifts of paracomplex geometric structures on paracomplex manifolds to its tangent bundle. Also, we obtain integrability on paracomplex tangent bundle.

Key Words: Paracomplex structure, paracomplex manifold, vertical lift, complete lift, horizontal lift, integrability.

1. Introduction

The method of lift has an important role in modern differentiable geometry. With the lift function it is possible to generalize to differentiable structures on any manifold to its extensions. Vertical, complete and horizontal lifts of functions, vector fields, 1-forms and other tensor fields defined on any manifold M to tangent manifold TM has been obtained by Yano and Ishihara [9, 11], Yano and Patterson [10]. Vertical, complete and horizontal lifts of geometric structures defined on any complex manifold M to its tangent bundle TM had been obtained by Tekkoyun [5, 6, 7] and Civelek [5, 6]. In this study, we obtain vertical, complete and horizontal lifts of differential geometric structures on paracomplex manifolds to its tangent bundles. Also, we conclude integrability conditions on paracomplex tangent bundle. Along this paper, all mappings and manifolds will be understood to be of class differentiable and the sum is taken over repeated indices.

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1.1. Paracomplex manifolds

An *almost product structure* J on manifold M of dimension $2m$ is a (1, 1) tensor field J on M such that $J^2 = I$. The pair (M, J) is called an *almost product manifold*. An *almost paracomplex manifold* is an almost product manifold (M, J) such that the two eigenbundles T^+M and T^-M associated to the eigenvalues +1 and -1 of J, respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even. Equivalently, a splitting of the tangent bundle TM of manifold M into the Whitney sum of two subbundles on $T^{\pm}M$ of the same fiber dimension is called an *almost paracomplex structure* on M. An almost paracomplex structure on manifold M may alternatively be defined as a G- structure on M with structural group $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$.

A *paracomplex manifold* is an almost paracomplex manifold (M, J) such that the G-structure defined by the tensor field J is integrable. $(x^\alpha, y^\alpha), 1 \le \alpha \le m$ is a real coordinate system on a neighborhood U of any point p of M, and $\{(\frac{\partial}{\partial x^\alpha})_p, (\frac{\partial}{\partial y^\alpha})_p\}$ and $\{(dx^\alpha)_p, (dy^\alpha)_p\}$ natural bases over $\mathbf{R}$ of the tangent space T_pM and the cotangent space T_p^*M of M, respectively. Then we can define

$$J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha},\ J(\frac{\partial}{\partial y^\alpha}) = \frac{\partial}{\partial x^\alpha}$$

and

$$J^*(dx^\alpha) = -dy^\alpha,\ J^*(dy^\alpha) = -dx^\alpha.$$

Let $z^\alpha = x^\alpha + \mathbf{j}y^\alpha$, $\overline{z}^\alpha = x^\alpha - \mathbf{j}y^\alpha$, $1 \le \alpha \le m$, $\mathbf{j}^2 = 1$, be a paracomplex local coordinate system on a neighborhood U of any point p of M. We define the vector fields as

$$(\frac{\partial}{\partial z^\alpha})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^\alpha})_p - \mathbf{j}(\frac{\partial}{\partial y^\alpha})_p\},\ (\frac{\partial}{\partial \overline{z}^\alpha})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^\alpha})_p + \mathbf{j}(\frac{\partial}{\partial y^\alpha})_p\},$$

and the dual covector fields as

$$(dz^\alpha)_p = (dx^\alpha)_p + \mathbf{j}(dy^\alpha)_p,\ (d\overline{z}^\alpha)_p = (dx^\alpha)_p - \mathbf{j}(dy^\alpha)_p,$$

which represent the bases of the tangent space T_pM and cotangent space T_p^*M of M, respectively. Then, using $\mathbf{j}^2 = 1$ we get

$$J(\frac{\partial}{\partial z^\alpha}) = -\mathbf{j}\frac{\partial}{\partial z^\alpha},\ J(\frac{\partial}{\partial \overline{z}^\alpha}) = \mathbf{j}\frac{\partial}{\partial \overline{z}^\alpha}.$$

The dual endomorphism J^* of the cotangent space T_p^*M at any point p of manifold M satisfies $J^{*2} = I$. Hence, using $\mathbf{j}^2 = 1$, it is found by

$$J^*(dz^\alpha) = -\mathbf{j}dz^\alpha,\ J^*(d\overline{z}^\alpha) = \mathbf{j}d\overline{z}^\alpha.$$

For each $p \in M$, let T_pM be set of tangent vectors $Z_p = Z^\alpha(\frac{\partial}{\partial z^\alpha})_p + \overline{Z}^\alpha(\frac{\partial}{\partial \overline{z}^\alpha})_p$, then TM is the union of these vector spaces. Thus, tangent bundle of a paracomplex manifold M is (TM, τ_M, M), where canonical projection τ_M is $\tau_M : TM \to M$ $(\tau_M(Z_p) = p)$ and, in addition this map are surjective submersion. Now, coordinates $\{z^\alpha, \overline{z}^\alpha, \acute{z}^\alpha, \acute{\overline{z}}^\alpha\}$, $1 \le \alpha \le m$, are taken into account as local coordinates for TM.

