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Second-Order Nonlinear Three Point Boundary-Value Problems on Time Scales

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Abstract

We consider a second order three point boundary value problem for dynamic equations on time scales and establish criteria for the existence of at least two positive solutions of an eigenvalue problem by an application of a fixed point theorem in cones. Existence result for non-eigenvalue problem is also given by the monotone method.

Key Words: Dynamic equations, cone, positive solutions, upper and lower solutions.

1. Introduction

We are concerned with the three point boundary value problem

$$-y^{\triangle\nabla}(t) = \lambda f(t, y(t)), \qquad t \in [a, b], \tag{1.1}$$

$$\alpha y(\rho(a)) - \beta y^{\triangle}(\rho(a)) = 0, \quad y(\sigma(b)) - \delta y(\eta) = 0, \tag{1.2}$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$, $\lambda > 0$, $0 < \delta < 1$, $\eta \in (\rho(a), \sigma(b))$. We likewise assume that $f : [a, b] \times R \to R$ is left-dense continuous.

Throughout this paper we let T be any time scale (nonempty closed subset of R) and $[a, b]$ be subset of T such that $[a, b] = \{t \in T : a \leq t \leq b\}$.

Related works on differential equations, difference equations and dynamic equations on time scale include [1–6, 9–15]. Three point boundary value problems on time scales was studied in the references [2, 3, 6, 12], in this study we also generalized the boundary condition. Some preliminary definitions and dynamic theorems on time scales can also be found in the books [7, 8] which are useful references for the time scales calculus.

Let $G(t,s)$ be the Green's function for the BVP. The

$$Ly \equiv -y^{\Delta\nabla}(t) = 0, \quad t \in [a,b], \tag{1.3}$$

$$\alpha y(\rho(a)) - \beta y^{\triangle}(\rho(a)) = 0, \quad y(\sigma(b)) - \delta y(\eta) = 0. \tag{1.4}$$

$G(t,s)$ is given by

$$G(t,s) = \frac{1}{D}\begin{cases} G_1(t,s), & \rho(a) < s \leq \eta; \\ G_2(t,s), & \eta < s < \sigma(b), \end{cases}$$

where

$$G_1(t,s) = \begin{cases} (\beta + \alpha(s-\rho(a)))(\sigma(b) - \delta\eta - t(1-\delta)), & s < t \\ (\beta + \alpha(t-\rho(a)))(\sigma(b) - \delta\eta - s(1-\delta)), & s \geq t \end{cases}$$

$$G_2(t,s) = \begin{cases} (\beta + \alpha(s-\rho(a)))(\sigma(b) - t) + (t-s)(\eta + \beta - \alpha\rho(a))\delta, & s \leq t \\ (\beta + \alpha(t-\rho(a)))(\sigma(b) - s), & s > t \end{cases}$$

and $D = \beta(1-\delta) + \alpha(\sigma(b) - \rho(a) - \delta(\eta - \rho(a)))$.

Lemma 1.1 *For $h(t) \in C[\rho(a), \sigma(b)]$, the BVP*

$$Ly \equiv -y^{\Delta\nabla}(t) = h(t), \quad t \in [a,b], \tag{1.5}$$

$$\alpha y(\rho(a)) - \beta y^{\triangle}(\rho(a)) = 0, \quad y(\sigma(b)) - \delta y(\eta) = 0 \tag{1.6}$$

has a unique solution

$$\begin{aligned} y(t) = \; & \tfrac{\beta+\alpha(t-\rho(a))}{D}\int_{\rho(a)}^{\sigma(b)}(\sigma(b)-s)h(s)\nabla s - \tfrac{\delta(\beta+\alpha(t-\rho(a)))}{D}\int_{\rho(a)}^{\eta}(\eta-s)h(s)\nabla s \\ & - \int_{\rho(a)}^{t}(t-s)h(s)\nabla s. \end{aligned}$$

Lemma 1.2 *Let* $0 < \delta < 1$*. If* $h(t) \in C[\rho(a), \sigma(b)]$*, and* $h \geq 0$*, then the unique solution* y *of the problem* $(1.5), (1.6)$ *satisfies*

$$y(t) \geq 0, \quad t \in (\rho(a), \sigma(b)).$$

Proof. From the fact that $y^{\triangle\nabla}(t) = -h(t) \leq 0$, we know that the graph of $y(t)$ is concave down on $[\rho(a), \sigma(b)]$ and $y^{\triangle}(t)$ monotone decreasing. Thus $y^{\triangle}(t) \leq y^{\triangle}(\rho(a)) = \frac{\alpha}{\beta} y(\rho(a))$, where $\beta \neq 0$.

