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A Survey on the Distribution of B-free Numbers

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Abstract

In this paper we present a survey of recent progress on the distribution of B-free numbers in short intervals and some of its applications.

Key Words: B-free numbers, linear forms.

1. Introduction

The distribution of special subsets of positive integers is a central topic of research in analytic number theory. In particular, extensive research has been done on the distribution of prime numbers. For an account on recent developments the reader may consult the nice survey of Yıldırım [25]. For the purpose of some applications one is motivated to consider the distribution of certain sequences of numbers defined by milder divisibility constraints. The notion of a B-free number, introduced by Erdös in [8], generalizes that of a square-free number. Given a sequence B of positive integers $1 < b_1 < b_2 < \dots$ such that

$$\sum_{k=1}^{\infty} \frac{1}{b_k} < \infty, \quad \text{and} \quad \gcd(b_k, b_j) = 1 \quad \text{for } k \neq j, \tag{1.1}$$

a number n is called B-free provided that no element b_k of B divides n. It is easy to see that a positive proportion of integers are B-free, as

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$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N \ : \ n \text{ is } B\text{-free}\} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{b_k}\right) > 0. \tag{1.2}$$

By taking B to be the sequence of squares of all the prime numbers, the set of B-free numbers coincides with the set of square-free numbers. Erdös [8] proved that for some $c < 1$ and all large enough N, the interval $[N, N + N^c]$ contains B-free numbers. He conjectured that for any $\epsilon > 0$ there exists $N_{B,\epsilon}$ such that for any $N \ge N_{B,\epsilon}$ the interval $[N, N + N^\epsilon]$ contains at least one B-free number. Szemerédi [23] proved Erdös' claim for all $c > \frac{1}{2}$. This was improved to $c > \frac{9}{20}$ by Bantle and Grupp [6], using the work of Iwaniec and Laborde [13]. A further improvement is due to Wu [24], who proved that

$$\#\{N \le n \le N + N^c \ : \ n \text{ is } B\text{-free}\} \gg_{B,c} N^c \tag{1.3}$$

for all $c > \frac{17}{41}$, using the work of Fouvry and Iwaniec [10] on exponential sums with monomials. Zhai [26] obtained the same result for $c > \frac{33}{80}$. The best result to date is due to Sargos and Wu [20], who reduced the value of c to $c > \frac{40}{97}$. Assuming the ABC-Conjecture Granville [11] proved Erdös' Conjecture in the case of square-free numbers. Unconditionally, Filaseta and Trifonov [9] proved that for N large enough and for all $c > \frac{1}{5}$, the interval $[N, N + N^c]$ contains a square-free number. In [4] the problem of finding B-free numbers in short arithmetic progressions was considered. In this connection the following generalization of Erdös' Conjecture for arithmetic progressions was suggested.

Conjecture 1. For any sequence B of positive integers satisfying (1), and any $\epsilon > 0$, there exists a number $N_{B,\epsilon}$ such that for any $N \ge N_{B,\epsilon}$, and any relatively prime integers a, b with $1 \le a, b \le N$, there exists an integer n, with $1 \le n \le N^\epsilon$, such that $an + b$ is B-free.

B-free numbers also have interesting applications to the problem of nonvanishing of Fourier coefficients of modular forms. Ramanujan first realized and studied many of the fascinating arithmetical properties of what is now known as the Ramanujan tau function $\tau(n)$. This is defined in terms of the Delta function, which is the unique normalized cusp form of weight 12 on $SL_2(\mathbb{Z})$, by

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad (q = e^{2\pi i z}, \ \operatorname{Im} z > 0). \tag{1.4}$$

