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Existence of Periodic Solutions for Second Order Rayleigh Equations With Piecewise Constant Argument

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Abstract

Based on a continuation theorem of Mawhin, periodic solutions are found for the second-order Rayleigh equation with piecewise constant argument.

Key Words: Rayleigh equation, deviating argument, piecewise constant argument, periodic solution, Mawhin's continuation theorem.

1. Introduction

Qualitative behaviors of first order delay differential equations with piecewise constant arguments are the subject of many investigations (see, e.g. [1–19]), while those of higher order equations are not.

However, there are reasons for studying higher order equations with piecewise constant arguments. Indeed, as mentioned in [10], a potential application of these equations is in the stabilization of hybrid control systems with feedback delay, where a hybrid system is one with a continuous plant and with a discrete (sampled) controller. As an example, suppose a moving particle is subjected to damping and a restoring controller $-\phi(x[t-k])$ which acts at sampled time $[t-k]$, then the equation of motion is of the form

$$x''(t) + a\,(t)\,x'\,(t) = -\phi(x[t-k]).$$

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In this paper we study a slightly more general second-order Rayleigh equation with piecewise constant argument of the form

$$x''(t) + f(t, x'(t)) + g(t, x([t-k])) = 0, \tag{1}$$

where $[\cdot]$ is the greatest-integer function, k is a positive integer, $f(t,x)$ and $g(t,x)$ are continuous on R^2 such that for $(t,x) \in R^2$,

$$f(t+\omega, x) = f(t,x)$$

and

$$g(t+\omega, x) = g(t,x),$$

for some positive integer ω. We also require $f(t,0) = 0$ for all t in R.

By a solution of (1) we mean a function $x(t)$ which is defined on R and which satisfies the conditions (i) $x'(t)$ is continuous on R, (ii) $x'(t)$ is differentiable at each point $t \in R$, with the possible exception of the points $[t] \in R$ where one-sided derivatives exist, and (iii) substitution of $x(t)$ into Eq. (1) leads to an identity on each interval $[n, n+1) \subset R$ with integral endpoints.

In this note, existence criteria for ω-periodic solutions of (1) will be established. For this purpose, we will make use of a continuation theorem of Mawhin. Let X and Y be two Banach spaces and $L : \mathrm{Dom}L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\,\mathrm{Im}\,L < +\infty$, and $\mathrm{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$. It follows that $L_{|\mathrm{Dom}L \cap \mathrm{Ker}P} : (I-P)X \to \mathrm{Im}L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Theorem A *(Mawhin's continuation theorem [20]). Let L be a Fredholm mapping of index zero, and let N be L-compact on $\bar{\Omega}$. Suppose*

(i) for each $\lambda \in (0,1)$, $x \in \partial\Omega, Lx \neq \lambda Nx$; and

(ii) for each $x \in \partial\Omega \cap \mathrm{Ker}L, QNx \neq 0$ and $\deg(JQN, \Omega \cap \mathrm{Ker}L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{dom}L$.

2. Existence Criteria

The main results of our paper are as follows.

Theorem 1 *Suppose there exist constants* $K > 0$, $D > 0$, $r_1 > 0$, $r_2 > 0$ *and* $r_3 \geqslant 0$ *such that*

(a_1) $|f(t,x)| \leqslant r_1 |x| + K$ *for* $(t,x) \in R^2$,

(b_1) $xg(t,x) > 0$ *and* $|g(t,x)| \geqslant r_2 |x|$ *for* $t \in R$ *and* $|x| > D$,

(c_1) $\lim_{x\to-\infty} \max_{0\leq t\leq\omega} \frac{g(t,x)}{x} \leq r_3$,

(d_1) $2\omega \left[r_1 + r_3 \left(\frac{r_1}{r_2} + \omega\right)\right] < 1.$

Then (1) has an ω*-periodic solution.*

Theorem 2 *Suppose there exist* $K > 0$, $D > 0$, $r_1 > 0$, $r_2 > 0$ *and* $r_3 \geqslant 0$ *such that*

(a_1) $| f(t,x) | \leqslant r_1 |x| + K$ *for* $(t,x) \in R^2$,

(b_1) $xg(t,x) > 0$ *and* $| g(t,x) | \geqslant r_2 |x|$, *for* $t \in R$ *and* $|x| > D$,

