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A Connected Sum of Knots and Fintushel-Stern Knot Surgery on 4-manifolds

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Abstract

We give some new examples of smooth 4-manifolds which are diffeomorphic although they are obtained by Fintushel-Stern knot surgeries on a smooth 4-manifold with different knots; the first such examples are given by Akbulut [1]. In the proof we essentially use the monodromy of a cusp.

1. Introduction

Let X be a smooth 4-manifold. In [4] a *cusp* in X is a PL embedded 2-sphere of self-intersection 0 with a single nonlocally flat point whose neighborhood is the cone on the right-hand trefoil knot. The regular neighborhood of a cusp is called a *cusp neighborhood.* It is fibered by smooth tori with one singular fiber, the cusp, and the monodromy is

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

If T is a smoothly embedded torus representing a nontrivial homology class $[T]$, we say that T is *c-embedded* if T is a smooth fiber in a cusp neighborhood.

Consider an oriented knot K in S^3, and let m denote an oriented meridional circle to K; see Figure 1. Let M_K be the 3-manifold obtained by performing 0-framed surgery on K. Then m can also be viewed as a circle in M_K. In $M_K \times S^1$ we have a smooth torus

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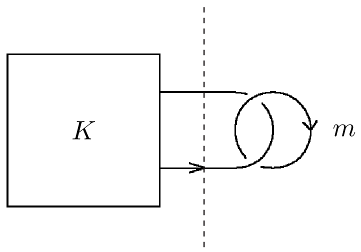

Figure 1

$T_m = m \times S^1$ of self-intersection 0. Since a tubular neighborhood of m has a canonical framing in M_K, a tubular neighborhood of the torus T_m in $M_K \times S^1$ has a canonical identification with $T_m \times D^2$. Let $X_{(K,\phi)}$ denote the fiber sum

$$X_{(K,\phi)} := [X \setminus (T \times D^2)] \cup_\phi [(M_K \times S^1) \setminus (T_m \times D^2)],$$

where $T \times D^2$ is a tubular neighborhood of the torus T in the manifold X and $\phi : \partial(T \times D^2) \to \partial(T_m \times D^2)$ is a homeomorphism. In general, the diffeomorphism type of $X_{(K,\phi)}$ depends on ϕ. If we fix an identification of T with $S^1 \times S^1$ and a homeomorphism $\phi : \partial(T \times D^2) \to \partial(T_m \times D^2)$ such that

$$\begin{aligned} \phi(S^1 \times * \times *) &= m \times * \times *, \\ \phi(* \times S^1 \times *) &= * \times S^1 \times *, \\ \phi(* \times * \times \partial D^2) &= * \times * \times \partial D^2, \end{aligned}$$

where $*$'s are points, then we shall simply denote $X_{(K,\phi)}$ by X_K. We call this operation *Fintushel-Stern knot surgery* on a 4-manifold X with K.

In case $H_1(X, \mathbf{Z})$ has no 2-torsion there is a natural identification of the *spin*^{c} structures of X with the characteristic elements of $H^2(X, \mathbf{Z})$. Recall that the Seiberg-Witten invariant SW_X is a function

$$SW_X : \{k \in H^2(X, \mathbf{Z}) | k \equiv w_2(TX) \bmod 2\} \to \mathbf{Z}.$$

The function SW_X has a compact support $B = \{\pm\beta_1, \dots, \pm\beta_n\}$ which is called *the set of basic classes*. By setting $t_\beta := \exp\beta$ for each $\beta \in H^2(X, \mathbf{Z})$, the function SW_X is usually written as a Laurent polynomial

$$SW_X = \sum_{\beta \in B} SW_X(\beta) t_\beta.$$

Then Fintushel and Stern [4] theorem says:

Theorem 1.1 *Let X be a simply connected smooth 4-manifold with $b^+ > 1$. Suppose that X contains a smoothly c-embedded torus T such that $\pi_1(X \setminus T) = 1$. Then*

$$SW_{X_K} = SW_X \cdot \Delta_K(t),$$

where $t = \exp 2[T]$ and $\Delta_K(t)$ is the Alexander polynomial of K.

To make sense of the statement of the theorem, we need to replace $[T]$ by its Poincaré dual.