*Case*1. If $y(\rho(a)) < 0$, then $y^{\triangle}(t) < 0$ for $t \in [\rho(a), \sigma(b)]$. Thus y is a monotone decreasing function, this is $y(t) \geq y(\sigma(b))$.
1. If $y(\sigma(b)) \geq 0$, then $y(t) > 0$. So this contradicts the assertion $y(t)$ is a monotone decreasing function.
2. If $y(\sigma(b)) < 0$, then we have that

$$y(\eta) = \frac{1}{\delta} y(\sigma(b)) < 0,$$

$$y(\sigma(b)) = \delta y(\eta) \geq y(\eta)$$

which contradicts the assertion that $y(t)$ is monotone decreasing.

*Case*2. If $y(\rho(a)) \geq 0$, then $y^{\triangle}(\rho(a))) \geq 0$. So $y(t)$ is monotone increasing on $[\rho(a), \rho(a) + \epsilon)$.
1. If $y(\sigma(b)) \geq 0$, then $y(t) \geq 0$ on $[\rho(a), \sigma(b)]$.
2. If $y(\sigma(b)) < 0$, then we have that

$$y(\eta) = \frac{1}{\delta} y(\sigma(b)) < 0,$$

$$y(\sigma(b)) = \delta y(\eta) \geq y(\eta)$$

which contradicts the assertion that the graph of $y(t)$ is concave down on $(\rho(a), \sigma(b))$.

If $\beta = 0$, from the boundary conditions we obtain $y(\rho(a)) = 0$.
1. If $y(\sigma(b)) \geq 0$, then the concavity of y implies that $y(t) \geq 0$ for $t \in [\rho(a), \sigma(b)]$.
2. If $y(\sigma(b)) < 0$, then

$$y(\eta) = \tfrac{1}{\delta} y(\sigma(b)) < 0,$$

$$y(\sigma(b)) = \delta y(\eta) \geq y(\eta).$$

This contradicts with the concavity of y. $\square$

Lemma 1.3 *If* $y^{\triangle\nabla}(t) \leq 0$*, then* $\frac{y(\sigma(b))}{\sigma(b)} \leq \frac{y(t)}{t} \leq \frac{y(\eta)}{\eta}$ *for all* $t \in [\eta, \sigma(b)]$*.*

Proof. Let $h(t) := y(t) - \dfrac{t}{\sigma(b) - \rho(a)} y(\sigma(b))$. Thus, we have $h(\eta) > 0$ and $h(\sigma(b)) = 0$. Since $h^{\triangle\nabla}(t) \leq 0$ then $h(t) \geq 0$ on $[\eta, \sigma(b)]$. So $\dfrac{y(\sigma(b))}{\sigma(b)} \leq \dfrac{y(t)}{t}$. For the function $h(t)$, since $h(\eta) > 0$, $h(\sigma(b)) = 0$ and $h^{\triangle\nabla}(t) \leq 0$ then the function $h(t)$ is decreasing on $[\eta, \sigma(b)]$. So $\dfrac{y(t)}{t} \leq \dfrac{y(\eta)}{\eta}$ for all $t \in [\eta, \sigma(b)]$. $\square$

Lemma 1.4 *Let* $0 < \delta < 1$*. If* $h(t) \in C[\rho(a), \sigma(b)]$*, and* $h \geq 0$*, then the unique solution* y *of* (1.5), (1.6) *satisfies*

$$\inf_{t \in [\eta, \sigma(b)]} y(t) \geq \gamma \|y\|,$$

where

$$\gamma := \min\{\frac{\delta(\sigma(b) - \eta)}{\sigma(b) - \delta\eta - \rho(a)(1 - \delta)}, \frac{\delta\eta}{\sigma(b)}\}.$$

