Turk J Math
30 (2006) , 25 – 31.

Quasi Separation Axioms

Mohammad S. Sarsak

Abstract

In [5], Maheshwari et al. introduced and studied some new separation axioms, namely, quasi semi T_i axioms where $i \in \{0, 1, 2\}$, the quasi semi $T_{1/2}$ axiom was then introduced and investigated by Gyu-Ihn et al. in [2]. In the present paper we introduce and study quasi T_i axioms, $i \in \{0, 1/2, 1, 2\}$ as a special variety of quasi semi T_i axioms, the class of quasi $T_{1/2}$ (respectively, quasi T_1) bitopological spaces is placed between quasi T_0 (respectively, quasi $T_{1/2}$) bitopological spaces and quasi T_1 (respectively, quasi T_2) bitopological spaces. Among several counter examples we introduce an example of a bitopological space which is quasi T_0 that fails to be quasi semi $T_{1/2}$, thus answering a question raised in [2].

Key words and phrases: bitopological spaces, quasi open sets, quasi semi-open sets, quasi T_i, quasi semi T_i, $i \in \{0, 1/2, 1, 2\}$.

1. Introduction

A bitopological space $(X; \tau_1, \tau_2)$ [3] is a non-empty set X with two topologies τ_1 and τ_2 on X. A subset A of a space (X, τ) is called semi-open in (X, τ) [4] if $A \subset \overline{IntA}$, the collection of all semi-open sets in a space (X, τ) will be denoted by SO(X, τ). A subset A of a bitopological space $(X; \tau_1, \tau_2)$ is called quasi semi-open in $(X; \tau_1, \tau_2)$ [5] if $A = U \cup V$ where $U \in SO(X, \tau_1)$, $V \in SO(X, \tau_2)$, A is called quasi semi-closed in $(X; \tau_1, \tau_2)$ if $X \backslash A$ is quasi semi-open in $(X; \tau_1, \tau_2)$ and the quasi semi-closure $qscl(A)$ of A is the intersection of all quasi semi-closed sets in $(X; \tau_1, \tau_2)$ that contain A. $QSO(X; \tau_1, \tau_2)$

2000 *AMS Mathematics Subject Classification:* Primary 54D10, 54E55.

(respectively, $QSC\,(X;\,\tau_1,\,\tau_2)$) will denote the class of all quasi semi-open (respectively, quasi semi-closed) sets in $(X;\,\tau_1,\,\tau_2)$.

A space $(X;\,\tau_1,\,\tau_2)$ is called quasi semi T_0 [5] if for any two distinct points x, y of X there exists $A \in QSO\,(X;\,\tau_1,\,\tau_2)$ such that $x \in A, y \notin A$ or $y \in A, x \notin A$, or equivalently, if $\text{qscl}\{x\} \neq \text{qscl}\{y\}$ for any two distinct points x, y of X, $(X;\,\tau_1,\,\tau_2)$ is called quasi semi T_1 if for any two distinct points x, y of X there exist $A, B \in QSO\,(X;\,\tau_1,\,\tau_2)$ such that $x \in A$, $y \notin A$ and $y \in B$, $x \notin B$, or equivalently, if the singleton subsets of X are quasi semi-closed in $(X;\,\tau_1,\,\tau_2)$ and $(X;\,\tau_1,\,\tau_2)$ is called quasi semi T_2 if for any two distinct points x, y of X there exist two disjoint sets $A, B \in QSO\,(X;\,\tau_1, \tau_2)$ such that $x \in A$ and $y \in B$.

A subset A is called quasi semi-generalized closed (briefly qsg-closed) in $(X;\,\tau_1,\,\tau_2)$ [2] if $qscl(A) \subset U$ whenever $A \subset U$ and $U \in QSO\,(X;\,\tau_1,\,\tau_2)$, a space $(X;\,\tau_1,\,\tau_2)$ is called quasi semi $T_{1/2}$ if every qsg-closed set in $(X;\,\tau_1,\,\tau_2)$ is quasi semi-closed in $(X;\,\tau_1,\,\tau_2)$, or equivalently, if every singleton subset of X is quasi semi-open or quasi semi-closed in $(X;\,\tau_1,\,\tau_2)$.