2. Lifts of Paracomplex Structures

2.1. Lifts of function

The *vertical lift* of paracomplex function $f \in \mathcal{F}(M)$ to TM is the function $f^v \in \mathcal{F}(TM)$ given by

$$f^v = f \circ \tau_M,$$

where $\tau_M : TM \to M$ canonical projection. We have $rang(f^v) = rang(f)$, since

$$f^v(Z_p) = f(\tau_M(Z_p)) = f(p), \quad \forall\, Z_p \in TM.$$

The *complete lift* of paracomplex function $f \in \mathcal{F}(M)$ to TM is the function $f^c \in \mathcal{F}(TM)$ given by

$$f^c = \acute{z}^\alpha(\frac{\partial f}{\partial z^\alpha})^v + \acute{\overline{z}}^\alpha(\frac{\partial f}{\partial \overline{z}^\alpha})^v,$$

where we denote by $(z^\alpha, \overline{z}^\alpha, \acute{z}^\alpha, \acute{\overline{z}}^\alpha)$ local coordinates of a chart-domain $TU \subset TM$. Furthermore, for $Z_p \in TM$ we have

$$f^c(Z_p) = \acute{z}^\alpha(Z_p)(\frac{\partial f}{\partial z^\alpha})^v(p) + \acute{\overline{z}}^\alpha(Z_p)(\frac{\partial f}{\partial \overline{z}^\alpha})^v(p).$$

The *horizontal lift* of $f \in \mathcal{F}(M)$ to TM is the function $f^h \in \mathcal{F}(TM)$ given by

$$f^h = f^c - \gamma(\nabla f), \quad (\gamma(\nabla f) = \nabla_\gamma f),$$

where ∇ is an affine linear connection on M with local components Γ_α^β, ∇f is gradient of f and γ is an operator given by

$$\gamma : \Im_s^r(M) \rightarrow \Im_{s-1}^r(TM).$$

Thus, $f^h = 0$ since

$$\nabla_\gamma f = \acute{z}^\alpha (\frac{\partial f}{\partial z^\alpha})^v + \acute{\overline{z}}^\alpha (\frac{\partial f}{\partial \overline{z}^\alpha})^v.$$

The vertical, complete and horizontal lifts of paracomplex functions obey the general properties

$$\begin{array}{ll} i) & (f.g)^v = f^v.g^v, (f+g)^v = f^v + g^v, \\ ii) & (f.g)^c = f^c.g^v + f^v.g^c, (f+g)^c = f^c + g^c, \\ iii) & (f.g)^h = 0, (f+g)^h = 0, \end{array}$$

for all $f, g \in \mathcal{F}(M)$.

2.2. Lifts of vector field

The *vertical lift* of a vector field $Z \in \chi(M)$ to TM is the vector field $Z^v \in \chi(TM)$ given by

$$Z^v(f^c) = (Zf)^v, \quad \forall f \in \mathcal{F}(M).$$

If $Z = Z^\alpha \frac{\partial}{\partial z^\alpha} + \overline{Z}^\alpha \frac{\partial}{\partial \overline{z}^\alpha}$ we have

$$Z^v = (Z^\alpha)^v \frac{\partial}{\partial \acute{z}^\alpha} + (\overline{Z}^\alpha)^v \frac{\partial}{\partial \acute{\overline{z}}^\alpha}, 1 \leq \alpha \leq m.$$

The *complete lift* of a vector field $Z \in \chi(M)$ to TM is the vector field $Z^c \in \chi(TM)$ given by

$$Z^c(f^c) = (Zf)^c, \quad \forall f \in \mathcal{F}(M).$$

Obviously, we obtain

$$Z^c = (Z^\alpha)^v \frac{\partial}{\partial z^\alpha} + (\overline{Z}^\alpha)^v \frac{\partial}{\partial \overline{z}^\alpha} + (Z^\alpha)^c \frac{\partial}{\partial \acute{z}^\alpha} + (\overline{Z}^\alpha)^c \frac{\partial}{\partial \acute{\overline{z}}^\alpha}, 1 \leq \alpha \leq m,$$

where $Z = Z^{\alpha}\frac{\partial}{\partial z^{\alpha}} + \overline{Z}^{\alpha}\frac{\partial}{\partial \overline{z}^{\alpha}}$.

The *horizontal lift* of a vector field $Z \in \chi(M)$ to TM is the vector field $Z^h \in \chi(TM)$ given by

$$Z^h f^v = (Zf)^v.$$

Obviously, we have

$$Z^h = Z^{\alpha}D_{\alpha} + \overline{Z}^{\alpha}\overline{D}_{\alpha}, 1 \leq \alpha, \beta \leq m$$

such that $D_{\alpha} = \frac{\partial}{\partial z^{\alpha}} - \Gamma^{\alpha}_{\beta}\frac{\partial}{\partial z'^{\alpha}}$ and $\overline{D}_{\alpha} = \frac{\partial}{\partial \overline{z}^{\alpha}} - \overline{\Gamma}^{\alpha}_{\beta}\frac{\partial}{\partial \overline{z}'^{\alpha}}, 1 \leq \alpha, \beta \leq m$, where $Z = Z^{\alpha}\frac{\partial}{\partial z^{\alpha}} + \overline{Z}^{\alpha}\frac{\partial}{\partial \overline{z}^{\alpha}}$.

