Turk J Math
30 (2006) , 101 – 116.

On Finitary Permutation Groups

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Abstract

In this work we give some sufficient conditions under which the structure of a transitive group of finitary permutations on an infinite set can be determined from the structure of a point stabilizer. Also, we give some sufficient conditions for the existence of a proper subgroup having an infinite orbit in a totally imprimitive p-group of finitary permutations. These results, with the help of some known results, give sufficient conditions for the nonexistence of a perfect locally finite minimal non FC - (p-group).

Key Words: Finitary permutation, primitive, almost primitive, totally imprimitive.

1. Introduction

Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. Many authors have investigated G by imposing suitable conditions on a point stabilizer (see, for example,[1], [2], [4], [7], [9]). Also another problem which might be interesting is finding sufficient conditions under which G can have a proper subgroup having an infinite orbit. In view of the important reduction theorems given in [3] and [8], any solution of the last problem means a solution for the following well known problem: Does there exist a perfect minimal non FC - (p-group)? The aim of this work is to obtain some sufficient conditions about the problems described above.

Let Ω be a (possibly infinite)set and let $Sym(\Omega)$ be the symmetric group on Ω. For each $x \in Sym(\Omega)$ the set $supp(x) = \{i \in \Omega : x(i) \neq i\}$ is called the *support* of x and if $supp(x)$ is finite, then x is called a *finitary permutation*. The set of all the finitary permutations on Ω forms a subgroup which is denoted by $FSym(\Omega)$. Let G be a subgroup

2000 *AMS Mathematics Subject Classification:* 20B 07 20B 35 20E 25

of $Sym(\Omega)$ and $a \in \Omega$. Then $G_a = \{g \in G : g(a) = a\}$ is called the *stabilizer* of a in G and $G(a) = \{g(a) : a \in G\}$ is called the *orbit* of G containing a. More generally if Δ is a nonempty subset of Ω, then $G_\Delta = \{g \in G : g(i) = i \text{ for all } i \in \Delta\}$ and $G_{\{\Delta\}} = \{g \in G : g(\Delta) = \Delta\}$ are called the *pointwise* and the *setwise* stabilizers of Δ. Furthermore if $g(\Delta) = \Delta$ or $g(\Delta) \cap \Delta = \emptyset$, for every $g \in G$, then Δ is called *a block* for G. A block Δ which is not equal to G and contains at least two elements is called *non-trivial.*

Finally a group G is called a *minimal non FC - group* if G is not an FC - group but every proper subgroup of G is an FC - group.

The main results of this work are stated below.

Theorem 1.1 *Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. Let F be a finite non-abelian subgroup of G and let Δ be a non-trivial block such that supp$(F) \subseteq \Delta$. Let S be a normal solvable subgroup of $G_{\{\Delta\}}$ of derived length $d \geq 0$. Then $\langle S^x : x \in G\rangle \neq G$.*

Theorem 1.2 *Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite.Then the following hold:*

(a) A point stabilizer cannot be solvable.

(b) If G satisfies the normalizer condition, then it is a p-group and G' is a minimal non FC - group.

If in Theorem 1.2(a) G is barely transitive (see[4] for the definition of a barely transitive group), then the result follows from [1, Theorem] or [4, Theorem 1]. Furthermore Theorem 1.2(b) is not true if the normalizer condition is satisfied only by a point stabilizer. Indeed in the example given at the end of Section 2 it is shown that in Wiegold's group[16, p.468] a point stabilizer satisfies the normalizer condition but the commutator subgroup of it, which is a perfect proper subgroup of the group, is not a minimal non FC - group.

Theorem 1.3 *Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. Then the following hold:*

(a) If a point stabilizer is locally solvable, then G is locally solvable.

(b) If a point stabilizer is locally nilpotent - by - solvable, then G is a p-group for some prime p.

Pinnock[12] shows that if a transitive subgroup of $FSym(\Omega)$ is either locally(nilpotent - by - abelian) or locally supersolvable, then it is a p - group. These results can be generalized as follows:

Corollary 1.4 *Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite.Then the following hold:*

(a) If a point stabilizer is locally (nilpotent - by - abelian), then G is a p-group for some prime p.

(b) If a point stabilizer is locally supersolvable then G is a p-group for some prime p.

Theorem 1.5 *Let G be a totally imprimitive p - subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that for every non-normal finite subgroup F of G there exists $y \in G \setminus N_G(F)$ such that $y^p \in C_G(F)$. Then G contains a proper subgroup that has an infinite orbit.*

Corollary 1.6 *Let G be a totally imprimitive p - subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that for every non-normal finite subgroup F of G there exists $y \in G \setminus N_G(F)$ such that $y^p \in FC_G(F)$. Then G contains a proper subgroup that has an infinite orbit.*

Corollary 1.7 *Let G be a totally imprimitive p - subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that there exists an infinite properly ascending chain of non-trivial blocks $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_k \subset \dots$ for G such that the following holds: For each $k \geq 1$ $< F_k^x : x \in G >$ is the largest normal subgroup of G that is contained in $G_{\{\Delta_k\}}$, where $F_k = \{x \in G : supp(x) \subseteq \Delta_k\}$. Then G contains a proper subgroup that has an infinite orbit.*

Corollary 1.8 *Let G be a totally imprimitive p - subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that there exists an infinite properly ascending chain of non-trivial blocks $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_k \subset \dots$ for G such that the following holds: For each $k \geq 1$ there exists $y_k \in G \setminus G_{\Delta_k}$ such that $< y_k > \cap G_{\{\Delta_k\}} \leq G_{\Delta_k}$. Then G contains a proper subgroup that has an infinite orbit.*

Corollary 1.8 can be used to prove the following:

Corollary 1.9 *([6, Theorem 3]) Let G be a totally imprimitive p - subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that $G =< x \in G : x^p = 1 >$. Then G cannot be a minimal non FC - group.*

Proof. Assume that G is a minimal non FC - group. Then every orbit of every proper subgroup of G is finite by [5, Lemma 8.3D] or [16, Theorem 1]. Let $X = \{x \in G : x^p = 1\}$. Assume that $G =< X >$. Let $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_k \subset \ldots$ be an infinite properly ascending chain of non-trivial blocks for G. By hypothesis for each $k \geq 1$ there exists an $x \in X$ such that $x \in G \setminus G_{\{\Delta_k\}}$ and $< x > \cap G_{\{\Delta_k\}} = 1$. But then G contains a proper subgroup that has an infinite orbit by Corollary 1.8, which is a contradiction. □

Remark 1.10 An easy induction shows that the group G of Corollary 1.9 cannot be generated by a subset of finite exponent.

Theorem 1.5 together with the important reduction theorems given in [3, Theorem 1] or [8, Theorem] gives the following:

Theorem 1.11 *Let G be a locally finite p - group that is also a minimal non FC- group. Assume that for every finite non-normal subgroup F of G there exists $y \in G \setminus N_G(F)$ such that $y^p \in FC_G(F)$. Then G cannot be perfect.*

The notation and the definitions are standard and may be found in [5], [10], [11] and [13]. Finally for a nonempty subset X we define $exp(X)$ to be the maximum of the set $\{o(x) : x \in X\}$, if it exists, otherwise we set it equal to ∞.

2. Proofs of Theorems 1, 2, 3

Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. If G has no non-trivial blocks, then it is called **primitive**, and if G has non-trivial blocks, then it is called **imprimitive**. If G is primitive, then it is isomorphc to $Alt(\Omega)$ or $FSym(\Omega)$ by [5, Lemma 8.3A] or [10, Theorem 2.3]. Next suppose that G is imprimitive. Then any non-trivial block is finite. If G has a maximal non-trivial block Δ, then $\Sigma = \{x(\Delta) : x \in G\}$ has a **system of blocks** for G. It is easy to see that G acts primitively on Σ, and so it has an epimorphic image which isomorphic to $Alt(\Omega)$ or $FSym(\Omega)$ by [5, Lemma 8.3A] or [10, Theorem 2.3] or [14, Proposition]. In the remaining case there exists a strictly increasing infinite ascending chain

$$\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_k \ \subset \ldots (1)$$

of finite blocks for G such that $\Omega = \bigcup_{k=1}^{\infty} \Delta_k$. In this case, both of Ω and G are countably infinite. Let $k \geq 1$ and put $\Sigma_k = \{x(\Delta_k) : x \in G\}$. For each $g \in G$ the equality $\bar{g}(x(\Delta_k)) = (gx)(\Delta_k)$ defines a permutation $\bar{g}$ on Σ_k and the correspondence $g \rightarrow \bar{g}$ defines a representation of G into $FSym(\Sigma_k)$. Let N_k denote the kernel of this representation. Then

$$N_1 \leq N_2 \leq \cdots \leq N_k \leq \ldots (2)$$

is an ascending chain of proper normal subgroups of G such that $G = \bigcup_{k=1}^{\infty} N_k$ and each N_k is isomorphic to a restricted direct product of isomorphic copies of a finite epimorphic image of itself. In particular, each N_k is an FC - group (see [5, Lemma 8.3B(i)] or [11, Theorem 2.4]). The two cases of the imprimitive case are called *almost primitive* and *totally imprimitive* by P.M. Neumann. In the rest of this work these will be used without further explanation.

Most of the basic properties of infinite finitary permutation groups can be found in [5], [10], [11] and [6]. Some of them are collected in Lemmas 1, 2, 3 for the convenience of the reader.

Lemma 2.1 *Let G be a transitive subgroup of $FSym(\Omega)$ for some set Ω. Let Δ be a non-trivial block for G and let H be a non-trivial subgroup of G such that $supp(H) \subseteq \Delta$. Then the following hold:*

(a) Let $\Gamma = supp(H)$ then $G_\Gamma \leq C_G(H)$;

(b) $N_G(H) \leq G_{\{\Delta\}}$;

(c) $H \cap G_\Delta = 1$ and $G_\Delta \leq C_G(H)$;

(d) $H^x \leq G_\Delta$ and so $[H^x, H] = 1$ for all $x \in G \setminus G_{\{\Delta\}}$;

(e) If $x \in G \setminus G_{\{\Delta\}}$, then $H' \leq [H, x]$

Proof. (a) Let $h \in H$ and $x \in G_\Gamma$. If $i \in \Gamma$ then $h(i) \in \Gamma$ and hence $x(h(i)) = h(x(i))$ since $x(i) = i$. If $i \notin \Gamma$ then $x(i) \notin \Gamma$ and hence $h(x(i)) = x(i) = x(h(i))$. (b) and (c) are left to the reader. (d) Let $h \in H$ and $x \in G \setminus G_{\{\Delta\}}$. Then $\Delta \cap x(\Delta) = \Delta \cap x^{-1}(\Delta) = \emptyset$.

Let $i \in \Delta$, then $x^{-1}hx(i) = x^{-1}h(x(i)) = x^{-1}x(i) = i$ and so $x^{-1}hx \in G_\Delta$. Hence it follows that $H^x \leq G_\Delta$ and so $[H, H^x] = 1$ by (c). (e) See the proof of [5, Lemma 8.3C(i)]. □

Lemma 2.2 *Let G be a totally imprimitive subgroup of $FSym(\Omega)$ and $a \in \Omega$, where Ω is infinite. Then the following hold:*

(a) Every orbit of G_a is finite;

(b) If $K \leq G$ and $K(a)$ is finite then $[K : K \cap G_a]$ is finite.