In the present paper, stronger axioms than quasi semi T_i , $i \in \{0,\,1/2,\,1,\,2\}$ are given, that will be called quasi T_i, $i \in \{0,\,1/2,\,1,\,2\}$. It is shown that every quasi T_i space is quasi semi T_i but not conversely, we also investigate some characterizations of quasi T_i spaces. Each of the implications quasi $T_2 \rightarrow$ quasi $T_1 \rightarrow$ quasi $T_{1/2} \rightarrow$ quasi T_0 is true while none of the reverse implications holds.

In [5], it was pointed out that every quasi semi T_2 space is quasi semi T_1 but not conversely. It was also pointed out in [2] that every quasi semi T_1 space is quasi semi $T_{1/2}$ but not conversely and that every quasi semi $T_{1/2}$ space is quasi semi T_0. In this paper the outhors asked for an example of a quasi semi T_0 space that fails to be quasi semi $T_{1/2}$. Such an example is given in this paper.

Throughout this paper no separation axiom is assumed unless stated explicitly, for the notions not defined here we refer the reader to [1].

2. Quasi Separation Axioms

Definition 1 *A subset A of a space $(X;\,\tau_1,\,\tau_2)$ is said to be quasi open in $(X;\,\tau_1,\,\tau_2)$ if $A = U \cup V$ for some $U \in \tau_1$ and $V \in \tau_2$. The complement of a quasi open set in $(X;\,\tau_1,\,\tau_2)$ is said to be quasi closed in $(X;\,\tau_1,\,\tau_2)$. $QO\,(X;\,\tau_1,\,\tau_2)$ (respectively, $QC\,(X;\,\tau_1,\,\tau_2)$) will denote the class of all quasi open (respectively, quasi closed) sets in $(X;\,\tau_1,\,\tau_2)$.*

Definition 2 *For a subset A of a space $(X;\, \tau_1,\, \tau_2)$, we define the quasi kernel of A (briefly, qker(A)) as follows: qker(A) $= \cap\{F : F \in QO(X;\, \tau_1,\, \tau_2),\, A \subset F\}$. A is said to be a quasi $\Lambda-$ set in $(X;\, \tau_1,\, \tau_2)$ if $A =$ qker(A), or equivalently, if A is the intersection of quasi open sets. A is said to be quasi λ-closed in $(X;\, \tau_1,\, \tau_2)$ if it is the intersection of a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$ and a quasi closed set in $(X;\, \tau_1,\, \tau_2)$, clearly, quasi Λ-sets and quasi closed sets are quasi λ-closed; complements of quasi λ-closed sets in $(X;\, \tau_1,\, \tau_2)$ are said to be quasi λ-open in $(X;\, \tau_1,\, \tau_2)$.*

Definition 3 *For a subset A of a space $(X;\, \tau_1,\, \tau_2)$, we define the quasi closure of A (briefly qcl(A)) as follows: qcl(A) $= \cap\{F : F \in QC(X;\, \tau_1,\, \tau_2),\, A \subset F\}$, or equivalently, qcl(A) is the smallest quasi closed set in $(X;\, \tau_1,\, \tau_2)$ that contains A. Obviously, A is quasi closed in $(X;\, \tau_1,\, \tau_2)$ if and only if $A =$ qcl(A) and $x \in$ qcl(A) if and only if every set $U \in QO(X;\, \tau_1,\, \tau_2)$ containing x meets A. A is said to be quasi generalized closed (briefly qg-closed) in $(X;\, \tau_1,\, \tau_2)$ if qcl(A) $\subset U$ whenever $A \subset U$ and $U \in QO(X;\, \tau_1,\, \tau_2)$, or equivalently, if qcl(A) $\subset q\ker(A)$. The complement of a quasi generalized closed set in $(X;\, \tau_1,\, \tau_2)$ is said to be quasi generalized open (briefly qg-open) in $(X;\, \tau_1,\, \tau_2)$. $QGO(X;\, \tau_1,\, \tau_2)$ (respectively, $QGC(X;\, \tau_1,\, \tau_2)$) will denote the class of all quasi generalized open (respectively, quasi generalized closed) sets in $(X;\, \tau_1,\, \tau_2)$. Obviously, $QC(X;\, \tau_1,\, \tau_2)$ is a subclass of $QGC(X;\, \tau_1,\, \tau_2)$.*

The following two propositions are analogous to Theorem 3.5 and Theorem 3.6 of [2], respectively; they have similar proofs.