The vertical, complete and horizontal lifts of paracomplex vector fields have the following general properties:

i) $(X+Y)^v = X^v + Y^v, (X+Y)^c = X^c + Y^c, (fX)^v = f^v X^v, (fX)^c = f^c X^v + f^v X^c$
ii) $X^v(f^v) = 0, X^c(f^v) = X^v(f^c) = (Xf)^v, X^c(f^c) = (Xf)^c,$
iii) $[X^v, Y^v] = 0, [X^v, Y^c] = [X^c, Y^v] = [X, Y]^v, [X^c, Y^c] = [X, Y]^c,$
iv) $\chi(U) = Sp\left\{\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \overline{z}^{\alpha}}\right\}, \chi(TU) = Sp\left\{\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \overline{z}^{\alpha}}, \frac{\partial}{\partial z'^{\alpha}}, \frac{\partial}{\partial \overline{z}'^{\alpha}}\right\}$
v) $(\frac{\partial}{\partial z^{\alpha}})^c = \frac{\partial}{\partial z^{\alpha}}, (\frac{\partial}{\partial \overline{z}^{\alpha}})^c = \frac{\partial}{\partial \overline{z}^{\alpha}}, (\frac{\partial}{\partial z^{\alpha}})^v = \frac{\partial}{\partial z'^{\alpha}}, (\frac{\partial}{\partial \overline{z}^{\alpha}})^v = \frac{\partial}{\partial \overline{z}'^{\alpha}},$
vi) $(Z+W)^h = Z^h + W^h, Z^h(f^v) = (Zf)^v, (\frac{\partial}{\partial z^{\alpha}})^h = D_{\alpha}, (\frac{\partial}{\partial \overline{z}^{\alpha}})^h = \overline{D}_{\alpha},$

for all $f \in \mathcal{F}(M)$, $X, Y, Z, W \in \chi(M)$. Where $1 \leq \alpha \leq m$ and $[,]$ is Lie bracket. The set of local vector fields $\left\{D_{\alpha}, \overline{D}_{\alpha}, V_{\alpha} = \frac{\partial}{\partial z'^{\alpha}}\overline{V}_{\alpha} = \frac{\partial}{\partial \overline{z}'^{\alpha}}\right\}$ is called an *adapted frame* to ∇.

2.3. Lifts of 1-form

The *vertical lift* of a 1-form $\omega \in \chi^*(M)$ to TM is the 1-form $\omega^v \in \chi^*(TM)$ given by

$$\omega^v(Z^c) = (\omega Z)^v, \quad \forall Z \in \chi(M).$$

The vertical lift of the paracomplex 1-form ω given by

$$\omega = \omega_{\alpha}dz^{\alpha} + \overline{\omega}_{\alpha}d\overline{z}^{\alpha}$$

is

$$\omega^v = (\omega_{\alpha})^v dz^{\alpha} + (\overline{\omega}_{\alpha})^v d\overline{z}^{\alpha}, 1 \leq \alpha \leq m.$$

The *complete lift* of a 1-form $\omega \in \chi^*(M)$ to TM is the 1-form $\omega^c \in \chi^*(TM)$ given by

$$\omega^c(Z^c) = (\omega Z)^c, \quad \forall Z \in \chi(M).$$

If $\omega = \omega_\alpha dz^\alpha + \overline{\omega}_\alpha d\overline{z}^\alpha$, we calculate

$$\omega^c = (\omega_\alpha)^c dz^\alpha + (\overline{\omega}_\alpha)^c d\overline{z}^\alpha + (\omega_\alpha)^v d\acute{z}^\alpha + (\overline{\omega}_\alpha)^v d\acute{\overline{z}}^\alpha, 1 \leq \alpha \leq m.$$

The *horizontal lift* of a 1-form $\omega \in \chi^*(M)$ to TM is the 1-form $\omega^h \in \chi^*(TM)$ given by

$$\omega^h(Z^h) = 0, \omega^h(Z^v) = (\omega Z)^v.$$

If $\omega = \omega_\alpha dz^\alpha + \overline{\omega}_\alpha d\overline{z}^\alpha$ we obtain

$$\omega^h = \omega_\alpha \eta^\alpha + \overline{\omega}_\alpha \overline{\eta}^\alpha, 1 \leq \alpha, \beta \leq m$$

such that $\eta^\alpha = \overline{d}\acute{z}^\alpha + \Gamma^\alpha_\beta \overline{d} z^\alpha$, $\overline{\eta}^\alpha = \overline{d}\acute{\overline{z}}^\alpha + \overline{\Gamma}^\alpha_\beta \overline{d}\overline{z}^\alpha$.