(c_2) $\lim_{x\to+\infty} \max_{0\leq t\leq\omega} \frac{g(t,x)}{x} \leq r_3$,

(d_1) $2\omega \left[r_1 + r_3 \left(\frac{r_1}{r_2} + \omega\right)\right] < 1.$

Then (1) has an ω*-periodic solution.*

Theorem 3 *Suppose there exist* $K > 0$, $D > 0$ *and* $r \geqslant 0$ *such that*

(a_2) $| f(t,x) | \leqslant K$ *for* $(t,x) \in R^2$,

(b_2) $xg(t,x) > 0$ *and* $| g(t,x) | > K$, *for* $t \in R$ *and* $|x| > D$,

(c_3) $\lim_{x\to-\infty} \max_{0\leq t\leq\omega} \frac{g(t,x)}{x} \leq r < \frac{1}{2\omega^2}$.

Then (1) has an ω*-periodic solution.*

Theorem 4 *Suppose there exist positive constants* $K > 0$, $D > 0$ *and* $r \geqslant 0$ *such that*

(a_2) $| f(t,x) | \leqslant K$ *for* $(t,x) \in R^2$,

(b_2) $xg(t,x) > 0$ *and* $| g(t,x) | > K$, *for* $t \in R$ *and* $|x| > D$,

(c_4) $\lim_{x\to+\infty} \max_{0\leq t\leq\omega} \frac{g(t,x)}{x} \leq r < \frac{1}{2\omega^2}$.

Then (1) has an ω*-periodic solution.*

In order to prove Theorem 1, we first make the simple observation that $x(t)$ is an ω-periodic solution of the following equation

$$x'(t) = x'(0) - \int_0^t (f(s, x'(s)) + g(s, x([s-k])))\, ds,\ t \in R, \tag{2}$$

if, and only if, $x(t)$ is an ω-periodic solution of (1).

Next, let X_ω be the Banach space of all real ω-periodic differentiable continuous functions of the form $x = x(t)$ which is defined on R and endowed with the usual linear structure as well as the norm $\|x\|_1 = \|x\|_0 + \|x'\|_0$ where $\|\cdot\|_0$ denotes the maximum norm. Let Y_ω be the Banach space of all real continuous functions of the form $y = \alpha t + h(t)$ such that $y(0) = 0$ where $\alpha \in R$ and $h(t) \in X_\omega$, and endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$. Let the zero element of X_ω and Y_ω be denoted by θ_1 and θ_2 respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$ respectively by

$$Lx(t) = x'(t) - x'(0),\ t \in R, \tag{3}$$

and

$$Nx(t) = -\int_0^t (f(s, x'(s)) + g(s, x([s-k])))\, ds, \quad t \in R. \tag{4}$$

Let

$$\bar{h}(t) = -\int_0^t f(s, x([s]))\, ds + \frac{t}{\omega}\int_0^\omega f(s, x([s]))\, ds, \quad t \in R. \tag{5}$$

Since $\bar{h} \in X_\omega$ and $\bar{h}(0) = 0$, N is a well-defined operator from X_ω to Y_ω. Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$ respectively by

$$Px(t) = x(0), \quad t \in R, \tag{6}$$

for $x = x(t) \in X_\omega$ and

$$Qy(t) = \alpha t,\ t \in R, \tag{7}$$

for $y(t) = \alpha t + h(t) \in Y_\omega$.

Lemma 1 *Let the mapping L be defined by (3). Then*

$$\mathrm{Ker}L = \{x \in X_\omega | x(t) = c, t \in R\}, \tag{8}$$

that is, the set of all real constant functions.

Indeed, it is easy to see from (3) that (8) holds.