He proved striking congruences satisfied by $\tau(n)$ (see his collected works [19]). Later, the existence of these congruences was explained in the more general context of modular forms and the theory of Galois representations by Deligne and Serre, and Swinnerton-Dyer. For a survey of Ramanujan's original approach to these congruences, with complete proofs and commentary, see the paper of Berndt and Ono [7]. A long standing conjecture of Lehmer [15] states that $\tau(n) \neq 0$ for any $n \geq 1$. In relation to this conjecture, Serre [23] initiated the general study of estimating the size of possible gaps in the Fourier expansion of modular forms via the gap function

$$i_f(n) = \min\{j \geq 0 \ : \ a_f(n+j) \neq 0\}. \tag{1.5}$$

Note that $i_f(n)$ counts the maximum number of consecutive vanishing Fourier coefficients starting with $a_f(n)$. To explain the connection between B-free numbers and the nonvanishing problem for Fourier coefficients of modular forms, let us recall some basic facts about newforms. First, newforms are eigenvectors for the Hecke operators (see Chapter 3 of [14]), and they form a basis for the vector space of cusp forms. Next, if $f(z)$ is a newform then (see [16] and [17]) its Fourier coefficients form a multiplicative arithmetic function, which is essential for producing nonzero Fourier coefficients. In particular, if we consider the set B consisting of all primes p for which $a_f(p) = 0$, then

$$a_f(n) = \prod_{p|n} a_f(p) \neq 0, \tag{1.6}$$

for any n which is B-free and square-free. Balog and Ono [5] obtained strong nonvanishing results on the Fourier coefficients of a newform $f(z)$ without complex multiplication concerning their short interval distribution. They proved that

$$i_f(n) \ll_{f,\epsilon} n^{\frac{17}{41}+\epsilon}. \tag{1.7}$$

In the case of newforms associated to elliptic curves without complex multiplication, the exponent $17/41$ was improved to $69/169$ and then to $51/134$ in [1] and [2] respectively. In [1] it is also proved that $i_f(n) \ll_{f,\phi} \phi(n)$ for almost all n where ϕ is a function monotonically tending to infinity. It follows that the gap function is very small most of the time. In [2] it was shown that for every $\epsilon > 0$ there is an $M = M(f, \epsilon)$ such that

$$\#\{n \leq x \ : \ i_f(n) \leq M\} \geq (1-\epsilon)x. \tag{1.8}$$

It was also shown in [2] that for almost all elliptic curves over $\mathbb{Q}$ without complex multiplication and for any $\epsilon > 0$,

$$\frac{1}{x}\sum_{n\leq x} i_{f_E}(n) \ll_E e^{\frac{8\log x}{\log\log x}} \ll_\epsilon x^\epsilon \tag{1.9}$$

where $f_E(z) = \sum_{n=1}^\infty a_E(n)q^n$ is the weight 2 newform associated to $E/\mathbb{Q}$. In [3] some nonvanishing results for the Fourier coefficients of newforms in arithmetic progressions have been proved. More precisely, for any $\sigma > \frac{9}{20}$ there exists an effectively computable $\eta > 0$ depending only on σ, such that for any newform

$$f(z) = \sum_{n=1}^\infty a_f(n)q^n \in S_k(\Gamma_0(N), \chi) \tag{1.10}$$

without complex multiplication, any large x, any $y \geq x^\sigma$, and any relatively prime integers b, a satisfying $1 \leq b < a \leq x^\eta$, one has

$$\#\{x - y < n \leq x \ : \ a_f(n) \neq 0 \text{ and } n \equiv b \pmod a\} \gg_{\sigma,f} \frac{y}{a}. \tag{1.11}$$

We remark that, the hypothesis that $f(z)$ has no complex multiplication, is essential for the above results, since in the case of complex multiplication one encounters naturally occuring large gaps in the Fourier expansion (see the Serre-Stark basis theorem [22] for weight 1/2 modular forms, which states that all such forms are linear combinations of theta series). For other arithmetic aspects of the properties of Fourier coefficients of modular forms, the reader may consult the recent monograph of Ono [18].

In the next two sections we sketch two different mechanisms of producing B-free numbers in certain small sets, which can be short intervals, or short arithmetic progressions, or values taken by a primitive linear form in two variables. The first approach has been recently used by the authors [4] in order to prove the existence of B-free numbers in short arithmetic progressions. This involves the application of a weighted sieve to count B-free numbers in short intervals. A crucial role in this method is played by the estimates of Fouvry and Iwaniec [10] on exponential sums with monomials, which are used to control the size of remainder terms that appear in the above weighted sieve. The second method described below, which can be used to find B-free numbers in small unions of short arithmetic progressions, also employs a weighted sieve. In this case the estimates of Fouvry and Iwaniec do not appear. Instead, a key ingredient in this approach is an averaging technique that produces a strong saving in the error terms.

