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On Cauchy's Bound for Zeros of a Polynomial

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Abstract

In this note, we improve upon Cauchy's classical bound, and upon some recent bounds for the moduli of the zeros of a polynomial.

Key Words: Zeros, polynomials, upper bound, moduli, refinement.

1. Introduction and Statement of Results

Let

$$\begin{aligned} f(z) &= z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0, \\ & a_i \neq 0, \text{ for at least one } i \in I, \\ & I = \{0, 1, 2, \ldots, n-1\}, \end{aligned}$$

be a polynomial of degree n, with complex coefficients. Then, according to Cauchy's classical result [1], we have the following theorem.

Theorem A

$$Z[f(z)] \subset \overline{B}(\eta) \subset B(1+a),$$

where η is the unique positive root of the equation

$$Q(x) = 0,$$

$$Q(x) = x^n - |a_{n-1}|x^{n-1} - |a_{n-2}|x^{n-2} - \ldots - |a_1|x - |a_0|, \tag{1}$$

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$$Z[f(z)] = \text{the set of all zeros of the polynomial } f(z),$$

$$B(r) = \{z : |z| < r\}, \bar{B}(r) = \{z : |z| \le r\}.$$

and

$$a = \max_{i \in I} |a_i|. \tag{2}$$

Sun and Hsieh [2] obtained certain refinements of Cauchy's classical bound. They proved the next theorem.

Theorem B

$$Z[f(z)] \subset \overline{B}(\eta) \subset B(1+\delta_1) \subset B(1+\delta_2) \subset B(1+a),$$

with

$$\eta < 1 + \delta_1 \le 1 + \delta_2 \le 1 + a,$$

where δ_1 is the unique positive root of the equation

$$Q_1(x) = 0,$$

$$Q_1(x) = x^3 + (2 - |a_{n-1}|)x^2 + (1 - |a_{n-1}| - |a_{n-2}|)x - a, \tag{3}$$

and

$$\delta_2 = \frac{1}{2}[(|a_{n-1}| - 1) + \sqrt{(|a_{n-1}| - 1)^2 + 4a}].$$

Theorem C Let

$$\begin{aligned}
g_1(z) &= (-1)^n f(z) f(-z), \\
h(z) &= g_1(\sqrt{z}) = \sum_{i=0}^{n} b_i z^i, \text{ say}, \\
b &= \max_{i \in I} |b_i|, \\
m &= \max\{i : i \in I \ \& \ |b_i| = b\}, \\
\widetilde{b} &= \max_{i \in I \sim \{m\}} |b_i|,
\end{aligned}$$

$$\alpha = \begin{cases} \min[(b/\widetilde{b})^{1/(2(n-m-1))}, \{b(n-m-1)\}^{1/2(n-m)}]; \widetilde{b} \neq 0, m \neq n-1 \ \& \ b \geq 1 \\ 1; \text{ otherwise} \end{cases},$$

$$\begin{aligned} g_2(z) &= \alpha^{-2n}(-1)^n f(\alpha z) f(-\alpha z), \\ C(z) &= g_2(\sqrt{z}) = \sum_{i=0}^{n} c_i z^i, \text{ say, (with } c_n = 1, \text{ obviously)}, \\ c &= \max_{i \in I} |c_i|. \end{aligned}$$

Then

$$Z[f(z)] \subset \bar{B}(\alpha\sqrt{\widetilde{\eta}}) \subset B(\alpha\sqrt{1+\widetilde{\delta_1}}) \subset B(\alpha\sqrt{1+\widetilde{\delta_2}}) \subset B(\alpha\sqrt{1+c}), \tag{4}$$

with

$$\widetilde{\eta} < 1 + \widetilde{\delta_1} \leq 1 + \widetilde{\delta_2} \leq 1 + c,$$

where $\widetilde{\eta}$ is the unique positive root of the equation

$$\widetilde{Q}(x) = 0,$$

$\widetilde{\delta_1}$ is the unique positive root of the equation

$$\widetilde{Q_1}(x) = 0,$$

$$\begin{aligned} \widetilde{Q}(x) &= x^n - |c_{n-1}|x^{n-1} - |c_{n-2}|x^{n-2} - \ldots - |c_1|x - |c_0|, \\ \widetilde{Q_1}(x) &= x^3 + (2 - |c_{n-1}|)x^2 + (1 - |c_{n-1}| - |c_{n-2}|)x - c, \end{aligned}$$

and

$$\widetilde{\delta_2} = \frac{1}{2}\{(|c_{n-1}| - 1) + \sqrt{(|c_{n-1}| - 1)^2 + 4c}\}.$$

In this note, we have also obtained a refinement of Cauchy's classical bound and then obtained certain other similar bounds also. More precisely, we have proved the following theorem.