Since the Seiberg-Witten invariant is a diffeomorphism invariant, if SW_X and $\Delta_K(t)$ are nontrivial, then X and X_K are not diffeomorphic. Fintushel and Stern conjectured that if X is the Kummer surface $K3$, then the association $K \to X_K$ gives an injective map from the set of isotopy classes of knots in S^3 to the set of diffeomorphism classes of smooth structures on X. In [1] Akbulut gave first counterexamples to this conjecture:

Theorem 1.2 *Let X be a smooth 4-manifold. Suppose that X contains a smoothly c-embedded torus T. Fix an identification of T with $S^1 \times S^1$. We denote the mirror reflection of an oriented knot K by K^*, see Figure 2. Then*

$$X_K = X_{K^*},$$

and this diffeomorphism leaves the core torus invariant.

We denote an oriented meridional circle to K^* by m'. In the Alexander polynomials K is equal to K^*, i.e., $\Delta_K(t) = \Delta_{K^*}(t)$. In Section 2 we give a simple proof of Theorem 1.2.

Next we give a relation between a connected sum of knots and Fintushel-Stern knot surgery; this observation is given by S. Finashin, see Lemma 3.1 in [3]. Let T_1 and T_2 be regular fibers in a cusp neighborhood in X. We fix common identifications of T_1 and T_2 with $S^1 \times S^1$ by holonomy. Let K_1 and K_2 be oriented knots in S^3. We construct X_{K_1} by using T_1 and K_1. Since X_{K_1} also has a cusp neighborhood which has T_2 as a smooth fiber, we can construct $(X_{K_1})_{K_2}$ by using T_2 and K_2.

Theorem 1.3

$$(X_{K_1})_{K_2} = X_{K_1 \sharp K_2},$$

where $K_1 \sharp K_2$ is the connected sum of K_1 and K_2.

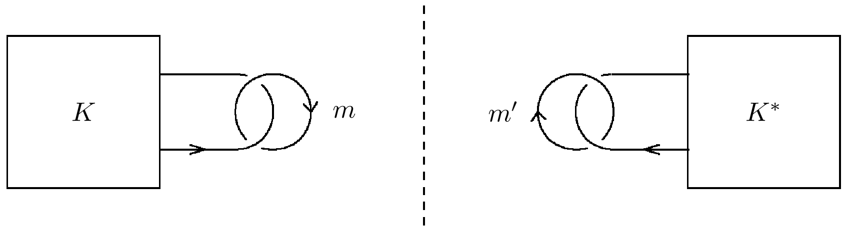

Figure 2

Note that $\Delta_{K_1}(t) \cdot \Delta_{K_2}(t) = \Delta_{K_1 \sharp K_2}(t)$. Because the core torus is invariant with respect to the diffeomorphism $X_{K_1} = X_{K_1^*}$, we obtain the following corollary:

Corollary 1.4

$$(X_{K_1})_{K_2} = X_{(K_1^*)\sharp K_2}.$$

Finally these claims give us new counterexamples to the conjecture:

Corollary 1.5

$$X_{K_1 \sharp K_2} = X_{(K_1^*)\sharp K_2}.$$

In section 3 we prove Theorem 1.3.

2. A simple proof of Theorem 1.2

In this section we give a simple proof of Theorem 1.2.

We denote the oppositely oriented circle to an oriented S^1 by $\overline{S^1}$. Let Y denote the fiber sum

$$Y := [X \setminus (T \times D^2)] \cup_\psi [(M_{K^*} \times S^1) \setminus (T_{m'} \times D^2)],$$

where $\psi : \partial(T \times D^2) \to \partial(T_{m'} \times D^2)$ is a homeomorphism such that

$$\begin{aligned} \psi(\overline{S^1} \times * \times *) &= m' \times * \times *, \\ \psi(* \times \overline{S^1} \times *) &= * \times S^1 \times *, \\ \psi(* \times * \times \partial D^2) &= * \times * \times \partial D^2. \end{aligned}$$

Since the third power of the monodromy of the cusp is $-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on a smooth fiber T, Y is diffeomorphic to X_{K^*}. Let $f : M_K \to M_{K^*}$ be an orientation reversing diffeomorphism which maps the points to their mirror reflection points and $f \times (-\mathrm{id}_{S^1}) : M_K \times S^1 \to M_{K^*} \times S^1$ an orientation preserving diffeomorphism, where $-\mathrm{id}_{S^1}$ is the orientation reversing diffeomorphism of S^1. Then we can construct a diffeomorphism $F : X_K \to Y$ by

$$F(x) := \begin{cases} (f \times (-\mathrm{id}_{S^1}))(x), & \text{for } x \in (M_K \times S^1) \setminus (T_m \times D^2) \\ x, & \text{for } x \in X \setminus (T \times D^2), \end{cases}$$

and F maps the core torus to itself. Hence $X_{K^*} = Y = X_K$ and we finish proving the theorem. $\square$