2. Distribution of B-free numbers. The first approach

We choose $\sigma, \eta > 0$ such that

$$\frac{1803}{10}\eta + \frac{9}{20} < \sigma \leq \frac{9}{19}\,. \tag{2.12}$$

Let a, b be relatively prime integers with $1 \leq a \leq N^{\eta}$. Our goal is to show that for $N \geq N_{B,\sigma,\eta}$, there exists many integer numbers n with $N \leq n \leq N + N^{\sigma}$ such that $n \equiv b (\text{mod } a)$ and n is B-free. For simplicity, let us put $x = N + N^{\sigma}$ and $y = N^{\sigma}$. We define the sets

$$P_1 = P_1(x, \delta_1, \mu) = \{x^{\delta_1} \leq p \leq x^{\delta_1+\mu} : p \text{ prime}\} \tag{2.13}$$

and

$$P_2 = P_2(x, \delta_2, \mu) = \{x^{\delta_2} \leq u \leq x^{\delta_2+\mu} : u \text{ prime,}\} \tag{2.14}$$

where $\mu > 0$ is as small as we want and $\delta_2 + \mu < \delta_1 < \delta_1 + \mu < \sigma$. For each $n \leq x$ we assign the sieving weight

$$w(n) = \sum_{p \in P_1} \sum_{\substack{u \in P_2 \\ n \equiv 0 \,(\text{mod } pu)}} 1. \tag{2.15}$$

Clearly $w(n) \leq C(\delta_1, \delta_2)$ independently of x. Next, we make the following reduction. For any element $b \in B$ which is prime we put $b' = b$. For any $b \in B$ which is not prime we let b' equal the product of the largest two prime factors of b. Here, if the largest prime factor p of b appears with multiplicity strictly greater than 1, we let $b' = p^2$. Let A be the sequence of numbers b' arranged in increasing order. It is easy to see that the sum of reciprocals of elements of A is convergent. Moreover, any A-free number is B-free. Hence it will be enough to find A-free numbers in the required arithmetic progressions. This specific structure of A, namely that A consists of a set of primes with sum of inverses finite, a set of squares of primes and a set of products of two distinct prime numbers with sum of inverses finite, will become important later. Let us re-denote the elements of A as $b_1 < b_2 < \ldots < b_s < \ldots$. Hence it suffices to show that

$$\sum_{\substack{x-y<n\leq x \\ n\equiv b(\bmod a) \\ n \text{ is A-free}}} w(n) \gg_\sigma \frac{y}{a}\,. \tag{2.16}$$

We consider the inequality

$$\sum_{\substack{x-y<n\leq x \\ n\equiv b(\bmod a) \\ n \text{ is } A\text{-free}}} w(n) \geq \sum_{\substack{x-y<n\leq x \\ n\equiv b(\bmod a) \\ n\not\equiv 0(\bmod b_s) \\ \text{for all } s\leq m}} w(n) - \sum_{\substack{x-y<n\leq x \\ n\equiv b(\bmod a) \\ b_m<b_s\leq \frac{y}{a} \\ n\equiv 0(\bmod b_s) \\ \text{for some } s>m}} w(n) - \sum_{\substack{x-y<n\leq x \\ n\equiv b(\bmod a) \\ \frac{y}{a}<b_s\leq x \\ n\equiv 0(\bmod b_s) \\ \text{for some } s>m}} w(n), \tag{2.17}$$

where m is a parameter to be fixed appropriately later. Using the definition of $w(n)$ and the inclusion-exclusion principle we obtain that the main term, which is the first sum on the right hand side of (2.17), is

$$M_0 = \sum_{\substack{x-y<n\leq x \\ n\equiv b(\bmod a) \\ n\not\equiv 0(\bmod b_s) \\ \text{for all } s\leq m}} w(n) = \frac{y}{a}\sum_{p\in P_1}\frac{1}{p}\sum_{u\in P_2}\frac{1}{u}\prod_{j\leq m}\left(1-\frac{1}{b_j}\right) + R_{a,b}(x,y), \tag{2.18}$$

where for any subset ω of $\{1,\ldots,m\}$, $d_\omega = \prod_{s\in\omega} b_s$ and