Proof. (a) Put $H = G_a$ and choose $b \in \Omega$. By (1) there exists a finite block Δ such that $a, b \in \Delta$. Then $H \leq G_{\{\Delta\}}$. Hence $H(b) \subseteq H(\Delta) = \Delta$ and so $H(b)$ is finite. (b) This follows from the fact that $[K : K_a] = |K(a)|$ and $K \cap G_a = K_a$. □

Lemma 2.3 *Let G be a subgroup of $FSym(\Omega)$, where F is a finite subgroup of G and Δ is a block for G such that $supp(F) \subseteq \Delta$. Then the following hold:*

(a) $U = \{u \in G_{\{\Delta\}} : supp(u) \subseteq \Delta\}$ is a normal subgroup of $G_{\{\Delta\}}$ with $F \subseteq U$.

(b) Let $y \in G$ and let t be the smallest positive integer such that $y^t \in G_{\{\Delta\}}$. Assume that y^t normalizes F. Then $F^{<y>} = F \times F^y \times \cdots \times F^{y^{t-1}}$.

Proof. (a) Clearly $F \subseteq U$. For any $u, v \in U$ and $x \in G_{\{\Delta\}}$ it is easy to check that $uv^{-1}, u^x \in U$.

(b) Since y^t normalizes F it follows that $F^{<y>} =< F^{y^k} : 0 \leq k \leq t-1 >$. Since $y, y^2, \ldots, y^{t-1}$ are not contained in $G_{\{\Delta\}}$, $F^{y^k} \leq G_\Delta$ and $[F, F^{y^k}] = 1$ for all $1 \leq k \leq t-1$ and so $F \cap < F^{y^k} : 1 \leq k \leq t-1 >= 1$ by Lemma 2.1. Also the above properties hold if F is replaced by F^{y^k} for any $1 \leq k \leq t-1$. Therefore $F^{<y>}$ is the direct product of $F, F^y, \ldots, F^{y^{t-1}}$. □

Lemma 2.4 *Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. Let F be a finite non-abelian subgroup of G and let Δ be a non-trivial block for G such that $supp(F) \subseteq \Delta$. If A is a normal abelian subgroup of $G_{\{\Delta\}}$, then $< A^x : x \in G >\leq G_{\{\Delta\}}$.*

Proof. Put $H = G_{\{\Delta\}}$. Let $x \in G \setminus H$. Then $F \leq H^x$ by Lemma2.1(d) and $A^x \lhd H^x$. If there is an $a \in A^x \setminus H$ then $F' \leq [F, a]$ and then $A^x \leq C_G(F') \leq H$ by Lemma 2.1, which is a contradiction. Hence it follows that $A^x \leq H$ for any $x \in G$ and so the assertion follows. □

Lemma 2.5 *Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite and K be an ascendant subgroup of G. If K has an infinite orbit on Ω, then K is a transitive normal subgroup of G.*

Proof. Assume that $K(i)$ is infinite for some $i \in \Omega$ but K is not normal in G. Then there exist ascendant subgroups K_1 and K_2 of G such that $K < K_1 < K_2$, $K \lhd K_1 \lhd K_2$ but K is not normal in K_2. Since $K \lhd K_1$ K(i) is a block for the action of K_1 on $K_1(i)$. So if $K(i) \neq K_1(i)$, then there exists $x \in K_1$ such that $x(K(i)) \cap K(i) = \emptyset$ and so $K(i) \subseteq supp(x)$ which is impossible since $supp(x)$ is finite. Hence it follows that $K(i) = K_1(i)$ and so K acts transitively on $K_1(i)$. Similarly K_1 acts transitively on K_2 and so $K_1(i) = K_2(i)$ which yields that $K(i) = K_2(i)$. Continuing in this way it follows that $K(i) = \Omega$ and so K is transitive on Ω. Now since $K_1(i)$ and $K_2(i)$ are transitive subgroups of $FSym(\Omega)$ having the property that $K \lhd K_1 \lhd K_2$ it follows from [11, Theorem 3.3] that $K \lhd K_2$ which is a contradiction. □

Proof of Theorem 1.1 Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. Let F be a non-abelian subgroup of G and let Δ be a non-trivial block such that $supp(F) \subseteq \Delta$. Then G cannot be primitive. First suppose that G has a maximal block Γ with $\Delta \subseteq \Gamma$. Put $\Sigma = \{x(\Gamma) : x \in G\}$. Let K be the kernel of the representation of G into $FSym(\Sigma)$. Then G/K is isomorphic to $Alt(\Sigma)$ or $FSym(\Sigma)$ by [5, Lemma 8.3(B)] or [14, Proposition]. Then also $(G/K)_\Gamma$ is isomorphic to $Alt(\Sigma \setminus \Gamma)$ or $FSym(\Sigma \setminus \Gamma)$. Moreover it is easy to see that $(G/K)_\Gamma = G_{\{\Gamma\}}/K$. Therefore $G_{\{\Gamma\}}/K$ contains a unique subgroup of index ≤ 2 which is isomorphic to the simple group $Alt(\Sigma \setminus \Gamma)$. Since $G_{\{\Delta\}}$ has finite index in $G_{\{\Gamma\}}$ it follows that $Alt(\Sigma \setminus \Gamma)$ is isomorphic to a subgroup of $G_{\{\Delta\}}K/K$. Hence it follows that any solvable normal subgroup of $G_{\{\Delta\}}$ is contained in K.