Lemma 2 *Let the mapping L be defined by (3). Then*

$$\mathrm{Im}L = \{y \in X_\omega \mid y(0) = 0\} \subset Y_\omega. \tag{9}$$

Proof. It suffices to show that for each $y = y(t) \in X_\omega$ that satisfies $y(0) = 0$, there is a $x = x(t) \in X_\omega$ such that

$$y(t) = x'(t) - x'(0), \quad t \in R. \tag{10}$$

But this is relatively easy, since we may let

$$x(t) = \int_0^t y(s)\,ds - \frac{t}{\omega}\int_0^\omega y(s)\,ds, \ t \in R. \tag{11}$$

Then it may easily be checked that (11) holds. The proof is complete. □

Lemma 3 *The mapping L defined by (3) is a Fredholm mapping of index zero.*

Indeed, from Lemma 1, Lemma 2 and the definition of Y_ω, $\dim \mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L = 1 < +\infty$. From (9), we see that $\mathrm{Im}L$ is closed in Y_ω. Hence L is a Fredholm mapping of index zero.

Lemma 4 *Let the mapping L, P and Q be defined by (3), (6) and (7) respectively. Then ImP* $= \mathrm{Ker}L$ *and ImL* $= \mathrm{Ker}Q$.

Indeed, from Lemma 1, Lemma 2 and the defining conditions (6) and (7), it is easy to see that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q$.

Lemma 5 *Let L and N be defined by (3) and (4) respectively. Suppose Ω is an open and bounded subset of X_ω. Then N is L-compact on $\overline{\Omega}$.*

Proof. It is easy to see that for any $x \in \overline{\Omega}$,

$$QNx(t) = -\frac{t}{\omega} \int_0^\omega \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds, \tag{12}$$

so,

$$\|QNx\|_2 = \left| \frac{1}{\omega} \int_0^\omega \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds \right|, \tag{13}$$

and

$$\begin{aligned} (I-Q)\,Nx(t) &= -\int_0^t \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds \\ &\quad + \frac{t}{\omega} \int_0^\omega \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds \end{aligned} \tag{14}$$

for $t \geq 0$. These lead us to

$$\begin{aligned} K_P\,(I-Q)\,Nx(t) &= -\int_0^t dv \int_0^v \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds \\ &\quad + \frac{t}{\omega} \int_0^\omega dv \int_0^v \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds \\ &\quad + \frac{t^2}{2\omega} \int_0^\omega \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds \\ &\quad - \frac{t}{2} \int_0^\omega \left(f\left(s, x'(s)\right) + g\left(s, x\left([s-k]\right)\right)\right) ds. \end{aligned} \tag{15}$$

By (13), we see that $QN\left(\bar{\Omega}\right)$ is bounded. Noting that (7) holds and N is a completely continuous mapping, by means of the Arzela-Ascoli theorem we know that $\overline{K_P\,(I-Q)\,N\left(\bar{\Omega}\right)}$ is relatively compact. Thus N is L-compact on $\overline{\Omega}$. The proof is complete. □

Lemma 6 *Suppose $g(t)$ is a real, bounded and continuous function on $[a,b)$ and $\lim_{t\to b^-} g(t)$ exists. Then there is a point $\xi \in (a,b)$ such that*

$$\int_a^b g(s)\,ds = g(\xi)(b-a). \tag{16}$$

The above result is only a slight extension of the integral mean value theorem and is easily proved.

We will need the integral equation

$$x'(t) = x'(0) - \lambda \int_0^t \left(f(s, x'(s)) + g(s, x([s-k]))\right) ds, \;\; t \in R, \tag{17}$$

where $\lambda \in (0,1)$.

We now turn to the proof of Theorem 1: Let L, N, P and Q be defined by (3), (4), (6) and (7) respectively. Let $x(t)$ be a ω-periodic solution of (9). By (9), we have

$$\int_0^\omega \left(f(s, x'(s)) + g(s, x([s-k]))\right) ds = 0, \tag{18}$$

that is

$$\int_0^\omega f(s, x'(s))\, ds = \sum_{i=1}^{\omega} \int_{i-1}^{i} g(s, x([s-k]))\, ds. \tag{19}$$

Using the integral mean value theorem and Lemma 6, there are $\xi_i \in [i-1, i]$, $i = 1, 2, ..., \omega$, and $\xi \in [0, \omega]$ such that

$$f(\xi, x'(\xi)) = -\frac{1}{\omega} \sum_{i=1}^{\omega} g(\xi_i, x([i-1-k])). \tag{20}$$