$$R_{a,b}(x,y) = \sum_{\omega}(-1)^{|\omega|}\sum_{p\in P_1}\sum_{u\in P_2} r_{pud_\omega,a,b}(x,y) \tag{2.19}$$

is a sum of remainder terms with $|r_{pud_\omega,a,b}(x,y)| \leq 1$. It is easy to eliminate the d_ω since

$$\sum_{p\in P_1}\sum_{u\in P_2} r_{pud_\omega,a,b}(x,y) = \sum_{p\in P_1}\sum_{u\in P_2} r_{pu,a,b}\Big(\frac{x}{d_\omega},\frac{y}{d_\omega}\Big), \tag{2.20}$$

and x, y are about the same size with $\frac{x}{d_\omega}$, $\frac{y}{d_\omega}$, respectively. The modern version of the linear sieve due to Iwaniec [12] requires nontrivial estimates for sums of remainder terms with two parameters as in $R_{a,b}(x,y)$. Using Fourier analysis, Poisson summation formula and a smooth approximating function, it is possible to show that

$$\Big|\sum_{p\in P_1}\sum_{u\in P_2} r_{pu,a,b}(x,y)\Big| \le \frac{1}{a}\int_{x-y}^{x}\Big|\sum_{|\nu|\le H}\sum_{p\le M}\sum_{u\le N}\frac{\chi_{P_1}(p)\chi_{P_2}(u)}{pu}e\Big(\frac{bp^*u^*\nu}{a}\Big)e\Big(-\frac{\nu t}{apu}\Big)\Big|dt \tag{2.21}$$

$$+O\Big(yx^{-\lambda}\Big),$$

where $M = x^{\delta_1+\mu}$, $N = x^{\delta_2+\mu}$, $H = \frac{aMNx^{2\lambda}}{y}$ for some small $\lambda > 0$, and p^* and u^* are the inverses of p and u modulo a. In order to estimate $R_{a,b}(x,y)$ it is sufficient to estimate the integrand in the above formula, which can be treated as an exponential sum with monomials. Strong upper bounds for such sums were obtained by Fouvry and Iwaniec [10]. In particular, one may use the following result from [10].

Let $\alpha \neq 0, 1$ and $H, M, N, X \ge 1$. Let $\chi(\nu)$ be an additive character and ϕ_p, ψ_u be complex numbers with absolute value ≤ 1. Then

$$\left|\sum_{\frac{H}{2}\le\nu\le H}\sum_{\frac{M}{2}\le p\le M}\sum_{\frac{N}{2}\le u\le N}\phi_p\psi_u\chi(\nu)e\left(X\frac{\nu u^{-1}p^{\alpha}}{HN^{-1}M^{\alpha}}\right)\right| \ll (HMN)^{\frac{1}{2}}\left(\log(2HMNX)\right)^4 \tag{2.22}$$

$$\times\Big[X^{\frac{1}{8}}(H+N)^{\frac{1}{2}}(X^{\frac{1}{8}}H^{-\frac{1}{6}}M^{\frac{1}{12}}N^{\frac{1}{6}} + X^{\frac{1}{8}}H^{-\frac{1}{8}}N^{\frac{3}{8}} + N^{\frac{1}{2}} + N^{\frac{1}{4}}M^{\frac{1}{8}}) + M^{\frac{1}{2}} + X^{-\frac{1}{4}}M^{\frac{1}{2}}N\Big].$$

In this way, it is possible to show that for $\eta > 0$ small enough, $|R_{a,b}(x,y)| = o\left(\frac{y}{a}\right)$. We now use Mertens estimates

$$\sum_{p\in P_1}\frac{1}{p} = \log\left(1+\frac{\mu}{\delta_1}\right) + O\left(\frac{1}{\log x}\right) \ge C_1, \tag{2.23}$$

and

$$\sum_{u\in P_2}\frac{1}{u} \ge C_2, \tag{2.24}$$

where C_1 and C_2 are positive absolute constants depending on μ, δ_1 and δ_2 only. Combining these with the fact that $\prod_{j=1}^{\infty}\left(1-\frac{1}{b_j}\right)>0$, we obtain that

$$M_0 \geq C\,\frac{y}{a}+o\left(\frac{y}{a}\right) \tag{2.25}$$

for $C>0$ and $a \leq x^{\eta}$. It is easy to see that

$$E_1=\sum_{\substack{x-y<n\leq x\\ n\equiv b(\operatorname{mod} a)\\ b_m<b_s\leq \frac{y}{a}\\ n\equiv 0(\operatorname{mod} b_s)\\ \text{for some } s>m}} w(n) \leq 2C(\delta_1,\delta_2)\,\frac{y}{a}\sum_{s>m}\frac{1}{b_s}\,. \tag{2.26}$$