3. Proof of Theorem 1.3

We define an oriented link as in Figure 3; let N be the 3-manifold obtained by

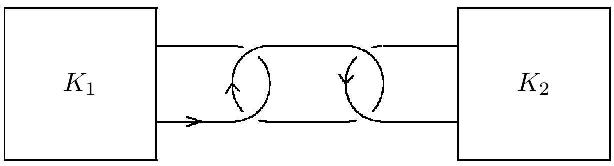

Figure 3

performing 0-framed surgery on each component of the link. Let W denote the fiber sum

$$W := [X \setminus (T \times D^2)] \cup_\phi [(N \times S^1) \setminus (T_{m_1} \times D^2)],$$

where m_1 is an oriented meridional circle to K_1. Because T_2 is ambient isotopic to $m_1 \times S^1$, we can easily see that W is diffeomorphic to $(X_{K_1})_{K_2}$. Now we shall play Kirby calculus on *the 3-manifold* N as in Figure 4. The last step of vanishing components can be found

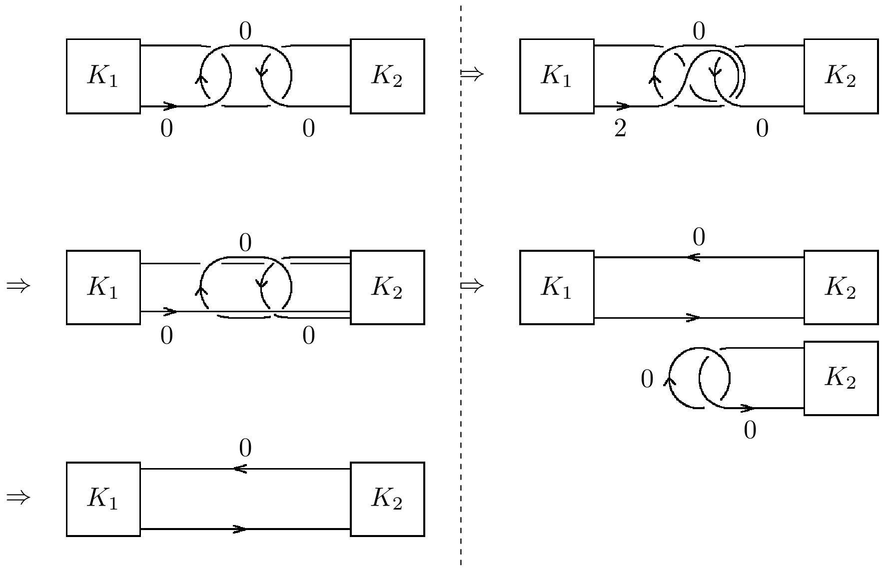

Figure 4

in example 5.2 of [6]. Hence $N \setminus (m_1 \times D^2)$ is diffeomorphic to $M_{K_1 \sharp K_2} \setminus (m_1 \times D^2)$, and W is diffeomorphic to $X_{K_1 \sharp K_2}$. We finish proving Theorem 1.3. □

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References

[1] S. Akbulut. *A Fake Cusp and a Fishtail*, Turkish J. Math. 23 (1999), no. 1, 19–31.

[2] S. Akbulut. *Variations on Fintushel-Stern Knot Surgery on 4-manifolds*, Turkish J. Math. 26 (2002), no. 1, 81–92.

[3] S. Finashin. *Knotting of algebraic curves in* CP^2, Topology 41 (2002), 47–55.

[4] R. Fintushel and R. Stern. *Knots, Links, and 4-manifolds*, Invent. Math. 134, no. 2 (1998), 363–400.

[5] R. E. Gompf and A. I. Stipsicz. *4-manifolds and Kirby Calculus*, Graduate Studies in Mathematics 20, (1999), American Mathematical Society.

[6] R. C. Kirby. *The Topology of 4-manifolds*, Lecture Notes in Mathematics 1374, (1989), Springer-Verlarg.

[7] M. Tange. Master's thesis, Kyoto University (2003).

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