The properties of vertical, complete and horizontal lifts of paracomplex 1-forms are

$i)$ $(\omega + \theta)^v = \omega^v + \theta^v, (\omega + \theta)^c = \omega^c + \theta^c, (f\omega)^v = f^v \omega^v, (f\omega)^v = f^v \omega^v,$

$ii)$ $\omega^v(Z^v) = 0, \omega^c(Z^v) = \omega^v(Z^c) = (\omega Z)^v, \omega^c(Z^c) = (\omega Z)^c,$

$iii)$ $\chi^*(U) = Sp\{dz^\alpha, d\overline{z}^\alpha\}, \chi^*(TU) = Sp\left\{dz^\alpha, d\overline{z}^\alpha, d\acute{z}^\alpha, d\acute{\overline{z}}^\alpha\right\},$

$iv)$ $(dz^\alpha)^c = \overline{d}\acute{z}^\alpha, (d\overline{z}^\alpha)^c = \overline{d}\acute{\overline{z}}^\alpha, (dz^\alpha)^v = \overline{d}z^\alpha, (d\overline{z}^\alpha)^v = \overline{d}\overline{z}^\alpha,$

$v)$ $(\omega + \theta)^h = \omega^h + \theta^h, \omega^h(Z^h) = 0, \omega^h(Z^v) = (\omega Z)^v, (dz^\alpha)^h = \eta^\alpha, (d\overline{z}^\alpha)^h = \overline{\eta}^\alpha,$

for all $f \in \mathcal{F}(M)$, $Z \in \chi(M)$, $\omega, \theta \in \chi^*(M)$, where $1 \leq \alpha \leq m$ and $\overline{d}$ denotes the differential operator on TM. The dual coframe $\left\{\theta^\alpha = dz^\alpha, \overline{\theta}^\alpha = d\overline{z}^\alpha, \eta^\alpha, \overline{\eta}^\alpha\right\}$ is called an *adapted coframe* to ∇.

2.4. Lifts of tensor fields of type (1,1)

The *vertical lift* of a paracomplex tensor field of type (1,1) $F \in \Im^1_1(M)$ to tangent bundle TM of a paracomplex manifold M is the tensor field $F^v \in \Im^1_1(TM)$ given by

$$F^v(Z^c) = (FZ)^v, \quad \forall Z \in \chi(M).$$

Clearly, we have

$$F^{v}=(F_{\alpha}^{\beta})^{v}\frac{\partial}{\partial z^{\beta}}\otimes dz^{\alpha}+(\overline{F}_{\alpha}^{\beta})^{v}\frac{\partial}{\partial \overline{z}^{\beta}}\otimes d\overline{z}^{\alpha},1\leq\alpha,\beta\leq m,$$

where $F=F_{\alpha}^{\beta}\frac{\partial}{\partial z^{\beta}}\otimes dz^{\alpha}+\overline{F}_{\alpha}^{\beta}\frac{\partial}{\partial \overline{z}^{\beta}}\otimes d\overline{z}^{\alpha}$.

The *complete lift* of a paracomplex tensor field of type (1,1) $F\in\Im_{1}^{1}(M)$ to TM is the tensor field $F^{c}\in\Im_{1}^{1}(TM)$ given by

$$F^{c}(Z^{c})=(FZ)^{c},\quad\forall Z\in\chi(M).$$

The complete lift of the paracomplex tensor field of type (1,1) F is

$$\begin{aligned}F^{c}=&(F_{\alpha}^{\beta})^{v}\tfrac{\partial}{\partial z^{\beta}}\otimes dz^{\alpha}+(F_{\alpha}^{\beta})^{c}\tfrac{\partial}{\partial \acute{z}^{\beta}}\otimes dz^{\alpha}+(F_{\alpha}^{\beta})^{v}\tfrac{\partial}{\partial \acute{z}^{\beta}}\otimes d\acute{z}^{\alpha}\\&+(\overline{F}_{\alpha}^{\beta})^{v}\tfrac{\partial}{\partial \overline{z}^{\beta}}\otimes d\overline{z}^{\alpha}+(\overline{F}_{\alpha}^{\beta})^{c}\tfrac{\partial}{\partial \acute{\overline{z}}^{\beta}}\otimes d\overline{z}^{\alpha}+(\overline{F}_{\alpha}^{\beta})^{v}\tfrac{\partial}{\partial \acute{\overline{z}}^{\beta}}\otimes d\acute{\overline{z}}^{\alpha}.\end{aligned}$$

The *horizontal lift* of a paracomplex tensor field of type (1,1) $F\in\Im_{1}^{1}(M)$ to TM is the tensor field $F^{h}\in\Im_{1}^{1}(TM)$ given by

$$F^{h}(Z^{h})=(FZ)^{h},F^{h}(Z^{v})=(F(Z))^{v}.$$

The horizontal lift of the paracomplex tensor field of type (1,1) F is

$$\begin{aligned}F^{h}=&F_{\alpha}^{\beta}\tfrac{\partial}{\partial z^{\beta}}\otimes dz^{\alpha}+(\Gamma_{\beta}^{\alpha}F_{\alpha}^{\beta}-\Gamma_{\alpha}^{\beta}F_{\beta}^{\alpha})\tfrac{\partial}{\partial \acute{z}^{\beta}}\otimes dz^{\alpha}+F_{\alpha}^{\beta}\tfrac{\partial}{\partial \acute{z}^{\beta}}\otimes d\acute{z}^{\alpha}\\&+\overline{F}_{\alpha}^{\beta}\tfrac{\partial}{\partial \overline{z}^{\beta}}\otimes d\overline{z}^{\alpha}+(\overline{\Gamma}_{\beta}^{\alpha}\overline{F}_{\alpha}^{\beta}-\overline{\Gamma}_{\alpha}^{\beta}\overline{F}_{\beta}^{\alpha})\tfrac{\partial}{\partial \acute{\overline{z}}^{\beta}}\otimes d\overline{z}^{\alpha}+\overline{F}_{\alpha}^{\beta}\tfrac{\partial}{\partial \acute{\overline{z}}^{\beta}}\otimes d\acute{\overline{z}}^{\alpha},\end{aligned}$$

where $1\leq\alpha,\beta\leq m$.