Proof. By the second boundary condition we know that $y(\eta) \geq y(\sigma(b))$. Pick $t_0 \in (\rho(a), \sigma(b))$ such that $y(t_0) = \|y\|$. If $t_0 < \eta < \sigma(b)$, then

$$\min_{t \in [\eta, \sigma(b)]} y(t) = y(\sigma(b))$$

and

$$\frac{y(\sigma(b)) - y(\eta)}{\sigma(b) - \eta} \leq \frac{y(\eta) - y(t_0)}{\eta - t_0}.$$

Therefore

$$\min_{t \in [\eta, \sigma(b)]} y(t) \geq \frac{\delta(\sigma(b) - \eta)}{\sigma(b) - \delta\eta - \rho(a)(1 - \delta)} \|y\|.$$

If $\eta \leq t_0 < \sigma(b)$, again we have $y(\sigma(b)) = \min_{t\in[\eta,\sigma(b)]} y(t)$. From the Lemma 1.3, we have $\frac{y(\eta)}{\eta} \geq \frac{y(t_0)}{t_0}$.
Combining with the boundary condition $\delta y(\eta) = y(\sigma(b))$, we conclude that

$$\frac{y(\sigma(b))}{\delta\eta} \geq \frac{y(t_0)}{t_0} \geq \frac{y(t_0)}{\sigma(b)} = \frac{\|y\|}{\sigma(b)}.$$

This is

$$\min_{t\in[\eta,\sigma(b)]} y(t) \geq \frac{\delta\eta}{\sigma(b)}\|y\|.$$

□

2. A Fixed Point Theorem

Let B be a Banach space, and P a closed, nonempty subset of B. P is a cone provided (i) $\alpha u + \beta v \in P$ for all $u, v \in P$ and all $\alpha, \beta \geq 0$, and (ii) $u, -u \in P$ imply $u = \theta(\theta$ is zero of $P)$.
We refer to [9] for a discussion of the fixed point index that we use below. In particular, we will make frequent use of the following lemma.

Lemma 2.1 *Let B be a Banach space, and let $P \subset B$ be a cone in B. Assume $r > 0$ and that $\Phi : P_r \to P$ is compact operator such that $\Phi x \neq x$ for $x \in \partial P_r := \{x \in P : \|x\| = r\}$. Then, the following assertions hold:*
(i) If $\|x\| \leq \|\Phi x\|$, for all $x \in \partial P_r$, then $i(\Phi, P_r, P) = 0$.
(ii)If $\|x\| \geq \|\Phi x\|$, for all $x \in \partial P_r$, then $i(\Phi, P_r, P) = 1$.

Thus, if there exist $r_1 > r_2 > 0$ such that condition (i) holds for $x \in \partial P_{r_1}$ and (ii) holds for $x \in \partial P_{r_2}$ (or (ii) and (i)), then, from the additivity properties of the index, we know that

$$i(\Phi, P_{r_1}, P) = i(\Phi, P_{r_1} \backslash Int(P_{r_2}), P) + i(\Phi, P_{r_2}, P).$$

As consequence $i(\Phi, P_{r_1} \backslash Int(P_{r_2}), P) \neq 0$, from where we assure the existence of a nonzero fixed point of operator Φ whose norm is between r_1 and r_2.

3. An Existence Theorem For $1.1 - 1.2$

We will assume that

$(A1)$ $f : [a, b] \times R$ is continuous with respect to ξ and $f(t, \xi)$ for $\xi \in R^+$, where R^+ denotes the set of nonnegative real numbers.

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$f_0 := \lim_{x \longrightarrow 0^+} \inf \min_{t \in [a,b]} \frac{f(t,x)}{x}$

$f^0 := \lim_{x \longrightarrow 0^+} \sup \max_{t \in [a,b]} \frac{f(t,x)}{x}$

$f_\infty := \lim_{x \longrightarrow \infty} \inf \min_{t \in [a,b]} \frac{f(t,x)}{x}$

$f^\infty := \lim_{x \longrightarrow \infty} \sup \max_{t \in [a,b]} \frac{f(t,x)}{x}$.