We may now fix the parameter m such that

$$C-2C(\delta_1,\delta_2)\sum_{s>m}\frac{1}{b_s}=C_0>0. \tag{2.27}$$

It follows that

$$M_0-E_1 \geq C_0\,\frac{y}{a}+o\left(\frac{y}{a}\right). \tag{2.28}$$

For estimating

$$E_2=\sum_{\substack{x-y<n\leq x\\ n\equiv b(\operatorname{mod} a)\\ \frac{y}{a}<b_s\leq x\\ n\equiv 0(\operatorname{mod} b_s)\\ \text{for some } s>m}} w(n), \tag{2.29}$$

we use the definition of $w(n)$ and the specific structure of A that was mentioned at the beginning of this section to see that $E_2=o\left(\frac{y}{a}\right)$ by imposing some conditions on σ, δ_1 and δ_2. Combining all these estimates we obtain

$$M_0-E_1-E_2 \geq C_0\,\frac{y}{a}+o\left(\frac{y}{a}\right). \tag{2.30}$$

The compatibility of all conditions imposed on $\delta_1, \delta_2, \sigma, \eta$ give us

$$\frac{1803}{10}\eta+\frac{9}{20}<\sigma\leq\frac{9}{19}\,. \tag{2.31}$$

By using the method explained above, one obtains the following result from [4].

Theorem 1 *Let η, σ be positive real numbers satisfying $20\sigma > 9 + 3606\eta$ and let B be a sequence of pairwise relatively prime positive integers with sum of inverses finite. Then there exists $N_{B,\sigma,\eta}$ such that for any $N \geq N_{B,\sigma,\eta}$ and any relatively prime integers a, b with $1 \leq a \leq N^\eta$, there exists $N \leq n \leq N + N^\sigma$ for which $n \equiv b\,(\bmod a)$ and n is B-free.*

3. Distribution of B-free numbers. The second approach

Let B be a sequence of positive integers $b_1 < b_2 < \dots$ with the sum of reciprocals convergent, and any two of them relatively prime. Let N be a large positive integer, choose relatively prime positive integers a and b smaller than N, and consider a small union of arithmetic progressions

$$\mathcal{M} = \{ax + by : 1 \leq x \leq N^{\theta_1}, 1 \leq y \leq N^{\theta_2}\} \tag{3.32}$$

for some $0 < \theta_1, \theta_2 < 1$. The question we address here is the existence of elements from $\mathcal{M}$ which are B-free numbers. Without any loss of generality, we may assume in what follows that $\theta_2 \leq \theta_1$. We first make the same reduction on the set B as in the first approach. Thus we obtain a set A, such that any A-free number is B-free. Denote the elements of A by $b_1 < b_2 < \dots < b_s < \dots$. We will look at numbers of form $n = ax + by$, where $x \leq N^{\theta_1}$, $y \leq N^{\theta_2}$. Clearly $n \equiv by \pmod a$. We put $z = N^{\theta_1} a$. We fix y temporarily, such that $(y, a) = 1$. Let

$$P = \{N^\mu < p < N^{\mu+\epsilon} : p \text{ prime}\} \setminus \{\text{prime divisors of } a\}, \tag{3.33}$$

where $\mu = \theta_1 - \epsilon$ with $\epsilon > 0$ fixed, but as small as we wish. Now we use the sieving weight

$$w(n) = \sum_{\substack{p \in P \\ n \equiv 0 (\bmod p)}} 1. \tag{3.34}$$

We have the inequality $w(n) \leq C(\mu)$ for all $n \leq 2N^{1+\theta_1}$ so that

$$\sum_{\substack{by + \frac{z}{2} \leq n \leq by + z \\ n \text{ is } A-\text{free} \\ n \equiv by (\bmod a)}} 1 \gg \sum_{\substack{by + \frac{z}{2} \leq n \leq by + z \\ n \text{ is } A-\text{free} \\ n \equiv by (\bmod a)}} w(n) \tag{3.35}$$

$$\geq \sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\not\equiv 0\,(\mathrm{mod}\,b_j)\text{ for all } j\leq k}} w(n) - \sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,b_j)\text{ for some}\\ b_k<b_j\leq \frac{z}{2a}}} w(n) - \sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,b_j)\text{ for some}\\ \frac{z}{2a}<b_j\leq by+z}} w(n).$$