Next suppose that G is totally imprimitive. Put $H = G_{\{\Delta\}}$. First suppose that $G = G'$. We use induction on d. By Lemma 2.4 the assertion is true for $d \leq 1$. So suppose that $d > 1$ and the assertion is true for smaller derived lengths. Put $L =< (S^{(d-1)})^x : x \in G >$. Again $L \leq H$ by Lemma 2.4. Let M be the kernel of the representation of G into $FSym(\Lambda)$, where $\Lambda = \{x(\Delta) : x \in G\}$ and put $\bar{G} = G/M$.

Then $L \leq M$. Clearly $\bar{G}$ is totally imprimitive and the derived length of $\bar{S}$ is less than d. Let $\overline{F_1}$ be a finite non-abelian subgroup of $\bar{G}$ and Γ be a non-trivial block for $\bar{G}$ such that $\Delta \subseteq \Gamma$ and $supp(\overline{F_1}) \subseteq \Gamma$. There exist $1 = x_1, x_2, \ldots, x_r \in G$ such that $\Gamma = \{x_1(\Delta), x_2(\Delta), \ldots, x_r(\Delta)\}$. Put $\Delta_1 = x_1(\Delta) \cup x_2(\Delta) \cup \cdots \cup x_r(\Delta)$. Then Δ_1 is a block for G and $\bar{G}_{\{\Gamma\}} = \overline{G_{\{\Delta_1\}}}$. Put $\bar{T} = \bar{G}_\Gamma$. Then $\bar{G}_{\{\Gamma\}}/\bar{T}$ is a finite group and $\bar{T} \leq \bar{G}_{x_i(\Delta)}$ for all $i \geq 1$. Thus $\bar{G}_{\{\Gamma\}} = \bar{X}\bar{T}$ for some finite subgroup $\bar{X}$ of $\bar{G}_{\{\Gamma\}}$. Since $\bar{G}_{\{\Gamma\}}$ is an $FC-group$ we may suppose that $\bar{X}$ is normal in $\bar{G}_{\{\Gamma\}}$.

Put $D_S = S \cap T$. Since $T \leq G_{\{\Delta\}}$, $D_S \lhd T$ and S/D_S is finite. Now $\overline{D_S} \cap C_{\bar{T}}(\bar{X})$ is normalized by $\bar{X}\bar{T} = \bar{G}_{\{\Gamma\}}$ and $[\overline{D_S} : \overline{D_S} \cap C_{\bar{T}}(\bar{X})]$ is finite since $C_{\bar{T}}(\bar{X}) \cap \overline{T}$ has finite index in $\overline{G}_{\{\Gamma\}}$. Since the derived length of $\overline{D_S} \cap C_{\bar{T}}(\bar{X})$ is less than d, it follows by induction hypothesis that $\bar{R} =< (\overline{D_S} \cap C_{\bar{T}}(\bar{X}))^{\bar{x}} : x \in G >\neq \bar{G}$. Clearly every orbit of R on Ω is finite by Lemma 2.5 and [11, Theorem 1] since $G = G'$. Since $\bar{T}$ and $C_{\bar{T}}(\bar{X})$ have finite index in $\bar{G}_{\{\Gamma\}}$, $[\bar{G}_{\{\Gamma\}} : C_{\bar{T}}(\bar{X}) \cap \overline{T}] = m$ for some $m \geq 1$. Thus $[\bar{S} : \bar{S} \cap (C_{\bar{T}}(\bar{X}) \cap \bar{T})] = [\bar{S} : \overline{D_S} \cap C_{\bar{T}}(\bar{X})] \leq m$ and hence $|\bar{S}\bar{R}/\bar{R}| \leq m$. Since G is totally imprimitive it is the union of an ascending chain of proper normal subgroups. Therefore there exists a proper normal subgroup N of G such that $\bar{R} \leq \bar{N}$ and $\bar{S}\bar{R}/\bar{R} \leq \bar{N}$, since every orbit of R is finite. Clearly then $< S^x : x \in G >\neq G$.

Now suppose that $G' < G$. If $G'S \neq G$ then we are done. So suppose that $G'S = G$. Let $W = G'$ and $S_1 = S \cap W$. Then $U =< S_1^x : x \in G' >\neq W$ by the above paragraph. Since every orbit of U is finite there exists a non-trivial block Π for G containing Δ such that $U \leq G_{\{\Pi\}}$. Let B be the kernel of the representation of G into $FSym(Y)$, where $Y = \{x(\Pi) : x \in G\}$. Put $\bar{G} = G/B$. Since $\bar{H} = \bar{S}(\bar{H}) \cap \bar{W}$ and $\bar{S} \cap \bar{W} = 1$, it follows that $\bar{S} \leq Z(\bar{H})$. Put $V = G_{\{\Pi\}}$. Then $[V : H]$ is finite. Let A be the largest normal subgroup of V contained in H. Then V/A is finite. Put $\bar{Z} = Z(\bar{A})$. Then $[\bar{S} : \bar{S} \cap \bar{Z}]$ is finite. Since $\bar{Z} \lhd \bar{V}$, if $X =< Z^x : x \in G >$ then $\bar{X} \neq \bar{G}$ by Lemma 2.4. Now consider $\bar{G}/\bar{X}$. Then $\bar{S}\bar{X}/\bar{X}$ is finite. Also $\bar{G}/\bar{X}$ is an ascending union of proper normal subgroups. Therefore $\bar{S}\bar{X}/\bar{X}$ is contained in a proper normal subgroup of $\bar{G}/\bar{X}$, which implies that S is contained in a proper normal subgroup of G. This completes the proof of the theorem. □

As an easy consequence of the above proof one can verify the following easily: Suppose that in Theorem 1.1 G is perfect. Let $W_d = \{S \leq G_{\{\Delta\}} : S \lhd G_{\{\Delta\}}$ and $S^{(d)} = 1\}$. Then $< S^x : S \in W_d$ and $x \in G >\neq G$.