It is not difficult to show that the eigenvalue problem $(1.1), (1.2)$ having a solution is equivalent to the fixed point equation

$$y = \Phi_\lambda(y), \quad y \in B = C[a, b], \tag{3.1}$$

having a solution, where the operator Φ_λ is defined by

$$\Phi_\lambda y(t) = \lambda \int_{\rho(a)}^{\sigma(b)} G(t, s) f(s, y(s)) \nabla s. \tag{3.2}$$

Now, consider the Banach space B with maximum norm and the cone P in B given by

$$P = \{y \in B : y(t) \geq 0, t \in [a, b] \text{ and } \inf_{t \geq \eta} y(t) \geq \gamma \|y\|\}.$$

It is obvious that P is a cone in B. Moreover, by Lemma 1.4 $\Phi_\lambda(P) \subset P$. It is also easy to see that $\Phi_\lambda : P \longrightarrow P$ is completely continuous.

Now, we are ready to obtain criteria for the existence of least two positive solutions of the eigenvalue problem $(1.1), (1.2)$.

Theorem 3.1 *If* $(A1)$ *holds and either*

(a) $\frac{1}{\gamma K f_\infty} < \lambda < \frac{1}{L f^0}$ *or*

(b) $\frac{1}{\gamma K f_0} < \lambda < \frac{1}{L f^\infty}$

is satisfied, where $K = \min_{t\in[a,b]} \int_{\rho(a)}^{\sigma(b)} G(t,s)\nabla s,\ L = \max_{t\in[a,b]} \int_{\rho(a)}^{\sigma(b)} G(t,s)\nabla s,$ *then the eigenvalue problem* (1.1), (1.2) *has two positive solutions on* $[\rho(a), \sigma(b)]$.

Proof. Assume (a) holds. Since

$$\lambda < \frac{1}{Lf^0},$$

there is an $\epsilon > 0$ so that

$$L(f^0 + \epsilon)\lambda \leq 1.$$

Using the definition of f^0, there is an $r_1 > 0$, sufficiently small, so that

$$\max_{t\in[a,b]} \frac{f(t,x)}{x} < f^0 + \epsilon$$

for $0 < x \leq r_1$.

It follows that $f(t,x) < (f^0+\epsilon)x$ for $0 < x \leq r_1$, $t \in [a,b]$.
Assume that $u \in \partial P_{r_1}$, then

$$\begin{aligned} \Phi_\lambda u(t) &< \lambda(f^0+\epsilon)\|u\| \int_{\rho(a)}^{\sigma(b)} G(t,s)\nabla s \\ &\leq \lambda(f^0+\epsilon)\|u\| L \\ &\leq \|u\|, \end{aligned}$$

for $t \in [a,b]$.

Next, we use the assumption

$$\frac{1}{\gamma K f_\infty} < \lambda.$$

First, we consider the case when $f_\infty < \infty$. In this case pick an $\epsilon_1 > 0$ so that

$$\gamma\lambda K(f_\infty - \epsilon_1) \geq 1.$$

Using the definition of f_∞, there is an $r > r_1$ sufficiently large , so that

$$\min_{t\in[a,b]} \frac{f(t,x)}{x} > f_\infty - \epsilon_1,$$

for $x \geq r$.

It follows that $f(t,x) > (f_\infty - \epsilon_1)x$ for $x \geq r$, $t \in [a,b]$. We now show that there is an $r_2 \geq r$ such that if $u \in \partial P_{r_2}$, then $\|\Phi_\lambda u\| > \|u\|$.

Pick $r_2 \geq \delta r > r_1$. Now assume $u \in \partial P_{r_2}$ and consider

$$\begin{aligned}
\Phi_\lambda u(t) &> \lambda(f_\infty - \epsilon_1)\int_{\rho(a)}^{\sigma(b)} G(t,s)u(s)\nabla s \\
&\geq \lambda(f_\infty - \epsilon_1)\gamma\|u\|\int_{\rho(a)}^{\sigma(b)} G(t,s)\nabla s \\
&\geq \lambda(f_\infty - \epsilon_1)\gamma K\|u\| \\
&\geq \|u\|,
\end{aligned}$$

for $t \in [a,b]$.

Finally, we consider the case $f_\infty = \infty$. In this case the hypothesis becomes $\lambda > 0$. Choose $M > 0$ sufficiently large so that

$$\lambda M\gamma K \geq 1,$$

for any $t \in [a,b]$.