For the main term we have by the inclusion-exclusion principle that

$$\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\not\equiv 0\,(\mathrm{mod}\,b_j)\text{ for all } j\leq k}} w(n) = \sum_{\Omega}(-1)^{|\Omega|}\sum_{p\in P}\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,b_\Omega)\\ n\equiv 0\,(\mathrm{mod}\,p)}} 1, \tag{3.36}$$

where $b_\Omega = \prod_{j\in\Omega} b_j$ for each subset Ω of $\{1,2,\ldots,k\}$. Since $(a,b) = 1 = (a,y)$ in the conditions of the right-hand side of (3.36), we have $(a,b_\Omega) = 1$ and $(a,p) = 1$. Clearly $(b_\Omega, p) = 1$ as soon as we fix k and let N tend to infinity. By the Chinese Remainder Theorem, the congruences $n \equiv by\,(\mathrm{mod}\,a)$, $n \equiv 0\,(\mathrm{mod}\,b_\Omega)$ and $n \equiv 0\,(\mathrm{mod}\,p)$ can be combined to $n \equiv c\,(\mathrm{mod}\,apb_\Omega)$. Note that

$$\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv c\,(\mathrm{mod}\,apb_\Omega)}} 1 = \frac{z}{2apb_\Omega} + r_{p,b_\Omega}(by,z), \tag{3.37}$$

where $|r_{p,b_\Omega}(by,z)| \leq 1$ is a remainder term. Combining (3.36) and (3.37), we obtain that the main term equals

$$\sum_{\Omega}(-1)^{|\Omega|}\sum_{p\in P}\left(\frac{z}{2apb_\Omega} + r_{p,b_\Omega}(by,z)\right) \tag{3.38}$$

$$= \frac{z}{2a}\sum_{p\in P}\frac{1}{p}\prod_{j\leq k}\left(1-\frac{1}{b_j}\right) + \sum_{\Omega}(-1)^{|\Omega|}\sum_{p\in P} r_{p,b_\Omega}(by,z).$$

Since $a \leq N$, the number $\omega(a)$ of distinct prime divisors of a satisfies $\omega(a) = O\left(\frac{\log N}{\log\log N}\right)$. It follows that

$$\sum_{p\in P}\frac{1}{p} \geq \log\left(1+\frac{\epsilon}{2\mu+\epsilon}\right) + O\left(\frac{1}{\log N}\right), \tag{3.39}$$

so that $\sum_{p\in P}\frac{1}{p}\geq C_1(\mu,\epsilon)$ as N tends to infinity. Since $\sum_s \frac{1}{b_s}<\infty$, we have

$$\prod_{j\geq k}\left(1-\frac{1}{b_j}\right)\geq\prod_{j=1}^{\infty}\left(1-\frac{1}{b_j}\right)=C>0. \tag{3.40}$$

On the other hand for the remainder terms, we have

$$\sum_{\Omega}(-1)^{|\Omega|}\sum_{p\in P}r_{p,b_\Omega}(by,z)\leq 2^k|P|=o\left(\frac{z}{a}\right), \tag{3.41}$$

if k is such that $2^k=o(\log N)$, which will be true as soon as we fix k. Hence the main term is

$$\geq CC_1(\mu,\epsilon)\frac{z}{2a}+o\left(\frac{z}{a}\right)=\frac{1}{2}CC_1(\mu,\epsilon)N^{\theta_1}+o\left(N^{\theta_1}\right). \tag{3.42}$$

We now analyze the first error term on the right side of (3.35). We have

$$\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,b_j)\text{ for some}\\ b_k<b_j\leq\frac{z}{2a}}}w(n)\leq C(\mu)\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,b_j)\text{ for some}\\ b_k<b_j\leq\frac{z}{2a}}}1. \tag{3.43}$$