Proof of Theorem 1.2 Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. (a) Let $a \in \Omega$ and suppose that G_a is solvable. By [5, Lemma 8.3B(i)]or [10, Theorem 2.3] G cannot be primitive. First suppose that G is almost primitive. Then G has a maximal non-trivial block Γ containing a. Then, as in the proof of Theorem 1.1, G contains a normal subgroup K such that G/K is isomorphic to $Alt(\Sigma)$ or $FSym(\Sigma)$, where $\Sigma = \{x(\Gamma) : x \in G\}$ and then $(G/K)_\Gamma$ is isomorphic to $Alt(\Sigma \setminus \{\Gamma\})$ or $FSym(\Sigma \setminus \{\Gamma\})$. But since G_aK/K has finite index in $(G/K)_\Gamma$, this gives a contradiction.

Next suppose that G is totally imprimitive. Then G contains a non-abelian finite subgroup F and a non-trivial block Δ such that $supp(F) \cup \{a\} \subseteq \Delta$. Clearly G_a has finite index in $G_{\{\Delta\}}$ and so $G_{\{\Delta\}}$ contains a normal solvable subgroup S of finite index. By Theorem 1.1 $M =< S^x : x \in G >\neq G$. Since $G_{\{\Delta\}}/M$ is finite $G_{\{\Delta\}}$ is contained in a proper normal subgroup N of G. In particular $G_a \leq N$. Since G_b is conjugate to G_a for any $b \in \Omega$ it follows that $G =< G_b : b \in \Omega >\leq N$, which is a contradiction. □

(b) Suppose that G satisfies the normalizer condition. Then G is locally nilpotent by [13, 12.2.2] and so it is a p - group for some prime p by [15, Theorem 1]. It is easy to see that G' is transitive and hence perfect by [11, Theorem 1]. Let H be a proper subgroup of G'. If every orbit of H is finite, then H is an FC - group by [5, Lemma 8.3(D)] or [16, Theorem 1]. So suppose that $H(a)$ is infinite for some $a \in \Omega$. Then H is normal in G by Lemma 2.5, since it is ascendant in G by assumption. This implies that $H(a)$ is a block for G and so $H(a) = \Omega$ since any non-trivial block for G is finite. In particular then H is transitive on Ω. But now $G' \leq H$ by [5, Lemma 8.3C] or [11, Theorem1], which is a contradiction. This completes the proof of the theorem.

Proof of Theorem 1.3 Let G be a transitive subgroup of $FSym(\Omega)$, where Ω is infinite. Let $a \in \Omega$ and suppose that G_a satisfies (a) or (b) of the theorem. Then it follows as in the proof of Theorem 1.2 that G can be neither primitive nor almost primitive. Therefore we may assume that G is totally imprimitive. Then $G = \bigcup_{k=1}^{\infty} N_k$ by (2). For each $k \geq 1$ $[N_k : N_k \cap G_a]$ is finite by Lemma 2.2(b). Let M_k be the largest normal subgroup of N_k contained in $N_k \cap G_a$. Then N_k/M_k is finite.

(a) Suppose that G_a is locally solvable. Then M_k is locally solvable. In fact then M_k is solvable since N_k is isomorphic to a subgroup of the direct product of isomorphic copies of a finite group as was explained in the introduction. Let S_k be the product of all the normal solvable subgroups of N_k for all $k \geq 1$. Then S_k is normal in G and N_kS_k/S_k is finite since $M_k \leq S_k$. Define $S =< S_k : k \geq 1 >$. Then S is locally solvable. Also $S \lhd G$ and G/S is an FC -group since $G/S = \bigcup_{k=1}^{\infty} N_kS/S$ and each N_kS/S is finite. In this case G/S cannot be represented as a group of finitary permutations on an infinite set by [5, Lemma 8.3D] or [16, Theorem 1] and so S must be transitive by [11, Lemma 2.1]. This implies that $G' \leq S$ and so G is locally solvable, which completes the proof of (a).

(b) Suppose that G_a is locally nilpotent-by-solvable of derived length $t \geq 0$, say. Then G is locally solvable by (a). Let $\eta(G_a)$ be the Hirsh - Plotkin radical of G_a. Then $G_a/\eta(G_a)$ is solvable of derived length $\leq t$. Clearly if G' is locally nilpotent then the assertion follows from [15, Theorem 1] and [12, Lemma 2.1], since G' is transitive on Ω. Therefore it suffices to show that G' is locally nilpotent. Without loss of generality we may suppose that $G = G'$.

Since G_a is locally nilpotent-by-solvable of derived length t and $M_k \leq G_a$ it follows that $M_k/\eta(M_k)$ is solvable of derived length $\leq t$. Also $\eta(M_k) \leq \eta(N_k)$ for all $k \geq 1$. Define $K =< \eta(N_k) : k \geq 1 >$. Then K is a locally nilpotent normal subgroup of G. So if K is transitive on Ω, then $G = G' \leq K$ and thus G is locally nilpotent. Assume if possible that K is not transitive. Then every orbit of K, being a block for G, must be finite.