Theorem 1

$$Z[f(z)] \subset \bar{B}(\eta) \subset B(1+\delta_0) \subset B(1+\delta_1),$$

with

$$\eta < 1+\delta_0 \leq 1+\delta_1,$$

where δ_0 *is the unique positive root of the equation*

$$\begin{aligned} Q_0(x) &= 0, && (5)\\ Q_0(x) &= x^4 + (3-|a_{n-1}|)x^3 + (3-2|a_{n-1}|-|a_{n-2}|)x^2 + \\ &\quad (1-|a_{n-1}|-|a_{n-2}|-|a_{n-3}|)x - a. && (6) \end{aligned}$$

Remark 1 It is obvious that Theorem 1 is a refinement of Theorem B and therefore, also a refinement of Cauchy's classical bound.

Theorem 2

$$Z[f(z)] \subset \bar{B}(\alpha\sqrt{\widetilde{\eta}}) \subset B(\alpha\sqrt{1+\widetilde{\delta_0}}) \subset B(\alpha\sqrt{1+\widetilde{\delta_1}}),$$

with

$$\widetilde{\eta} < 1+\widetilde{\delta_0} \leq 1+\widetilde{\delta_1},$$

where $\widetilde{\delta_0}$ *is the unique positive root of the equation*

$$\begin{aligned} \widetilde{Q_0}(x) &= 0, \\ \widetilde{Q_0}(x) &= x^4 + (3-|c_{n-1}|)x^3 + (3-2|c_{n-1}|-|c_{n-2}|)x^2 \\ &\quad +(1-|c_{n-1}|-|c_{n-2}|-|c_{n-3}|)x - c. \end{aligned}$$

Remark 2 It is obvious that Theorem 2 is a refinement of Theorem C. Therefore, thinking of Theorem 1 and Theorem 2 together, we can say that we have got upper bounds for the moduli of the zeros of the polynomial $f(z)$, better than those obtained by Sun and Hsieh [2], and hence, also better than those obtained by Zilovic et al. [3], as suggested by Sun and Hsieh [2].

2. Proofs of the Theorems

Proof of Theorem 1 That equation (5) has a unique positive root δ_0, follows by the use of Descartes' rule of signs. Further,

$$\begin{aligned}
Q(1+\delta_0) &= (1+\delta_0)^n - |a_{n-1}|(1+\delta_0)^{n-1} - |a_{n-2}|(1+\delta_0)^{n-2} - \\
&\quad |a_{n-3}|(1+\delta_0)^{n-3} - |a_{n-4}|(1+\delta_0)^{n-4} - \ldots \\
&\quad \ldots - |a_0|, \text{ (by (1))}, \\
&\geq (1+\delta_0)^n - |a_{n-1}|(1+\delta_0)^{n-1} - |a_{n-2}|(1+\delta_0)^{n-2} \\
&\quad -|a_{n-3}|(1+\delta_0)^{n-3} - a(1+\delta_0)^{n-4} - \ldots \\
&\quad \ldots - a(1+\delta_0) - a, \text{(by (2))}, \\
&= (1+\delta_0)^n - |a_{n-1}|(1+\delta_0)^{n-1} - |a_{n-2}|(1+\delta_0)^{n-2} \\
&\quad -|a_{n-3}|(1+\delta_0)^{n-3} - a\left\{\frac{(1+\delta_0)^{n-3}-1}{\delta_0}\right\}, \\
&> (1+\delta_0)^{n-3}\left\{(1+\delta_0)^3 - |a_{n-1}|(1+\delta_0)^2 - \right. \\
&\quad \left. |a_{n-2}|(1+\delta_0) - |a_{n-3}| - \frac{a}{\delta_0}\right\}, \\
&= \frac{(1+\delta_0)^{n-3}}{\delta_0} Q_0(\delta_0), \\
&= 0,
\end{aligned}$$

which implies

$$\eta < 1 + \delta_0.$$

Again,

$$\begin{aligned}
Q_0(\delta_1) &= Q_0(\delta_1) - \delta_1 Q_1(\delta_1) - Q_1(\delta_1) \\
&= \delta_1(a - |a_{n-3}|), \text{ (by (3) and (6))}, \\
&\geq 0,
\end{aligned}$$

thereby implying that

$$\delta_0 \leq \delta_1.$$

And now Theorem 1 follows, by using the fact that η is unique positive root of the equation

$$Q(x) = 0.$$

Proof of Theorem 2. We can prove, as in the proof of Theorem 1, that

$$\begin{aligned} \widetilde{\eta} &< 1+\widetilde{\delta_0}, \\ \widetilde{\delta_0} &\leq \widetilde{\delta_1}, \end{aligned}$$

and then Theorem 2 follows by using

$$Z[f(z)] \subset \bar{B}(\alpha\sqrt{\widetilde{\eta}}) \text{ (by (4))}.$$

References

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