Let $\Phi = \max_{0 \le t \le \omega} x(t)$, $\Psi = \min_{0 \le t \le \omega} x(t)$,

$$M = \max_{0 \le t \le \omega, \Psi \le x \le \Phi} g(t, x)$$

and

$$m = \min_{0 \le t \le \omega, \Psi \le x \le \Phi} g(t, x).$$

Since $x(t)$ is ω-periodic, we see that

$$m \le \frac{1}{\omega} \sum_{i=1}^{\omega} g(\xi_i, x([i-1-k])) \le M. \tag{21}$$

By (21), the continuity of $g(t,x)$, and the intermediate value theorem, there are η and $t_1 \in [0,\omega]$ such that

$$\frac{1}{\omega}\sum_{i=1}^{\omega} g(\xi_i, x([i-1-k])) = g(\eta, x(t_1)). \tag{22}$$

From (20) and (22) we have

$$f(\xi, x'(\xi)) = g(\eta, x(t_1)). \tag{23}$$

We assert that

$$|x(t_1)| \le \frac{r_1}{r_2}\|x'\|_0 + D + \frac{K}{r_2}. \tag{24}$$

Indeed our assertion is true if $|x(t_1)| \le D$. Otherwise, by (a_1), (b_1) and (23), we have

$$\begin{aligned} r_2|x(t_1)| &\le |g(\eta, x(t_1))| = |f(\xi, x'(\xi))| \\ &\le r_1|x'(\xi)| + K \le r_1\|x'\|_0 + K, \end{aligned} \tag{25}$$

which implies (24).

For for any $t \in [0,\omega]$, we now have

$$\begin{aligned} |x(t)| &\le |x(t_1)| + \left|\int_{t_1}^{t} x'(s)\,ds\right| \\ &\le |x(t_1)| + \int_0^{\omega}|x'(s)|\,ds \le \left(\frac{r_1}{r_2}+\omega\right)\|x'\|_0 + D + \frac{K}{r_2}, \end{aligned} \tag{26}$$

so that

$$\|x\|_0 \le \left(\frac{r_1}{r_2}+\omega\right)\|x'\|_0 + D + \frac{K}{r_2}. \tag{27}$$

By condition (d_1), we know that there is a positive number ε such that

$$\eta_1 = 2\omega\left[r_1 + (r_3+\varepsilon)\left(\frac{r_1}{r_2}+\omega\right)\right] < 1. \tag{28}$$

From condition (c_1) ,we see that there is an $\rho > D$ such that for $t \in R$ and $x < -\rho$,

$$\frac{g(t,x)}{x} < r_3 + \varepsilon. \tag{29}$$

Let

$$E_1 = \{t \mid t \in [0,\omega], x([t-k]) < -\rho\}, \tag{30}$$

$$E_2 = \{t \mid t \in [0,\omega], |x([t-k])| \leq \rho\}, \tag{31}$$

$$E_3 \backslash (E_1 \cup E_2) \tag{32}$$

and

$$M_0 = \max_{0 \leq t \leq 2\pi, |x| \leq \rho} |G(t,x)|. \tag{33}$$

By (27), (29) and (30), we have

$$\begin{aligned} \int_{E_1} |g(t, x([t-k]))|\, dt &\leq \int_{E_1} (r_3+\varepsilon)\, |x([t-k])|\, dt \\ &\leq \omega (r_3+\varepsilon) \max_{0 \leq t \leq 2\pi} |x(t)| = \omega (r_3+\varepsilon) \|x\|_0 \\ &\leq \omega (r_3+\varepsilon) \left[\left(\frac{r_1}{r_2} + \omega \right) \|x'\|_0 + D + \frac{K}{r_2} \right]. \end{aligned} \tag{34}$$

From (31) and (33), we have

$$\int_{E_2} |g(t, x([t-k]))|\, dt \leq \omega M_0. \tag{35}$$

It follows from condition (a_1) that

$$\int_0^\omega |f(t, x'(t))|\, dt \leqslant \omega (r_1 \|x'\|_0 + K). \tag{36}$$