With respect to the adapted frame, we have

$$F^{h}=F_{\alpha}^{\beta}D_{\beta}\otimes\theta^{\alpha}+F_{\alpha}^{\beta}V_{\beta}\otimes\eta^{\alpha}+\overline{F}_{\alpha}^{\beta}\overline{D}_{\beta}\otimes\overline{\theta}^{\alpha}+\overline{F}_{\alpha}^{\beta}\overline{V}_{\beta}\otimes\overline{\eta}^{\alpha}.$$

The general properties of vertical, complete and horizontal lifts of paracomplex tensor fields of type (1,1) are

$$\begin{aligned}&i)&&F^{v}(Z^{v})=0,\\&ii)&&F^{v}(Z^{c})=F^{c}(Z^{v})=(F(Z))^{v},\\&iii)&&F^{c}(Z^{c})=(F(Z))^{c},\\&iv)&&F^{h}(Z^{h})=(FZ)^{h},\\&v)&&F^{h}(Z^{v})=(F(Z))^{v},\end{aligned}$$

for all $F \in \Im_1^1(M)$ and $Z \in \chi(M)$.

2.5. Lifts of tensor fields of type (0,2)

The *vertical lift* of a paracomplex tensor field of type (0,2) $g \in \Im_2^0(M)$ to TM of a paracomplex manifold M is the tensor field $g^v \in \Im_2^0(TM)$ given by

$$g^v(F^c Z^c, W^c) + g^v(Z^c, F^c W^c) = 0, \forall Z, W \in \chi(M).$$

The vertical lift of a paracomplex tensor field g given by

$$g = g_{\overline{\alpha}\beta} d\overline{z}^\alpha \otimes dz^\beta + g_{\alpha\overline{\beta}} dz^\alpha \otimes d\overline{z}^\beta$$

is

$$g^v = (g_{\overline{\alpha}\beta})^v d\overline{z}^\alpha \otimes dz^\beta + (g_{\alpha\overline{\beta}})^v dz^\alpha \otimes d\overline{z}^\beta,\ 1 \leq \alpha, \beta \leq m.$$

The *complete lift* of a paracomplex tensor field of type (0,2) $g \in \Im_2^0(M)$ to TM is the tensor field $g^c \in \Im_2^0(TM)$ given by

$$g^c(F^c Z^c, W^c) + g^c(Z^c, F^c W^c) = 0, \forall Z, W \in \chi(M).$$

For the complete lift of a paracomplex tensor field g, we obtain

$$\begin{aligned} g^c = {} & (g_{\overline{\alpha}\beta})^c d\overline{z}^\alpha \otimes dz^\beta + (g_{\overline{\alpha}\beta})^v d\acute{\overline{z}}^\alpha \otimes dz^\beta + (g_{\overline{\alpha}\beta})^v d\overline{z}^\alpha \otimes d\acute{z}^\beta \\ & + (g_{\alpha\overline{\beta}})^c dz^\alpha \otimes d\overline{z}^\beta + (g_{\alpha\overline{\beta}})^v d\acute{z}^\alpha \otimes d\overline{z}^\beta + (g_{\alpha\overline{\beta}})^v dz^\alpha \otimes d\acute{\overline{z}}^\beta. \end{aligned}$$

The *horizontal lift* of a paracomplex tensor field of type (0,2) $g \in \Im_2^0(M)$ to TM is the tensor field $g^h \in \Im_2^0(TM)$ given by

$$g^h(F^v Z^v, W^v) = 0, g^h(Z^v, F^h W^h) = 0, g^h(F^h Z^h, W^h) + g^h(Z^h, F^h W^h) = 0,$$

for all $Z, W \in \chi(M)$.

We obtain

$$g^h = g_{\overline{\alpha}\beta} \overline{\theta}^\alpha \otimes \eta^\beta + g_{\alpha\overline{\beta}} \theta^\alpha \otimes \overline{\eta}^\beta, 1 \leq \alpha, \beta \leq m,$$

for the horizontal lift of paracomplex tensor field of type (0,2) g with respect to the adapted coframe.

The general properties of vertical, complete and horizontal lifts of paracomplex tensor fields of type (0,2) are

$$
\begin{array}{ll}
i) & g^v(F^cZ^c, W^c) + g^v(Z^c, F^cW^c) = 0 \\
ii) & g^c(F^cZ^c, W^c) + g^c(Z^c, F^cW^c) = 0, \\
iii) & g^h(F^vZ^v, W^v) = 0, \\
iv) & g^h(Z^v, F^hW^h) = 0, \\
v) & g^h(F^hZ^h, W^h) + g^h(Z^h, F^hW^h) = 0,
\end{array}
$$

for all $g \in \Im_2^0(M)$ and $Z, W \in \chi(M)$.