So there exists $r > r_1$ so that $f(t,x) > Mx$ for $x \geq r$ and for all $t \in [a,b]$. Now define r_2 as before and assume $u \in \partial P_{r_2}$

$$\begin{aligned}
\Phi_\lambda u(t) &> \lambda M\int_{\rho(a)}^{\sigma(b)} G(t,s)u(s)\nabla s \\
&\geq \lambda M\gamma\|u\|K \\
&\geq \|u\|,
\end{aligned}$$

for $t \in [a,b]$.

Therefore by Lemma 2.1, Φ_λ has a fixed point u with $r_1 < \|u\| < r_2$, and, in consequence, condition (a) yields the existence of a positive solution on $[a,b]$ of such problem.

The proof of part (b) is similar. □

Theorem 3.2 *Let the assumption* $(A1)$ *hold.*

(a) *If* $f^0 = 0$ *or* $f^\infty = 0$, *then there is a* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$ *the problem* $(1.1), (1.2)$ *has a positive solution.*

(b) *If* $f^0 = f^\infty = 0$, *then there is a* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$ *the problem* $(1.1), (1.2)$ *has two positive solutions.*

(*c*) *If* $f^0 = \infty$ *or* $f^\infty = \infty$, *then there is a* λ_0 *such that for all* $0 < \lambda \leq \lambda_0$ *the problem* (1.1), (1.2) *has a positive solution.*

(*d*) *If* $f^0 = f^\infty = \infty$, *then there is a* λ_0 *such that for all* $0 < \lambda \leq \lambda_0$ *the problem* (1.1), (1.2) *has two positive solutions.*

Proof.

(*a*) Let $t_0 \in (\rho(a), \sigma(b))$ and for all $p > 0$ define

$$m(p) = \min\{\int_{\rho(a)}^{\sigma(b)} G(t_0, s) f(s, u(s)) \nabla s, u \in \partial P_p\}.$$

It can be shown that $m(p) > 0$ for all $p > 0$. We now show that for any $p_0 > 0$ that for all $\lambda \geq \lambda_0$, where $\lambda_0 := \frac{p_0}{m(p_0)}$, we have that if $u \in \partial P_{p_0}$, then $\|\Phi_\lambda u\| \geq \|u\|$. To prove this let $u \in \partial P_{p_0}$. Then for $\lambda \geq \lambda_0$,

$$\begin{aligned} \Phi_\lambda u(t_0) &= \lambda \int_{\rho(a)}^{\sigma(b)} G(t_0, s) f(s, u(s)) \nabla s \\ &\geq \lambda m(p_0) \geq \lambda_0 m(p_0) = p_0 = \|u\|. \end{aligned}$$

Hence it follows that $\|\Phi_\lambda u\| \geq \|u\|$ for all $u \in \partial P_{p_0}$ and $\lambda \geq \lambda_0$.

We now show that the condition $f_0 = 0$ implies that given any $p_0 > 0$ there is an h_0 such that $0 < h_0 < p_0$ and for any $u \in \partial P_{h_0}$ it follows that $\|\Phi_\lambda u\| \leq \|u\|$, for all $\lambda \geq \lambda_0$. To prove this fix $\lambda \geq \lambda_0$ and pick $\nu_0 > 0$ so that

$$\nu_0 \lambda L \leq 1 \tag{3.3}$$

Since

$$f^0 := \lim_{x \longrightarrow 0^+} \sup \max_{t \in [a,b]} \frac{f(t,x)}{x} = 0,$$

there is an $h_0 < p_0$ such that

$$\max_{t \in [a,b]} \frac{f(t,x)}{x} \leq \nu_0,$$

for $0 < x \leq h_0$. Hence we have that

$$f(t,x) \leq \nu_0 x, \tag{3.4}$$

for $t \in [\rho(a), \sigma(b)]$, $0 \le x \le h_0$.

Let $u \in \partial P_{h_0}$ and consider

$$\begin{aligned}\Phi_\lambda u(t) &= \lambda \int_{\rho(a)}^{\sigma(b)} G(t,s) f(s,u(s)) \nabla s \\ &\le \lambda \int_{\rho(a)}^{\sigma(b)} G(t,s) \nu_0 u(s) \nabla s \\ &\le \lambda \nu_0 \|u\| L \le \|u\|.\end{aligned}$$

It follows that if $u \in \partial P_{h_0}$, then $\|\Phi_\lambda u\| \le \|u\|$ and hence, the problem (1.1), (1.2) has a positive solution and the first part of (a) has been proven.