Let $a \in \Omega$, $\Delta = K(a)$ and $\Sigma = \{x(\Delta) : x \in G\}$. Let L be the kernel of the representation of G into $FSym(\Sigma)$. Then $K \leq L$. Put $\bar{G} = G/L$. Then $\bar{G}$ is transitive on Σ, in fact it is totally imprimitive since G is locally solvable. Now $\bar{G} = \bigcup_{k=1}^{\infty} \overline{N_k}$, and for each $k \geq 1$, $\overline{M_k}$ is a solvable normal subgroup of finite index of $\overline{N_k}$ with derived length $\leq t$. Therefore each $\overline{N_k}$ contains a characteristic subgroup of finite index with derived length $\leq t^2$ by [4, Lemma 4]. But since $\bar{G}$ is perfect, it follows that each $\overline{N_k}$ is solvable of derived length $\leq t^2 + 1$ for all $k \geq 1$ and thus $\overline{G}$ is solvable, which is a contradiction.□

Proof of Corollary 1.4 (a) Let $a \in \Omega$ and put $H = G_a$. Suppose that H is locally (nilpotent-by-abelian). Then G is locally solvable by Theorem 1.3(a), which implies that G is totally imprimitive and hence countably infinite. Let $F_1 \leq F_2 \leq \cdots \leq F_i \leq \ldots$ be an ascending chain of finite subgroups of H whose union is equal to H. By hypothesis F' is nilpotent for each $i \geq 1$. Since

$$H' = (\bigcup_{i=1}^{\infty} F_i)' = \bigcup_{i=1}^{\infty} F_i',$$

and $(F_i)' \leq (F_{i+1})'$ for all $i \geq 1$, it follows that H' is locally nilpotent and so H is locally nilpotent-by-abelian. Therefore by Theorem 1.3(b) G is a p-group for some prime p.

(b) Suppose that G_a is locally supersolvable. Since a supersolvable group is nilpotent-by-abelian by [13, 5.4.10], G_a is locally (nilpotent-by-abelian) and so G is a p - group for some prime p by (a), which was to be shown. $\square$

Example Let G be the 2 - group constructed by Wiegold in [16, p. 468]. Then G is a totally imprimitive subgroup of $FSym(\mathbb{N})$. By Lemma 2.2(a) every orbit of a point stabilizer is finite and so it is an FC-group. We show that G' is not a minimal non FC-group, which will show that the condition of Theorem 1.2(b) cannot be restricted to a point stabilizer. We will adopt the description of Wiegold's group given in [5, Exercise 8.3.1]. Thus for each $k \geq 1$ let T_k be the subset of $FSym(\mathbb{N})$ defined by

$$T_k = \{x_{k,n} : n = 1, 2, 3, \dots\}, \text{ where } x_{k,n} = \prod_{i=0}^{2^{k-1}-1}(i + n2^k, i + 2^k + 2^k - 1)$$

for each $n \geq 0$. Thus for example,

$$\begin{aligned} T_1 &= \{(01), (23), (45), \dots\}, T_2 = \{(02)(13), (46)(57), (8\,10)(9\,11), \dots\} \\ T_3 &= \{(04)(15)(26)(37), (8\,12)(9\,13)(10\,14)(11\,15), \dots\}. \end{aligned}$$

For each $k \geq 1$ let $G_k =< T_1, \dots, T_k >$. Then $G = \bigcup_{k=1}^{\infty} G_k$. Clearly G is a transitive 2 -subgroup of $FSym(\mathbb{N})$. It is easy to see that each G_k is normal in G and every orbit of it is finite. Furthermore each T_k is a set of disjoint involutions which are conjugate in G (see [16, p. 468]).

Next we show that G' is not a minimal non FC- group. Let $X =< x \in G' : x^2 = 1 >$. Then $X \lhd G$. Let $k \geq 1$. Then for each $n \geq 0$ there exists $g \in G$ such that $x_{k,n} = g^{-1}x_{k,0}g \in T_k$, and hence $x_{k,0}x_{k,n} = x_{k,0}g^{-1}x_{k,0}g \in X$ for all $n \geq 0$ since any two elements of T_k commute which implies that $x_{k,n} \in x_{k,0}X$. Clearly it follows form this that $G/X =< x_{k,0}X : k \geq 1 >$. We claim that G/X is abelian. Let $k, t \geq 1$. Clearly $x_{t,0}x_{k,0}x_{t,0}x_{k,0} \in X$ as above. Hence it follows that $x_{k,0}x_{t,0}X = (x_{k,0}x_{t,0})(x_{t,0}x_{k,0}x_{t,0}x_{k,0}X = x_{t,0}x_{k,0}X$ which shows that G/X is abelian and so $G' \leq X$. Since $X \leq G'$ it follows that $G' = X$ and so G' cannot be a minimal non FC -group by Corollary 1.9. Note that $G' \neq G$ since G' is generated by even permutations and so $T_1 \cap G' = \emptyset$.

3. Proofs of Theorems 4, 5

Lemma 3.1 *Let G be a totally imprimitive subgroup of $FSym(\Omega)$, where Ω is infinite, and let A be a generating subset for G. Then there exists $a \in A$ such that $< a >^G$ is not abelian.*

Proof. Assume that $< a >^G$ is abelian for all $a \in A$. Let F be a non-abelian finite subgroup of G and let Δ be a non-trivial block for G such that $supp(F) \subseteq \Delta$. Since $< A >= G$ and $G_{\{\Delta\}} \neq G$, there exists $a \in A \setminus G_{\{\Delta\}}$. Then $F' \leq [F, a] \leq < a >^G$ and so $< a >^G \leq G_{\{\Delta\}}$ by Lemma 2.1(b), which is a contradiction. □

Lemma 3.2 *Let G be a subgroup of $FSym(\Omega)$, F a subgroup of G and Δ be a non-trivial block for G such that $supp(F) \subseteq \Delta$. Let $y \in G \setminus G_{\{\Delta\}}$ and let t be the smallest positive integer such that $y^t \in G_{\{\Delta\}}$. Assume that $y^t \in FC_G(F)$. Then there exists $x \in F$ such that $(yx^{-1})^t \in C_G(F)$, and t is the smallest positive integer with this property.*