Proposition 4 *For a subset A of a space $(X; \tau_1, \tau_2)$, the following are equivalent:*

(i) A is quasi $\lambda-$closed in $(X;\, \tau_1,\, \tau_2)$.

(ii) $A = L \cap$ qcl(A), where L is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$.

(iii) $A =$ qker(A) $\cap$ qcl(A).

Proposition 5 *A subset A of a space $(X;\, \tau_1,\, \tau_2)$ is quasi closed in $(X;\, \tau_1,\, \tau_2)$ if and only if A is both qg-closed and quasi $\lambda-$closed in $(X;\, \tau_1,\, \tau_2)$.*

Definition 6 *A space $(X;\, \tau_1,\, \tau_2)$ is said to be quasi T_0 if for any two distinct points x, y of X, there exists $A \in QO\ (X;\, \tau_1,\, \tau_2)$ such that $x \in A$, $y \notin A$ or $y \in A$, $x \notin A$, or equivalently, if $(X,\, \tau_1 \vee \tau_2)$ is T_0, where $\tau_1 \vee \tau_2$ is the topology having for a subbase $\tau_1 \cup \tau_2$.*

Definition 7 *A space* $(X;\, \tau_1,\, \tau_2)$ *is said to be quasi* $T_{1/2}$ *if* $QC(X;\, \tau_1,\, \tau_2) = QGC(X;\, \tau_1,\, \tau_2)$.

Definition 8 *A space* $(X;\, \tau_1,\, \tau_2)$ *is said to be quasi* T_1 *if for any two distinct points x, y of X, there exist A,* $B \in QO(X;\, \tau_1,\, \tau_2)$ *such that* $x \in A$, $y \notin A$ *and* $y \in B$, $x \notin B$, *or equivalently, if* $(X,\, \tau_1 \vee \tau_2)$ *is* T_1.

Definition 9 *A space* $(X;\, \tau_1,\, \tau_2)$ *is said to be quasi* T_2 *if for any two distinct points x, y of X, there exist two disjoint sets A,* $B \in QO(X;\, \tau_1,\, \tau_2)$ *such that* $x \in A$ *and* $y \in B$.

The following proposition can be easily verified.

Proposition 10 *For a space* $(X;\, \tau_1,\, \tau_2)$*, the following are equivalent:*

(i) X is quasi T_0.

(ii) $qcl\{x\} \neq qcl\{y\}$ *for any two distinct points x, y of X.*

(iii) $q\ker\{x\} \neq q\ker\{y\}$ *for any two distinct points x, y of X.*

(iv) For any two distinct points x, y of X, there exists $A \in QO(X; \tau_1, \tau_2) \cup QC(X; \tau_1, \tau_2)$ *such that* $x \in A$, $y \notin A$.

Theorem 11 *For a space* $(X;\, \tau_1,\, \tau_2)$*, the following are equivalent:*

(i) X is quasi T_0.

(ii) Every singleton subset of X is quasi $\lambda-$ *closed in* $(X;\, \tau_1,\, \tau_2)$.

Proof. (i)→(ii): Let $x \in X$. By (i), it follows from Proposition 2.10 that for each $y \in X$, $y \neq x$, there exists $A_y \in QO(X; \tau_1, \tau_2) \cup QC(X; \tau_1, \tau_2)$ such that $x \in A_y$, $y \notin A_y$. Let $L = \cap\{A_y \in QO(X; \tau_1, \tau_2)\}, A = \cap\{A_y \in QC(X; \tau_1, \tau_2)\}$. Then L is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$, A is quasi closed in $(X;\, \tau_1,\, \tau_2)$ and $\{x\} = L \cap A$ or $\{x\} = L$ or $\{x\} = A$. Thus $\{x\}$ is quasi $\lambda-$ closed in $(X;\, \tau_1,\, \tau_2)$.