If there is an n with $by+\frac{z}{2}\leq n\leq by+z$ such that $n\equiv by\,(\mathrm{mod}\,a)$ and $n\equiv 0\,(\mathrm{mod}\,b_j)$ then $(a,b_j)=1$ since $(a,b)=1$ and $(y,a)=1$. So these congruences can be combined to $n\equiv c\,(\mathrm{mod}\,ab_j)$ for some integer c. The number of numbers n with $by+\frac{z}{2}\leq n\leq by+z$ such that $n\equiv c\,(\mathrm{mod}\,ab_j)$ is $\leq\frac{z}{ab_j}$. Hence the right side of (3.43) is

$$\leq\frac{C(\mu)z}{a}\sum_{j>k}\frac{1}{b_j}=C(\mu)N^{\theta_1}\sum_{j>k}\frac{1}{b_j}. \tag{3.44}$$

We may now fix k large enough in terms of μ and ϵ only such that the right side of (3.43) is less than or equal to half of the main term.

We now proceed to estimate the second error term on the right side of (3.35). We have

$$\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,b_j)\text{ for some}\\ \frac{z}{2a}<b_j\leq by+z}}w(n)=\sum_{\frac{z}{2a}<v_1v_2\leq by+z}\sum_{p\in P}\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,v_1v_2)\\ n\equiv 0\,(\mathrm{mod}\,p)}}1 \tag{3.45}$$

$$+\sum_{\frac{z}{2a}<r\leq by+z}\sum_{p\in P}\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,r)\\ n\equiv 0\,(\mathrm{mod}\,p)}} 1=\Sigma_1+\Sigma_2,$$

say, where r is a prime in A and $v_1 \leq v_2$ are primes such that v_1v_2 is in A. Note that there is at most one n with $by+\frac{z}{2} \leq n \leq by+z$ such that $n \equiv by\,(\mathrm{mod}\,a)$ and $n \equiv 0\,(\mathrm{mod}\,v_1v_2)$, since these congruences can be combined into one congruence $n \equiv c'\,(\mathrm{mod}\,av_1v_2)$ for some integer c', and $av_1v_2 > \frac{z}{2}$. Similarly, there is at most one n with $by+\frac{z}{2} \leq n \leq by+z$ such that $n \equiv by\,(\mathrm{mod}\,a)$ and $n \equiv 0\,(\mathrm{mod}\,r)$. Consequently we have

$$\mathcal{V}_{v_1,v_2}:=\sum_{p\in P}\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,v_1v_2)\\ n\equiv 0\,(\mathrm{mod}\,p)}} 1=O(1), \tag{3.46}$$

and

$$\mathcal{R}_r:=\sum_{p\in P}\sum_{\substack{by+\frac{z}{2}\leq n\leq by+z\\ n\equiv by\,(\mathrm{mod}\,a)\\ n\equiv 0\,(\mathrm{mod}\,r)\\ n\equiv 0\,(\mathrm{mod}\,p)}} 1=O(1), \tag{3.47}$$

where the O-constants are independent of v_1, v_2 and r. It follows from (3.45), (3.46) and (3.47) that

$$\Sigma_1+\Sigma_2 \ll \sum_{\substack{\frac{z}{2a}<v_1v_2\leq by+z\\ \mathcal{V}_{v_1,v_2}>0}} 1+\sum_{\substack{\frac{z}{2a}<r\leq by+z\\ \mathcal{R}_r>0}} 1. \tag{3.48}$$

We may assume that $(v_1v_2, p) = 1$ and $(r, p) = 1$, since the number of elements r or v_1v_2 of A having a prime divisor in P is bounded by the number of elements of P (recall that the elements of A are relatively prime, so two distinct elements of A can not have a common prime divisor), which is $O\left(\frac{N^{\mu+\epsilon}}{\log N}\right) = o\left(N^{\theta_1}\right)$. Hence we may assume in what follows that $(v_1v_2, p) = 1$ and $(r, p) = 1$. If $\mathcal{V}_{v_1,v_2} > 0$ then the congruences $n \equiv 0\,(\mathrm{mod}\,v_1v_2)$ and $n \equiv 0\,(\mathrm{mod}\,p)$ give $n \equiv 0\,(\mathrm{mod}\,v_1v_2p)$. Since $p \geq N^{\mu}$, we obtain that $v_1v_2 \leq \frac{by+z}{N^{\mu}}$, so that $v_1 \leq \sqrt{\frac{by+z}{N^{\mu}}}$. Since the elements of A are relatively prime to each other, there is a one to one correspondence between elements of A of form v_1v_2 and the set of v_1's. It is then easy to see that