Proof. By hypothesis there exists $h \in F$ and $c \in C_G(F)$ such that $y^t = ch$. Also $y(\Delta), \ldots, y^{t-1}(\Delta)$ are pairwise disjoint by the choice of t. Let $x \in F$ and $i \in \Delta$. We claim that $(yx)^k(i) = y^k(x(i))$ for all $1 \leq k \leq t$. Assume that it holds for $1 \leq k < t$. Then

$$(yx)^{k+1}(i) = (yx)((yx)^k(i)) = (yx)((y^k(x(i))) = y(y^k(x(i))) = y^{k+1}(x(i))$$

since $y^k(x(i)) \notin \Delta$ and $supp(F) \subseteq \Delta$. Thus the induction is complete. Now letting $k = t$ and $x = h^{-1}$ we get

$$(yh^{-1})^t(i) = y^t(h^{-1}(i)) = (y^t h^{-1})(i) = c(i).$$

Since i is any element of Δ it follows that $(yh^{-1})^t \mid_\Delta = c \mid_\Delta$. Hence $(yh^{-1})^t = (c \mid_\Delta)d$ where $d \in FSym(\Omega \setminus \Delta)$ which implies that $(yh^{-1}) \in C_G(F)$. Also it is clear from the induction that t is the smallest such number. □

Proof of Theorem 1.5 Let G be a totally imprimitive p - subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that every orbit of every proper subgroup of G is finite. Assume that for every non-normal finite subgroup F of G there exists $y \in G \setminus N_G(F)$ such that $y^p \in C_G(F)$. By Lemma 3.1 there exists $c_1 \in G$ such that $< c_1 >^G$ is not abelian. Put $F_1 =< c_1 >$ and let Λ_1 be a member of (1) containing $supp(F_1)$. Let U_1 be the normal closure of F_1 in $G_{\{\Delta\}}$. Then $supp(U_1) \subseteq \Delta_1$ by Lemma 2.3(a) and also $N_G(U_1) = G_{\{\Lambda_1\}}$ by Lemma 2.1(a). By the hypothesis there exists $c_2 \in G \setminus G_{\{\Lambda_1\}}$ such that $c_2^p \in C_G(U_1)$. In particular, c_2^p centralizes F_1 since $F_1 \subseteq U_1$. Therefore

$$F_1^{<c_2>} =< c_1 > \times < c_1 >^{c_2} \times \cdots \times \langle c_1 \rangle^{c_2^{p-1}},$$

by Lemma 2.3(b). Next, put $F_2 = F_1^{<c_2>} < c_2 >=< F_1, c_2 >$. Let Λ_2 be a member of (1) containing $supp(F_2)$ and U_2 be the normal closure of $supp(F_2)$ in $G_{\{\Lambda_2\}}$. Then, as in the first case, there exists $c_3 \in G \setminus G_{\{\Lambda_2\}}$ such that $c_3^p \in C_G(U_2)$ and so $F_2^{<c_3>} = F_2 \times F_2^{c_3} \times \cdots \times F_2^{c_3^{p-1}}$. Put $F_3 = F_2^{<c_3>} < c_3 >=< F_2, c_3 >$. Continuing in this way we obtain properly increasing infinite chains

$$\Lambda_1 \subset \Lambda_2 \subset \ldots \text{ and } F_1 \subset F_2 \subset \ldots$$

of non-trivial blocks Λ_i and finite subgroups $F_i =< c_1, c_2, \ldots c_i >$ of G such that $c_{i+1} \in G \setminus G_{\{\Lambda_i\}}$for every $i \geq 1$. Obviously $\Omega = \bigcup_{k=1}^{\infty} \Lambda_k$. Put $F = \bigcup_{i=1}^{\infty} F_i$. Clearly $F = G$ since the orbit of F containing any element of $supp(c_1)$ is infinite by the choice of the c_i.

We claim that $< c_1 >^F$ is abelian. It suffices to show that $< c_1 >^{F_i}$ is abelian for all $i \geq 1$. This is obvious for $i = 1$ since $F_1 =< c_1 >$. Assume that it holds for some $i \geq 1$.We must show that it holds for $i + 1$. Thus $< c_1 >^{F_i}$ is abelian by assumption. Now

$$F_{i+1} = (F_i^{<c_{i+1}>}) < c_{i+1} > \text{ and } F_i^{<c_{i+1}>} = F_i \times F_i^{c_{i+1}} \times \cdots \times F_i^{c_{i+1}^{p-1}}.$$

Hence we can write F_{i+1} as $F_{i+1} = (F_i \times F_i^{c_{i+1}} \times \cdots \times F_i^{c_{i+1}^{p-1}}) < c_{i+1} >$. Let $x, y \in F_{i+1}$. It suffices to show that $[c_1^x, c_1^y] = 1$. Now there exists $a_0, a_1, \ldots, a_{p-1}$ and $b_0, b_1, \ldots, b_{p-1} \in F_i$ and $0 \leq r, s \leq p-1$ such that

$$x = (a_0 a_1^{c_{i+1}} \ldots a_{p-1}^{c_{i+1}^{p-1}}) c_{i+1}^r \text{ and } y = (b_0 b_1^{c_{i+1}} \ldots b_{p-1}^{c_{i+1}^{p-1}}) c_{i+1}^s.$$