In view of (b_1), (18), (30), (31), (32), (34), (35) and (36), we get

$$\begin{aligned}
\int_{E_3} |g(t, x([t-k]))|\, dt &= \int_{E_3} g(t, x([t-k]))\, dt \\
&= -\int_0^{\omega} f(t, x'(t))\, dt - \int_{E_1} g(t, x([t-k]))\, dt \\
&\quad -\int_{E_2} g(t, x(t-\tau(t)))\, dt \\
&\leq \int_0^{\omega} |f(t, x'(t))|\, dt + \int_{E_1} |g(t, x([t-k]))|\, dt \\
&\quad + \int_{E_2} |g(t, x([t-k]))|\, dt \\
&\leq \omega(r_1 \|x'\|_0 + K) + \omega M_0 \\
&\quad + \omega(r_3+\varepsilon)\left[\left(\frac{r_1}{r_2}+\omega\right)\|x'\|_0 + D + \frac{K}{r_2}\right] \\
&\leq \omega\left[r_1 + (r_3+\varepsilon)\left(\frac{r_1}{r_2}+\omega\right)\right]\|x'\|_0 + M_1, \qquad (37)
\end{aligned}$$

for some positive number M_1. Thus it follows from (9), (34), (35), (36) and (37) that

$$\begin{aligned}
\int_0^{2\pi} |x''(t)|\, dt &\leq \int_0^{2\pi} |f(t, x'(t-\sigma(t)))|\, dt + \int_{E_1} |g(t, x([t-k]))|\, dt \\
&\quad + \int_{E_2} |G(t, x([t-k]))|\, dt + \int_{E_3} |G(t, x([t-k]))|\, dt + 2\pi \|p\|_0 \\
&\leq 2\pi(r_1\|x'\|_0 + K) + 2\pi(r_3+\varepsilon)\left[\left(\frac{r_1}{r_2}+2\pi\right)\|x'\|_0 + D + \frac{K}{r_2}\right] \\
&\quad + 2\pi M_0 + 2\pi\left[r_1 + (r_3+\varepsilon)\left(\frac{r_1}{r_2}+2\pi\right)\right]\|x'\|_0 + M_1 \\
&\quad + 2\pi\|p\|_0 \\
&= \eta_1\|x'\|_0 + M_2, \qquad (38)
\end{aligned}$$

for some positive number M_2. Note that $x(0) = x(2\pi)$, therefore there is a $t_2 \in [0, \omega]$ such that $x'(t_2) = 0$. Hence, for any $t \in [0, \omega]$, we have

$$|x'(t)| = \left| \int_{t_2}^{t} x''(s)\, ds \right| \le \int_0^{\omega} |x''(t)|\, dt, \tag{39}$$

that is

$$\|x'\|_0 \le \int_0^{\omega} |x''(t)|\, dt. \tag{40}$$

By (38) and (40), we see that

$$\|x'\|_0 \le \eta_1 \|x'\|_0 + M_2, \tag{41}$$

so that

$$\|x'\|_0 \le D_1, \tag{42}$$

where $D_1 = M_2 / (1 - \eta_1)$. From (27) and (42),we get

$$\|x\|_0 \le D_0 \tag{43}$$

where $D_0 = \left(\frac{r_1}{r_2} + \omega\right) D_1 + D + \frac{K}{r_2}$. Take a positive number $\overline{D} > \max\{D_0, D_1\} + D$, and let

$$\Omega = \left\{x \in X \mid \|x\|_1 < \overline{D}\right\}. \tag{44}$$

From Lemma 1 and Lemma 2, we know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. In view of the bounds found above for periodic solutions, we see that for any $\lambda \in (0,1)$ and any $x \in \partial\Omega$, $Lx \ne \lambda N x$. Since for any $x \in \partial\Omega \cap \mathrm{Ker} L$, $x = \overline{D}$ $(> D)$ or $x = -\overline{D}$, thus in view of (b_1) and (7) we have

$$\begin{aligned} QNx(t) &= -\frac{t}{\omega} \int_0^{\omega} \left(f(s, x'(s)) + g(s, x([s-k]))\right) ds \\ &= -\frac{t}{\omega} \int_0^{\omega} \left(f(s, 0) + g(s, x(([t-k])))\right) ds \\ &= -\frac{t}{\omega} \int_0^{2\pi} g(s, x)\, ds, \end{aligned}$$