2.6. Lift of para-Hermitian metric

The *vertical lift* of paracomplex structure J given by $J = \mathbf{j}\frac{\partial}{\partial z^\alpha} \otimes dz^\alpha - \mathbf{j}\frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha$ is

$$J^v = \mathbf{j}\frac{\partial}{\partial \acute{z}^\alpha} \otimes dz^\alpha - \mathbf{j}\frac{\partial}{\partial \overline{\acute{z}}^\alpha} \otimes d\overline{z}^\alpha.$$

Since $(J^v)^2 = 0$, J^v is an almost tangent structure for tangent bundle TM. The *complete lift* of J being a paracomplex tensor field of type (1,1) is

$$J^c = \mathbf{j}\frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + \mathbf{j}\frac{\partial}{\partial \acute{z}^\alpha} \otimes d\acute{z}^\alpha - \mathbf{j}\frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha - \mathbf{j}\frac{\partial}{\partial \overline{\acute{z}}^\alpha} \otimes d\overline{\acute{z}}^\alpha.$$

Because of $(J^c)^2 = I$, J^c is an almost paracomplex structure for tangent bundle TM. The *horizontal lift* of paracomplex structure J is

$$
\begin{array}{l}
J^h = \mathbf{j}\frac{\partial}{\partial z^\alpha} \otimes d\acute{z}^\alpha + \mathbf{j}\Gamma_\alpha^\beta\frac{\partial}{\partial z^\alpha} \otimes dz^\alpha - \mathbf{j}\Gamma_\alpha^\beta\frac{\partial}{\partial \acute{z}^\alpha} \otimes d\acute{z}^\alpha - \mathbf{j}\delta_\alpha^\beta\frac{\partial}{\partial \acute{z}^\alpha} \otimes dz^\alpha \\
-\mathbf{j}\frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{\acute{z}}^\alpha - \mathbf{j}\Gamma_\alpha^\beta\frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha + \mathbf{j}\Gamma_\alpha^\beta\frac{\partial}{\partial \overline{\acute{z}}^\alpha} \otimes d\overline{\acute{z}}^\alpha + \mathbf{j}\delta_\alpha^\beta\frac{\partial}{\partial \overline{\acute{z}}^\alpha} \otimes d\overline{z}^\alpha,
\end{array}
$$

where δ_α^β is the Kronecker delta. On account of $(J^h)^2 = 0$, J^h is an almost tangent structure for TM.

Suppose that let g is an almost para-Hermitian metric on a paracomplex manifold M. Thus, from vertical and complete lift properties we may obtain the equality

$$g^v(Z^c, W^c) + g^v(Z^c, J^cW^c) = 0.$$

For almost an para-Hermitian metric g defined on $M,$ since g^v is an almost para-Hermitian metric on tangent bundle $TM,$ the metric tensor g^v is called a *vertical lift* of g on M. The triple (TM, g^v, J^c) is called a *vertical lift* of almost para-Hermitian manifold (M, g, J).

Similarly, from vertical and complete lift properties, it is possible to obtain equality

$$g^c(Z^c, W^c) + g^c(Z^c, J^c W^c) = 0.$$

Because g^c is an almost para-Hermitian metric on $TM,$ we call g^c *complete lift* of g on M. We say that the triple (TM, g^c, J^c) is the *complete lift* of almost para-Hermitian manifold (M, g, J).

On account of g^h being an almost para-Hermitian metric on tangent bundle $TM,$ the metric tensor g^h is called a *horizontal lift* of g on M. From lift properties, one can easily obtain the equality

$$g^h(J^v Z^v, W^v) = 0, g^h(Z^v, J^h W^h) = 0, g^h(J^h Z^h, W^h) + g^h(Z^h, J^h W^h) = 0,$$

for all $Z, W \in \chi(M)$.The *horizontal lift* of almost para-Hermitian manifold (M, g, J) is the triple (TM, g^h, J^h).

2.7. Lift of para-Kähler metric

Let M be a paracomplex manifold and TM its tangent bundle. The *vertical lift* of almost para-Kähler form Φ on a paracomplex manifold M to TM is the paracomplex tensor field Φ^v defined by

$$\Phi^v(Z^c, W^c) = g^v(Z^c, J^c W^c), \text{ for all } Z, W \in \chi(M).$$

The *complete lift* of almost para-Kähler form Φ on M to TM is the paracomplex tensor field Φ^c defined by

$$\Phi^c(Z^c, W^c) = g^c(Z^c, J^c W^c), \text{ for all } Z, W \in \chi(M).$$

The *horizontal lift* of almost para-Kähler form Φ on a paracomplex manifold M to TM is the paracomplex tensor field Φ^h given by

$$\Phi^h(Z^h, W^h) = g^h(Z^h, J^h W^h) = (g(Z, JW))^h$$

for all $g \in \Im_2^0(M)$ and $Z, W \in \chi(M)$.

Vertical, complete and horizontal lifts of almost para-Kähler form Φ have the generic properties

$$
\begin{array}{ll}
i) & \Phi^v(Z^c, W^c) = g^v(Z^c, J^c W^c) \\
ii) & \Phi^c(Z^c, W^c) = g^c(Z^c, J^c W^c), \\
iii) & \Phi^h(Z^v, W^c) = g^h(Z^v, J^c W^c) = g^h(Z^c, J^c W^v) = (g(Z, JW))^v, \\
iv) & \Phi^h(Z^v, W^h) = g^h(Z^v, J^h W^h) = g^h(Z^h, J^c W^v),
\end{array}
$$

for all $Z, W \in \chi(M)$.