$$\sum_{\substack{\frac{z}{2a}<v_1v_2\leq by+z\\ \mathcal{V}_{v_1,v_2}>0}} 1 \leq \sqrt{\frac{by+z}{N^{\mu}}} = o\left(N^{\theta_1}\right), \tag{3.49}$$

provided $\theta_1 > 1/2$, $3\theta_1 > 1+\theta_2$, and ϵ is small enough. Next, we estimate the sum

$$\sum_{\substack{\frac{by+z}{2a}<r\leq by+z\\ \mathcal{R}_r>0}} 1$$

on average over all the admissible arithmetic progressions. To this end we consider all $y \leq N^{\theta_2}$ such that $(y,a)=1$, and repeat the above argument for each such y. The number of such y is $\#\{y \leq N^{\theta_2} : (y,a)=1\}$, which is easily seen to be $\gg \frac{N^{\theta_2}}{\log\log N}$. Hence the overall contribution of main terms in each of these arguments is $\gg \frac{N^{\theta_1+\theta_2}}{\log\log N}$. All the error terms are under control since they have been controlled individually for each y, except for the error term involving r. In order to control the error term involving r, we partition the interval $\left[\frac{N^{\theta_1}}{2}, 2N^{1+\theta_1}\right]$ into dyadic intervals of form $[T,2T]$. We have

$$\sum_{\substack{y\leq N^{\theta_2}\\ (y,a)=1}} \sum_{\substack{\frac{z}{2a}<r\leq by+z\\ \mathcal{R}_{r,y}>0}} 1 \leq \sum_{\frac{z}{2a}<r\leq 2N^{1+\theta}} \sum_{\substack{y\leq N^{\theta_2}\\ \mathcal{R}_{r,y}>0}} 1 \tag{3.50}$$

$$= \sum_{T} \sum_{T\leq r\leq 2T} \sum_{\substack{y\leq N^{\theta_2}\\ \mathcal{R}_{r,y}>0}} 1.$$

Note that

$$\sum_{\substack{y\leq N^{\theta_2}\\ \mathcal{R}_{r,y}>0}} 1 = \sum_{\substack{y\leq N^{\theta_2}\\ n\equiv by \,(\mathrm{mod}\, a)\\ n\equiv 0\,(\mathrm{mod}\, r)\\ n\equiv 0\,(\mathrm{mod}\, p)}} 1 \leq \sum_{\substack{n\leq 2N^{1+\theta_1}\\ n\equiv 0\,(\mathrm{mod}\, rp)}} 1 \leq \frac{2N^{1+\theta_1}}{rp} \leq \frac{2N^{1+\theta_1}}{TN^{\mu}}. \tag{3.51}$$

Combining (3.50) and (3.51) we see that the left side of (3.50) is

$$\ll \frac{2N^{1+\theta_1}}{TN^{\mu}} T\log N \ll N^{1+\theta_1-\mu}\log N. \tag{3.52}$$

One easily sees that

$$N^{1+\theta_1-\mu}\log N = o\left(\frac{N^{\theta_1+\theta_2}}{\log\log N}\right), \tag{3.53}$$

provided that $\theta_1+\theta_2>1$, and ϵ is small enough. Taking the above into account, one obtains the following result.

Theorem 2 *Let $0<\theta_1,\theta_2<1$ with $\theta_1+\theta_2>1$. Let B be a sequence of positive integers with the sum of inverses convergent, and any two of them relatively prime. Then there exists a number N_{B,θ_1,θ_2} such that for any $N\ge N_{B,\theta_1,\theta_2}$ and any coprime positive integers a, b with $\max\{a,b\}\le N$, the number of pairs positive integers x,y with $x\le N^{\theta_1}$ and $y\le N^{\theta_2}$ for which $ax+by$ is B-free, is $\gg \frac{N^{\theta_1+\theta_2}}{\log\log N}$.*

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