We now prove the second part of (a) of this theorem. Fix $\lambda \ge \lambda_0$, where $\lambda_0 = \frac{p_0}{m(p_0)}$. Pick ν_0 so that (3.3) holds. Since $f^\infty = 0$, there is a $H_0 > p_0$ so that

$$\max_{t \in [a,b]} \frac{f(t,x)}{x} \le \nu_0$$

for $x \ge H_0$. Hence we have that

$$f(t,x) \le \nu_0 x$$

for $t \in [\rho(a), \sigma(b)]$.

We consider two cases. The first case is that $f(t,u)$ is bounded on $[\rho(a), \sigma(b)] \times R^+$. In this case there is a positive number N such that

$$|f(t,u)| \le N$$

for $t \in [\rho(a), \sigma(b)]$, $u \in \mathbf{R}^+$. Choose $H_1 \ge H_0$ so that

$$N \lambda L \le H_1.$$

Then $u \in \partial P_{H_1}$, we have

$$\begin{aligned}\Phi_\lambda u(t) &= \lambda \int_{\rho(a)}^{\sigma(b)} G(t,s) f(s,u(s)) \nabla s \\ &\le \lambda N L \le H_1 = \|u\|.\end{aligned}$$

It follows that if $u \in \partial P_{H_1}$, then $\|\Phi_\lambda u\| \le \|u\|$. Since at the beginning of the proof of this theorem we proved that if $u \in \partial P_{p_0}$, then $\|\Phi_\lambda u\| \ge \|u\|$, and since $p_0 < H_1$ it follows from Lemma 2.1 that Φ_λ has a fixed point and hence the problem (1.1), (1.2) has a positive solution.

Next we consider the case where $f(t,u)$ is unbounded on $[\rho(a),\sigma(b)]\times R^+$. Let

$$g(h):=\max\{f(t,y): t\in[\rho(a),\sigma(b)], 0\le y\le h\}.$$

The function $g(h)$ is non-decreasing and

$$\lim_{h\to\infty} g(h)=\infty.$$

Choose $H_2\ge H_0$ so that

$$g(H_2)\ge g(h), \text{ for } 0\le h\le H_2.$$

Then for $u\in\partial P_{H_2}$, we have

$$\begin{aligned}\Phi_\lambda u(t) &= \lambda\int_{\rho(a)}^{\sigma(b)} G(t,s)f(s,u(s))\nabla s\\ &\le \lambda g(H_2)\int_{\rho(a)}^{\sigma(b)} G(t,s)\nabla s\\ &\le \lambda\nu_0 H_2 L\le H_2=\|u\|.\end{aligned}$$

It follows that the problem (1.1), (1.2) has a positive solution $u_0(t)$ satisfying $p_0\le\|u_0\|\le H_2$ and the proof of part (a) of this theorem is complete.

(b) Clearly if $f^0=f^\infty=0$, then by the proof of part (a) we get for any $p_0>0$ that for each fixed $\lambda\ge\lambda_0:=\frac{p_0}{m(p_0)}$ there are numbers $h_0<p_0<H_2$ such that there are two positive solutions of the problem (1.1), (1.2) satisfying $h_0\le\|u_1\|\le p_0\le\|u_2\|\le H_2$.

The proof of part (c) will be easy to see when we prove part (d) so we will only prove part (d) here.