An almost para-Hermitian manifold (resp. para-Hermitian manifold) (M, g, J) is said to be an *almost para-Kähler manifold*(resp. *para- Kähler manifold*) if $d\Phi = 0$. Similar to the above definition, we can give as follows. An almost para-Hermitian manifold (resp. para-Hermitian manifold) (TM, g^c, J^c) is called an *almost para-Kähler manifold (*resp. *para- Kähler manifold)* if $d\Phi^c = 0$.

3. Integrability on Paracomplex Tangent Bundle

Let M be almost paracomplex manifold and TM its tangent bundle. Let Z, W be vector fields and J almost paracomplex structure on M. The Nijenhuis tensor N_J endowed with paracomplex structure J on M is defined as

$$N_J(Z, W) = -J\,[JZ, W] - J\,[Z, JW] + [JZ, JW]\,.$$

Then vertical and complete lifts of the tensor fields given above are given by Z^c, W^c vector fields and by J^v almost tangent structure on TM. The *vertical lift* of N_J is the tensor $N^v_{J^v}$ being Nijenhuis tensor of J^v and given by

$$N^v_{J^v}(Z^c, W^c) = -J^v\,[J^v Z^c, W^c] - J^v\,[Z^c, J^v W^c] + [J^v Z^c, J^v W^c]$$

where N_J is a Nijenhuis tensor of almost paracomplex structure J on M. Let J^c be almost paracomplex structure on TM. The *complete lift* of N_J is the tensor $N^c_{J^c}$ being Nijenhuis tensor of J^c and given by

$$N^c_{J^c}(Z^c, W^c) = [Z^c, W^c] - J^c\,[J^c Z^c, W^c] - J^c\,[Z^c, J^c W^c] + [J^c Z^c, J^c W^c]$$

where N_J is a Nijenhuis tensor of J on M. Let J^h be an almost tangent structure on TM. The *horizontal lift* of N_J is the tensor $N^h_{J^h}$ being Nijenhuis tensor of J^h and is given by

$$N^h_{J^h}(Z^h, W^h) = \left[Z^h, W^h\right] - J^h\left[J^h Z^h, W^h\right] - J^h\left[Z^h, J^h W^h\right] + \left[J^h Z^h, J^h W^h\right]$$

where N_J is a Nijenhuis tensor of almost paracomplex structure J on M.

Lemma 3.1 *Let M be almost para-Hermitian manifold and TM its tangent bundle. Given the $\chi(M)$ Lie algebra and N^c Nijenhuis tensors of J^c on TM, then TM is paracomplex manifold if and only if $N^c_{J^c}(Z^c, W^c) = 0,\ \ Z, W \in \chi(M)$.*

Proof. Let TM be a paracomplex manifold. Then, we have

$$\begin{aligned} N^c_{J^c}(Z^c, W^c) &= [Z^c, W^c] - J^c\,[J^c Z^c, W^c] - J^c\,[Z^c, J^c W^c] + [J^c Z^c, J^c W^c] \\ &= \bigtriangledown^c_{Z^c} W^c - \bigtriangledown^c_{W^c} Z^c - J^c\,\bigtriangledown^c_{J^c Z^c} W^c + J^c\,\bigtriangledown^c_{W^c}\,J^c Z^c \\ &\quad -J^c\,\bigtriangledown^c_{Z^c}\,J^c W^c + J^c\,\bigtriangledown^c_{J^c W^c}\,Z^c + \bigtriangledown^c_{J^c Z^c} J^c W^c \\ &\quad -\bigtriangledown^c_{J^c W^c}\,J^c Z^c. \end{aligned}$$

Using $(J^c)^2 = I$, we get

$$\begin{aligned} N^c_{J^c}(Z^c, W^c) &= J^c(J^c \bigtriangledown^c_{Z^c}) W^c - J^c(J^c \bigtriangledown^c_{W^c}) Z^c - (J^c \bigtriangledown^c_{J^c Z^c}) W^c \\ &\quad + J^c(\bigtriangledown^c_{W^c} J^c) Z^c - J^c(\bigtriangledown^c_{Z^c} J^c) W^c + (J^c \bigtriangledown^c_{J^c W^c}) Z^c \\ &\quad + (\bigtriangledown^c_{J^c Z^c} J^c) W^c - (\bigtriangledown^c_{J^c W^c} J^c) Z^c \\ &= -J^c\left[\nabla^c_{Z^c}, J^c\right] W^c + J^c\left[\nabla^c_{W^c}, J^c\right] Z^c + \left[\nabla^c_{J^c Z^c}, J^c\right] W^c \\ &\quad - \left[\nabla^c_{J^c W^c}, J^c\right] Z^c. \end{aligned}$$

$\left[\nabla^c_{J^c Z^c}, J^c\right] W^c = J^c\left[\nabla^c_{Z^c}, J^c\right] W^c$ if and only if $N^c_{J^c}(Z^c, W^c) = 0$. Thus, the proof is complete. □

Hence one can easily obtain the following paracomplex extension of Newlander-Niremberg theorem [2].