so

$$QNx \neq \theta_2.$$

The isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is defined by $J(t\alpha) = \alpha$ for $\alpha \in R$ and $t \in R$. Then

$$JQNx = -\frac{1}{\omega}\int_0^\omega g(s,,x)\,ds \neq 0.$$

In particular, we see that if $x = \overline{D}$, then

$$JQNx = -\frac{1}{\omega}\int_0^\omega g\left(s,\overline{D}\right) ds < 0, \tag{45}$$

and if $x = -\overline{D}$, then

$$JQNx = -\frac{1}{\omega}\int_0^\omega g\left(s,-\overline{D}\right) ds > 0. \tag{46}$$

Consider the mapping

$$H(x,s) = -sx + (1-s)\,JQNx,\ \ 0 \leq s \leq 1. \tag{47}$$

From (45) and (47), for each $s \in [0,1]$ and $x = \overline{D}$, we have

$$H(x,s) = -s\overline{D} + (1-s)\frac{-1}{\omega}\int_0^\omega g\left(s,\overline{D},\overline{D}\right) ds < 0, \tag{48}$$

Similarly, from (46) and (47), for each $s \in [0,1]$ and $x = -\overline{D}$, we have

$$H(x,s) = s\overline{D} + (1-s)\frac{-1}{\omega}\int_0^\omega g\left(s,-\overline{D}\right) ds > 0. \tag{49}$$

By (48) and (49), $H(x,s)$ is a homotopy. This shows that

$$\deg\left(JQNx, \Omega \cap \mathrm{Ker}L, \theta_1\right) = \deg\left(-x, \Omega \cap \mathrm{Ker}L, \theta_1\right) \neq 0.$$

By Theorem A, we see that equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{Dom}L$. In other words, (1) has an ω-periodic solution $x(t)$. The proof is complete. □

The proof of Theorem 2 is similar to that of Theorem 1, and so we omit the details here.

Proof of Theorem 3 Let $x(t)$ be a ω-periodic solution of (9). Then (18) and (23) hold. We will prove that there are positive numbers D_2 and D_3 such that

$$\|x\|_0 \le D_2 \text{ and } \|x'\|_0 \le D_3. \tag{50}$$

By (a_2) and (23) we see that

$$|g(\eta, x(t_1))| = |f(\xi, x'(\xi))| \le K. \tag{51}$$

It follows from (b_2) and (51) that

$$|x(t_1)| \le D. \tag{52}$$

Thus for any $t \in [0, \omega]$, we have

$$\begin{aligned} |x(t)| &\le |x(t_1)| + \int_0^\omega |x'(s)|\, ds \\ &\le D + \omega \|x'\|_0 , \end{aligned}$$

so that

$$\|x\|_0 \le D + \omega \|x'\|_0 . \tag{53}$$

In view of condition (c_3), we can take a positive number ε_1 such that $\eta_2 = 2\omega^2 (r + \varepsilon_1) < 1$. Furthermore, we see that there is an $\rho_1 > D$ such that for $t \in R$ and $x < -\rho_1$,

$$\frac{g(t,x)}{x} < r + \varepsilon_1. \tag{54}$$

Let

$$E_1' = \{t \mid t \in [0, \omega], x([t-k]) < -\rho_1\}, \tag{55}$$

$$E_2' = \{t \mid t \in [0, \omega], |x([t-k])| \le \rho_1\}, \tag{56}$$

$$E_3' \backslash (E_1' \cup E_2') \tag{57}$$

and

$$M_3 = \max_{0 \le t \le \omega, |x| \le \rho_1} |g(t,x)| . \tag{58}$$

By (53), (54) and (55), we have

$$\begin{aligned}\int_{E_1'} |g(t, x([t-k]))|\, dt &\leq \int_{E_1'} (r+\varepsilon_1) |x([t-k])|\, dt \\ &\leq \omega (r+\varepsilon_1) \max_{0\leq t\leq \omega} |x(t)| = \omega (r+\varepsilon) \|x\|_0 \\ &\leq \omega (r+\varepsilon_1) [D + \omega\ \|x'\|_0]. \end{aligned} \tag{59}$$