(d) Assume $f_0=f_\infty=\infty$ and $0<r_1<r_2$ are given numbers. Let

$$M_i:=max\{f(t,y):(t,y)\in[\rho(a),\sigma(b)]\times[0,r_i]\} \text{ for } i=1,2.$$

Then if $u\in\partial P_{r_i}$, it follows that

$$\Phi_\lambda u(t)\le M_i\lambda\int_{\rho(a)}^{\sigma(b)} G(t,s)\nabla s.$$

It follows that we can pick $\lambda_0>0$ sufficiently small so that for all $0<\lambda\le\lambda_0$

$$\|\Phi_\lambda u\|\le\|u\|, \text{ for all } u\in\partial P_{r_i}, i=1,2.$$

Fix $\lambda\le\lambda_0$. Choose $M>0$ sufficiently large so that

$$\gamma M\lambda K \geq 1 \tag{3.5}$$

where $t_0 \in (\rho(a), \sigma(b))$. Since $f_0 = \infty$, there is $x_1 < r_1$ such that

$$\min_{t\in[a,b]} \frac{f(t,x)}{x} \geq M$$

for $0 < x \leq x_1$. Hence we have that

$$f(t,x) \geq Mx$$

for $t \in [\rho(a), \sigma(b)]$. We next show that if $u \in \partial P_{x_1}$, then $\|\Phi_\lambda u\| \geq \|u\|$. To show this assume $u \in \partial P_{x_1}$. Then

$$\begin{aligned} \Phi_\lambda u(t_0) &= \lambda \int_{\rho(a)}^{\sigma(b)} G(t_0,s) f(s,u(s)) \nabla s \\ &\geq \lambda M \int_{\rho(a)}^{\sigma(b)} G(t_0,s) u(s) \nabla s \\ &\geq \lambda M \gamma \|u\| K \geq \|u\|. \end{aligned}$$

Hence we have shown that if $u \in \partial P_{x_1}$, then $\|\Phi_\lambda u\| \geq \|u\|$.

Next, we use the assumption that $f_\infty = \infty$. Since $f_\infty = \infty$ there is $x_2 > r_2$ such that

$$\min_{t\in[a,b]} \frac{f(t,x)}{x} \geq M$$

for $x \geq x_2$ and M is chosen so that (3.5) holds. It follows that

$$f(t,x) \geq Mx$$

for $t \in [\rho(a), \sigma(b)]$. We show that if $u \in \partial P_{x_3}$, then $\|\Phi_\lambda u\| \geq \|u\|$. To show this assume $u \in \partial P_{x_1}$. Then

$$\begin{aligned} \Phi_\lambda u(t_0) &= \lambda \int_{\rho(a)}^{\sigma(b)} G(t_0,s) f(s,u(s)) \nabla s \\ &\geq \lambda M \int_{\rho(a)}^{\sigma(b)} G(t_0,s) u(s) \nabla s \\ &\geq \lambda M \gamma \|u\| K \geq \|u\|. \end{aligned}$$

Hence we have shown that if $u \in \partial P_{x_3}$, then $\|\Phi_\lambda u\| \geq \|u\|$. It follows from Lemma 2.1 that the operator Φ_λ has two fixed point $u_1(t)$ and $u_2(t)$ satisfying

$$x_1 < \|u_1\| < r_1 < r_2 < \|u_2\| < x_3.$$

□

4. Lower and Upper Solutions

We define the set

$$D := \{y : y^{\triangle\nabla} \text{ is continuous on } \quad [\rho(a), b]\}.$$

For any $u, v \in D$, we define the sector $[u, v]$ by

$$[u, v] := \{w \in D : u \leq w \leq v\}.$$

Definition 4.1 A real valued function $u(t) \in D$ on $[\rho(a), \sigma(b)]$ is a lower solution for $(1.1), (1.2)$ if

$$-u^{\triangle\nabla}(t) \leq \lambda f(t, u(t)) \text{for } t \text{ in } [a, b]$$

$$\alpha u(\rho(a)) - \beta u^{\triangle}(\rho(a)) = 0 \text{ and } u(\sigma(b)) \leq \delta u(\eta).$$

Similarly, a real valued function $v(t) \in D$ on $[\rho(a), \sigma(b)]$ is an upper solution for $(1.1), (1.2)$ if

$$-v^{\triangle\nabla}(t) \geq \lambda f(t, v(t)) \text{for } t \text{ in } [a, b]$$

$$\alpha v(\rho(a)) - \beta v^{\triangle}(\rho(a)) = 0 \text{ and } v(\sigma(b)) \geq \delta v(\eta).$$

We will prove that when the lower and upper solutions are given in the well order, i. e. $u \leq v$, problem $(1.1), (1.2)$ admits lying between both functions.