Theorem 3.1 *An almost paracomplex structure J^c is integrable if and only if its Nijenhuis tensor $N^c_{J^c}$ vanishes.*

Proposition 3.2 *Let M be almost para-Hermitian manifold and TM its tangent bundle. Given $\bigtriangledown^c_{Z^c}$ covariant derivative, Φ^c para-Kähler form and N^c Nijenhuis tensor of J^c on TM, then one obtains the equality*

$$2g^c((\bigtriangledown^c_{X^c}J^c)Y^c,Z^c)+3d\Phi^c(X^c,J^cY^c,Z^c)+3d\Phi^c(X^c,Y^c,Z^c)+g^c(N^c(Y^c,Z^c),J^cX^c)=0,$$

where $X,Y,Z\in\chi(M)$.

Proof. We get

$$2g^c((\bigtriangledown^c_{X^c}J^c)Y^c,Z^c)=2g^c(\bigtriangledown^c_{X^c}(JY)^c,Z^c)+2g^c(\bigtriangledown^c_{X^c}Y^c,J^cZ^c).$$

Then we have the equalities

$$\begin{aligned}2g^c(\bigtriangledown^c_{X^c}(JY)^c,Z^c) &= X^cg^c(J^cY^c,Z^c)+J^cY^cg^c(X^c,Z^c)-Z^cg^c(X^c,J^cY^c)\\ &\quad +g^c([X^c,J^cY^c],Z^c)+g^c([Z^c,X^c],J^cY^c)+g^c(X^c,[Z^c,J^cY^c]),\end{aligned}$$

and

$$\begin{aligned}2g^c(\bigtriangledown^c_{X^c}Y^c,J^cZ^c) &= X^cg^c(Y^c,J^cZ^c)+Y^cg^c(X^c,J^cZ^c)-J^cZ^cg^c(X^c,Y^c)\\ &\quad +g^c([X^c,Y^c],J^cZ^c)+g^c([J^cZ^c,X^c],Y^c)+g^c(X^c,[J^cZ^c,Y^c]).\end{aligned}$$

On the other hand we obtain

$$\begin{aligned}3d\Phi^c(X^c,Y^c,Z^c) &= X^c\Phi^c(Y^c,Z^c)+Y^c\Phi^c(Z^c,X^c)+Z^c\Phi^c(X^c,Y^c)\\ &\quad -\Phi^c([X^c,Y^c],Z^c)-\Phi^c([Y^c,Z^c],X^c)-\Phi^c([Z^c,X^c],Y^c),\end{aligned}$$

and

$$\begin{aligned}3d\Phi^c(X^c,J^cY^c,J^cZ^c) &= X^c\Phi^c(J^cY^c,J^cZ^c)+J^cY^c\Phi^c(J^cZ^c,X^c)\\ &\quad +J^cZ^c\Phi^c(X^c,J^cY^c)-\Phi^c([X^c,J^cY^c],J^cZ^c)\\ &\quad -\Phi^c([J^cY^c,J^cZ^c],X^c)-\Phi^c([J^cZ^c,X^c],J^cY^c),\end{aligned}$$

and also

$$\begin{aligned} g^c(N^c(Y^c, Z^c), J^cX^c) &= \Phi^c([Y^c, Z^c], X^c) - \Phi^c(J^c[J^cY^c, Z^c], X^c) \\ &\quad -\Phi^c(J^c[Y^c, J^cZ^c], X^c) + \Phi^c([J^cY^c, J^cZ^c], X^c). \end{aligned}$$

□

From the above equalities, proof finishes.

References

[1] Brickell, F., Clark, R.S.: Differentiable Manifolds, VRN Company, London, 1970

[2] De Leon M., Rodrigues P.R.: Methods of Differential Geometry in Analytical Mechanics, North-Hol. Math. St.,152, Elsevier Sc. Pub. Com., Inc.,Amsterdam, 1989

[3] Dodson, C.T.J., Poston, T.: Tensor Geometry, Pitman XIII., 598, London, 1979

[4] Etoya, J.J.: On a Complete Lifting of Derivations, Tensor, Vol. 38, pp.169-178(1982)

[5] Tekkoyun M., Civelek Ş.: First Order Lifts of Complex Structures, Algebras Groups and Geometries (AGG), Vol.19, No.3, pp.373-382(2002)

[6] Tekkoyun M., Civelek Ş.: On Lifts of Structures on Complex Manifolds, Differential Geometry-Dynamics Systems Vol.5, No.1, pp.59-64(2003)

[7] Tekkoyun M.: On Horizontal Lifts of Complex Structures, Hadronic Journal Supplement (HJS), Vol.18, No:4, pp. 411-424(2003)

[8] Cruceanu V., Gadea, P.M., Masqué, J.M.: Para-Hermitian and Para- Kähler Manifolds, Supported by the commission of the European Communities Action for Cooperation in Sciences and Technology with Central Eastern European Countries n. ERB3510PL920841

[9] Yano, K., Kon, M.: Structures on Manifolds, Volume 3, Singapore, 1984

[10] Yano, K., Patterson, Y.: Horizontal Lifts of a Manifold to its Cotangent Bundle, Jour. Math. Soc. Japan., Vol. 19, pp.185-198(1967)

[11] Yano, K., Ishihara, S.: Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 1973

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