(ii)→ (i): Let x, y be two distinct points of X. By (ii), $\{x\} = L \cap A$, where L is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$ and A is a quasi closed set in $(X;\, \tau_1,\, \tau_2)$. If $y \notin A$, then $X \backslash A$ is a quasi open set that contains y but not x. If $y \notin L$, then $y \notin A_y$ for some quasi open set A_y containing x. Thus X is quasi T_0. □

The proof of the following theorem is similar to that of Theorem 4.3 of [2].

Theorem 12 *For a space* $(X;\, \tau_1,\, \tau_2)$*, the following are equivalent:*

(i) X is quasi $T_{1/2}$.

(ii) Every singleton subset of X is quasi open or quasi closed in $(X;\, \tau_1,\, \tau_2)$.

(iii) Every subset of X is quasi $\lambda-$closed in $(X;\, \tau_1,\, \tau_2)$.

Definition 13 *A subset A of a space $(X;\, \tau_1,\, \tau_2)$ is said to be a generalized quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$ if $q\ker(A) \subset qcl(A)$.*

Obviously, every quasi Λ-set is a generalized quasi $\Lambda-$set. However, the following result asserts that the converse holds only for spaces that are quasi $T_{1/2}$.

Corollary 14 *For a space $(X;\, \tau_1,\, \tau_2)$, the following are equivalent:*

(i) X is quasi $T_{1/2}$.

(ii) Every generalized quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$ is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$.

Proof. (i)→(ii): Let A be a generalized quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$. Since X is quasi $T_{1/2}$, it follows from Theorem 2.12 that A is quasi $\lambda-$closed in $(X;\, \tau_1,\, \tau_2)$. Thus by Proposition 2.4, $A = qker(A) \cap qcl(A)$, but A is a generalized quasi Λ-set, so $A = qker(A)$, that is, A is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$.

(ii)→(i): Suppose that every generalized quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$ is a quasi $\Lambda-$set in $(X;\, \tau_1,\, \tau_2)$ and that X is not quasi $T_{1/2}$. Then by Theorem 2.12 there exists a point x of X such that $\{x\}$ is neither quasi open nor quasi closed in $(X;\, \tau_1,\, \tau_2)$. Let $A = X \backslash \{x\}$. Since $\{x\}$ is not quasi closed, we have $q\ker(A) = X$. Since $\{x\}$ is not quasi open, we have $qcl(A) = X$. Thus $q\ker(A) \subset qcl(A)$, that is, A is a generalized quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$. By assumption, A is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$, that is, $A = q\ker(A)$ which is a contradiction. □

Theorem 15 *For a space $(X;\, \tau_1,\, \tau_2)$, the following are equivalent:*

(i) X is quasi T_1.

(ii) Every singleton subset of X is quasi closed in $(X;\, \tau_1,\, \tau_2)$.

(iii) Every subset of X is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$.

(iv) Every singleton subset of X is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$.

Proof. (i)→(ii): Let x be a point of X. Since X is quasi T_1, it follows that for each $y \in X$, $y \neq x$, then $y \notin qcl\{x\}$, i.e. $qcl\{x\} \subset \{x\}$, but $x \in qcl\{x\}$, so $qcl\{x\} = \{x\}$, that is, $\{x\}$ is quasi closed in $(X;\, \tau_1,\, \tau_2)$.

(ii) → (iii): Let A be a subset of X. By (ii), $X \backslash \{x\}$ is quasi open in $(X;\, \tau_1,\, \tau_2)$ for each $x \notin A$ and therefore $A \subset qker(A) \subset \bigcap_{x \notin A} X \backslash \{x\} = A$. Thus $A = qker(A)$, that is, A is a quasi Λ-set in $(X;\, \tau_1,\, \tau_2)$.

(iii)→(iv): Clear.