From (56) and (58), we have

$$\int_{E_2'} |g(t, x([t-k]))|\, dt \leq \omega M_3. \tag{60}$$

It follows from condition (a_2) that

$$\int_0^\omega |f(t, x'(t))|\, dt \leqslant \omega K. \tag{61}$$

In view of (b_2), (18), (59), (60) and (61), we get

$$\begin{aligned}\int_{E_3'} |g(t, x([t-k]))|\, dt &= \int_{E_3'} g(t, x([t-k]))\, dt \\ &= -\int_0^\omega f(t, x'(t))\, dt - \int_{E_1} g(t, x([t-k]))\, dt \\ &\quad - \int_{E_2'} g(t, x([t-k]))\, dt \\ &\leq \int_0^\omega |f(t, x'(t))|\, dt + \int_{E_1} |g(t, x([t-k]))|\, dt \\ &\quad + \int_{E_2'} |g(t, x([t-k]))|\, dt \\ &\leq \omega K + \omega (r+\varepsilon_1) [D + \omega\ \|x'\|_0] + \omega M_3 \\ &\leq \omega^2 (r+\varepsilon_1) \|x'\|_0 + M_4, \end{aligned} \tag{62}$$

for some positive number M_4. It follows from (9), (59), (60), (61) and (62) that

$$\begin{aligned}\int_0^\omega |x''(t)|\,dt &\le \int_0^\omega |f(t,x'(t))|\,dt + \int_{E_1'} |g(t,x([t-k]))|\,dt \\ &\quad + \int_{E_2'} |g(t,x([t-k]))|\,dt + \int_{E_3'} |g(t,x([t-k]))|\,dt \\ &\le \omega K + \omega(r+\varepsilon_1)\left[D+\omega\,\|x'\|_0\right] + \omega M_3 + \omega^2(r+\varepsilon_1)\|x'\|_0 + M_4 \\ &= \eta_2\|x'\|_0 + M_5, \end{aligned} \tag{63}$$

for some positive number M_5. Since $x(0) = x(\omega)$, there is a $t_3 \in [0,\omega]$ such that $x'(t_3) = 0$. Hence, for any $t \in [0,\omega]$, we have

$$|x'(t)| = \left|\int_{t_3}^t x''(s)\,ds\right| \le \int_0^\omega |x''(t)|\,dt, \tag{64}$$

that is

$$\|x'\|_0 \le \int_0^\omega |x''(t)|\,dt. \tag{65}$$

By (63) and (65), we see that

$$\|x'\|_0 \le \eta_2\|x'\|_0 + M_5, \tag{66}$$

so that

$$\|x'\|_0 \le D_3, \tag{67}$$

where $D_3 = M_5/(1-\eta_2)$. From (53) and (67), we get

$$\|x\|_0 \le D_2 \tag{68}$$

where $D_2 = D + \omega D_3$. From (67) and (68), we see that there are positive numbers D_2 and D_3 such that (50) hold. The remaining proof is the same as that of Theorem 1. The proof is complete. □

The proof of Theorem 4 is similar to that of Theorem 3, and so we omit the details here.

Example. *Consider a Rayleigh equation of the form*

$$x''(t) + \frac{1+\cos \pi t}{48\pi(1+\pi)} x'(t) + \exp\left(-(x'(t))^2\right) + \frac{\exp\left((\sin \pi t)^2\right) h\left(x\left([t-k]\right)\right)}{25\pi(\pi+1)} = 1, \quad (69)$$

where k is a positive integer and

$$h(x) = \begin{cases} x^3 & x \geqslant 0 \\ x & x < 0 \end{cases}.$$

Take

$$f(t,x) = \frac{1+\cos \pi t}{200} x + \exp\left(-x^2\right) - 1,$$

and

$$g(t,x) = \frac{\exp\left((\sin \pi t)^2\right) h(x)}{101},$$

it is then easy to verify that all the assumptions in Theorem 1 are satisfied with $K = 2$, $D = 1$, $r_1 = \frac{1}{100}$, $r_2 = \frac{1}{101}$ and $r_3 = \frac{e}{101}$. Thus (69) has a 2-periodic solution. Furthermore, this solution is nontrivial since $y(t) \equiv 0$ is not a solution of (69).

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