Theorem 4.1 *Assume that the condition $(A1)$ is satisfied and u and v are respectively lower and upper solutions for the BVP (1.1)–(1.2) such that $u \leq v$ on $[\rho(a), \sigma(b)]$. Then the BVP (1.1), (1.2) has a solution $y \in [u, v]$ on $[\rho(a), \sigma(b)]$.*

Proof. Consider the BVP

$$-y^{\Delta\nabla}(t) = \lambda F(t, y(t)), \quad t \in [a, b], \tag{4.1}$$

$$\alpha y(\rho(a)) - \beta y^{\Delta}(\rho(a)) = 0, \quad y(\sigma(b)) = \delta y(\eta), \tag{4.2}$$

where,

$$F(t,\xi) = \begin{cases} f(t, v(t)) - \dfrac{\xi - v(t)}{1+|\xi|}, & \xi \geq v(t), \\ f(t,\xi), & u(t) \leq \xi \leq v(t), \\ f(t, u(t)) + \dfrac{\xi - u(t))}{1+|\xi|}, & \xi \leq u(t) \end{cases}$$

for $t \in [a, b]$.

Clearly, the function F is bounded for $t \in [a, b]$ and $\xi \in R$, and is continuous in ξ. Thus, by Theorem 3.1, there exists a solution $y(t)$ of the three point BVP (4.1)–(4.2).

We claim $y(t) \leq v(t)$ for $t \in [a, b]$. If not, from the boundary conditions we know that $y(t) - v(t)$ has a positive maximum at some $c \in [a, b]$. Consequently, we must have $(y - v)^{\Delta}(c) \leq 0$ and $(y - v)^{\Delta\nabla}(c) \leq 0$. On the other hand,

$$\begin{aligned} -y^{\Delta\nabla}(c) = \quad & F(c, y(c)) = \lambda f((c), \beta(c)) - \lambda \tfrac{y(c)-v(c)}{1+|y(c)|} \\ & < \lambda(f(c), v(c)) \leq -v^{\Delta\nabla}(c). \end{aligned}$$

Hence, we have

$$(y - v)^{\Delta\nabla}(c) > 0$$

which is a contradiction. It follows that $y(t) \leq v(t)$ on $[a, b]$.

Since $y(\sigma(b)) = \delta y(\eta) \leq \delta v(\eta) \leq v(\sigma(b))$, we have that $y(\sigma(b)) \leq v(\sigma(b))$. If a is left scattered, then $(y-v)(\rho(a)) = \frac{\beta}{\alpha(a-\rho(a))+\beta}(y-v)(a)$. So, we get $(y-v)(\rho(a)) \leq 0$. Thus we have $y(t) \leq v(t)$ on $[\rho(a), \sigma(b)]$.

Similarly, $u \leq y$ on $[\rho(a), \sigma(b)]$. Thus $y(t)$ is a solution of $(1.1), (1.2)$ and lies between u and v. □

Theorem 4.2 *Assume that the condition* $(A1)$ *is satisfied and* u *and* v *are lower and upper solutions of the* $(1.1), (1.2)$ *on* $[\rho(a), \sigma(b)]$. *If* $f(t, y)$ *is strictly decreasing* y *for each* $t \in [a, b]$. *Then* $u \le v$ *on* $[\rho(a), \sigma(b)]$.

Proof. We claim that $u \le v$ for $t \in [\rho(a), \sigma(b)]$. If not from the boundary conditions we know that $u - v$ has a positive maximum at some c in $[\rho(a), b]$. Consequently, we know that $(u - v)^{\triangle}(c) \le 0$ and $(u - v)^{\triangle\nabla}(c) \le 0$. On the other hand,

$$-u^{\triangle\nabla}(c) = f(c, u(c)) < f(c, v(c)) \le -v^{\triangle\nabla}(c).$$

Hence, we have

$$(u - v)^{\triangle\nabla}(c) > 0.$$

which is a contradiction. It follows that $u(t) \le v(t)$ on $[\rho(a), b]$.
Since $u(\sigma(b)) \le \delta u(\eta) \le \delta v(\eta) \le v(\sigma(b))$, we have that $u(\sigma(b)) \le v(\sigma(b))$. So, we get $u(t) \le v(t)$ on $[\rho(a), \sigma(b)]$. □

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