(iv)→(i): Let x, y be two distinct points of X. Then by (iii), $\{x\} = qker\{x\}$ and $\{y\} = qker\{y\}$. Thus, there exist A, $B \in$ QO$(X;\, \tau_1,\, \tau_2)$ such that $x \in A$, $y \notin A$ and $y \in B$, $x \notin B$, that is, X is quasi T_1. □

Remark 16 From the definitions, Theorem 2.11, Theorem 2.12 and Theorem 2.15, the following implications seem obvious: quasi T_2 → quasi T_1 → quasi $T_{1/2}$ → quasi T_0, on the other hand, since every quasi open set is quasi semi open, the implication quasi T_i → quasi semi T_i holds for each $i \in \{0,\, 1/2,\, 1,\, 2\}$. However, none of the above seven implications is reversible as the following examples show.

Example 17 *Let* $X = \{a,\, b,\, c\}$, $\tau_1 = \{X,\, \phi,\, \{a,\, b\}\}$ *and* $\tau_2 = \{X,\, \phi,\, \{a,\, c\}\}$. *Then* $(X;\, \tau_1,\, \tau_2)$ *is quasi* T_0*; it is not quasi semi* $T_{1/2}$ *since* $\{a\}$ *is neither quasi semi open nor quasi semi closed in* $(X;\, \tau_1,\, \tau_2)$*.*

Example 18 *Let* $X = \{a,\, b,\, c\}$, $\tau_1 = \{X,\, \phi,\, \{a\}\}$ *and* $\tau_2 = \{X,\, \phi,\, \{b\}\}$. *Then* $(X;\, \tau_1,\, \tau_2)$ *is quasi* $T_{1/2}$ *and quasi semi* T_2*; however, it is not quasi* T_1 *since* $\{a\}$ *and* $\{b\}$ *are not quasi closed in* $(X;\, \tau_1,\, \tau_2)$*.*

Example 19 *Let* $X = \{a,\, b,\, c\}$, $\tau_1 = \{X,\, \phi,\, \{a\}\}$ *and* $\tau_2 = \{X,\, \phi,\, \{a, b\}\}$. *Then* $(X;\, \tau_1,\, \tau_2)$ *is quasi semi* $T_{1/2}$; *it is not quasi* $T_{1/2}$ *since* $\{b\}$ *is neither quasi open nor quasi closed in* $(X;\, \tau_1,\, \tau_2)$*.*

Example 20 *Let* $X = \{a,\, b,\, c\}$, $\tau_1 = \{X,\, \phi,\, \{a\}\}$, *and* $\tau_2 = \{X,\, \phi,\, \{b, c\}\}$. *Then* $(X;\, \tau_1,\, \tau_2)$ *is quasi semi* T_0; *it is not quasi* T_0 *since* $qcl\{b\} = qcl\{c\} = \{b,\, c\}$*.*

Example 21 *Let* Z *be the set of integers* $\tau_1 = \{Z,\, \phi\} \cup \{Z \backslash A : A$ *is a finite subset of the nonnegative integers*$\}$, *and* $\tau_2 = \{Z,\, \phi\} \cup \{Z \backslash A : A$ *is a finite subset of the negative integers*$\}$. *Then* $(Z;\, \tau_1,\, \tau_2)$ *is quasi* T_1*; it is obviously not quasi* T_2*.*

References

[1] R. Engelking, General Topology, Heldermann (Berlin, 1989).

[2] Chae Gyu-Ihn, H. Maki, K. Aoki and Y. Mizuta, More on quasi semi open sets, Q & A in General Topology, 19 (2001), 11-16.

[3] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (1963), 71-89.

[4] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

[5] S. N. Maheshwari, Chae Gyu-Ihn and S. S. Thakur, Quasi semi-open sets, Univ. Ulsan Rep. 17(1) (1986), 133-137.

Mohammad S. SARSAK
Received 14.07.2004
Department of Mathematics,
Faculty of Science & Arts,
The Hashemite University,
P.O. Box 150459, Zarqa 13115, JORDAN
e-mail: sarsak@hu.edu.jo