Since $c_1^{a_0} \in F_i$ and $[F_i, F_i^{c_{i+1}^u}] = 1$ for $1 \leq u \leq p-1$ by Lemma 2.1(d), it is easy to see that $c_1^x = c_1^{a_0 c_{i+1}^r}$. Similarly $c_1^y = c_1^{b_0 c_{i+1}^s}$. Now $[c_1^x, c_1^y] = 1$ if $r \neq s$ by Lemma 2.1(d), since $[F_i^{c_{i+1}^r}, F_i^{c_{i+1}^s}] = 1$ due to the fact that $0 \leq r, s \leq p-1$. If $r = s$, then $[c_1^x, c_1^y] = [c_1^{a_0}, c_1^{b_0}]^{c_{i+1}^r} = 1$ since $< c_1 >^{F_i}$ is abelian. This completes the induction and so it follows that $< c_1 >^F$ is abelian. But since $< c_1 >^G$ is not abelian this contradicts the fact that $F = G$ and so the proof is complete. $\square$

Proof of Corollary 1.6 Let F be a non-normal finite subgroup of G and Δ be a minimal block such that $supp(F) \subseteq \Delta$. By hypothesis there exists $y \in G \setminus G_{\{\Delta\}}$ such that $y^p \in FC_G(F)$. By Lemma 3.2 there exists $x \in F$ such that $(yx)^p \in C_G(F)$. Also $yx \notin G_{\{\Delta\}}$ since $x \in G_{\{\Delta\}}$. Thus it follows that $yx \notin N_G(F)$ but $(yx)^p \in C_G(F)$ since $N_G(F) \leq G_{\{\Delta\}}$ by Lemma 2.1. Since F is any finite non-normal subgroup of G applying Theorem 1.5 yields a proper subgroup of G that has an infinite orbit. $\square$

Proof of Corollary 1.7 Let $k \geq 1$ and $H = G_{\{\Delta_k\}}$. Let M_k be the kernel of the representation of G into $FSym(\Sigma_k)$, where $\Sigma_k = \{x(\Delta_k) : x \in G\}$. Then M_k is the largest normal subgroup of G contained in H and so $M_k =< F_k^G >$ by hypothesis. Put $\bar{G} = G/M_k$. Let R be a proper normal subgroup of G such that $\bar{R} \neq 1$. Since $\bar{R}$ is nilpotent $\Omega_1(Z(\bar{R})) \neq 1$ and so it is not contained in $\bar{H}$ by definition of M_k. Choose $\bar{z} \in \Omega_1(Z(\bar{R})) \setminus \bar{H}$ such that $o(\bar{z}) = p$. Then $z \notin H$ but $z^p \in M_k$. Since $M_k =< F_k^G >= \prod_{x\in G} F^x \leq F_k G_{\Delta_k}$ it follows that $z^p \in F_k C_G(F_k)$.

Let now F be any finite non - normal subgroup of G. There exists $k \geq 1$ such that $supp(F) \subseteq \Delta_k$ and so by the above paragraph there there exists $y \in G \setminus N_G(F)$ such that $y^p \in FC_G(F)$. Therefore Corollary 1.6 yields a proper subgroup of G that has an infinite orbit. $\square$

Proof of Corollary 1.8 Let F be a non - normal finite subgroup of G. There exists a $k \geq 1$ such that $supp(F) \subseteq \Delta_k$. By hypothesis there exists $y \in G \setminus G_{\{\Delta_k\}}$ such that $y^p \in G_{\Delta_k} \leq C_G(F)$. Since $N_G(F) \leq G_{\{\Delta_k\}}$, the hypothesis of Theorem 1.5 is satisfied and so G contains a proper subgroup having an infinite orbit. $\square$

Proof of Theorem 1.11 Let G be a locally finite p - group that is also a minimal no FC - group. Then every proper subgroup of G is an FC-group. Assume that G is perfect and satisfies the hypothesis of the theorem. Then $Z(G/Z(G)) = 1$. Put $\bar{G} = G/Z(G)$. Then $Z(\bar{G}) = 1$. Since G is locally nilpotent, it has a proper normal subgroup $\bar{N} \neq 1$. Then since $Z(\bar{N}) \neq 1$, we can choose $\bar{a} \in Z(\bar{N})$ with $o(\bar{a}) = p$. Put $\Omega = \{(\bar{a})^{\bar{x}} : x \in G\}$. Clearly Ω is infinite and the conjugation action of $\bar{G}$ on Ω defines a finitary permutation group on Ω by [3, Theorem1] or [8, Theorem]. Let K be the kernel of this representation. Then

$\bar{G}/\bar{K}$ is isomorphic to a totally imprimitive p -subgroup of $FSym(\Omega)$. Since $\bar{G}/\bar{K} \cong G/K$ we may consider G/K instead of $\bar{G}/\bar{K}$.

Now we show that G/K satisfies the hypothesis of Corollary 1.6. By hypothesis for every non-normal finite subgroup F of G there exists $y \in G \setminus N_G(F)$ such that $y^p \in FC_G(F)$. Let X/K be a finite non - normal subgroup of G/K and let T be a finite subgroup of X such that $X = TK$. Let Δ be a non-trivial block for G/K such that $supp(X/K) \subseteq \Delta$. Put $L/K = (G/K)_{\{\Delta\}}$. Let V be the normal closure of T in L. Then V is a finite normal subgroup of L and $X \leq VK$. Thus $L/K = N_{L/K}(VK/K)$ since $supp(VK/K) \subseteq \Delta$. In particular $N_G(V) = L$. By hypothesis there exists $y \in G \setminus L$ such that $y^p \in VC_G(V)$. Then $yK \notin L/K$ but $y^pK \in (VK/K)C_{G/K}(X/K)$ since $X/K \leq VK/K$. Thus it follows that G/K satisfies the hypothesis of Corollary 1.6. But then G/K contains a proper subgroup having an infinite orbit, which is impossible by [5, Lemma 3.8D] or [16, Theorem 1] since G/K is a minimal non FC - group. This contradiction completes the proof of the theorem. □

The author is grateful to the referee for a careful reading of the manuscript and pointing out some errors.

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Received 17.02.2005