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On Groups with the Weak Wide Commensurable Property

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Dedicated to Prof. Dr. Cemal Koç on his 61^{st} birthday

Abstract

An infinite group with the weak wide commensurable property is shown to be abelian, provided that it is locally finite or locally graded or non-perfect or linear. We also investigate the properties of infinite non-abelian groups with the weak wide commensurable property. Moreover, we describe completely the structure of infinite locally finite groups whose p-subgroups have the weak wide commensurable property. (AMS MSC: 20F50, 20E34).

1. Introduction

If a group-theoretical property of groups is common to all finite groups, then it is called a finiteness property. Some well-known examples of finiteness properties are: being finitely generated, locally finite, residually finite, FC, min and max conditions. For details, the reader might like to see [7, Chapter 14].

We study infinite groups that satisfy a particular finiteness property, namely the *weak wide commensurable property.* To be precise, we consider infinite groups in which any two non-trivial proper subgroups have commensurable conjugates. Recall that two subgroups are called commensurable if their intersection is of finite index in both subgroups.

Clearly all finite groups, quasi-cyclic p-groups for every prime p and the additive group of integers satisfy the weak wide commensurable property. A more interesting class is quasi-finite groups (these are groups whose proper subgroups are all finite). Probably the most well-known non-abelian quasi-finite group is the Tarski group which was constructed by Ol'shanskii, answering the question of Tarski on the existence of infinite groups whose non-trivial proper subgroups are of order a fixed prime. For details, see [6, Theorem 28.1].

There is also a non-abelian torsion-free group that satisfies the weak wide commensurable property which was constructed by Ol'shanskii again. In this group all proper subgroups are (infinite) cyclic and any two proper subgroups intersect non-trivially [6, Theorem 31.4].

Through private communication with Alexander Ol'shanskii, we learned that by using Guba's method in [1], it is possible to construct a torsion-free simple group that satisfies the weak wide commensurable property. In this group, all proper subgroups are (infinite) cyclic and any pair of proper subgroups intersect trivially.

In this work, first we prove that a *nice* infinite group with the weak wide commensurable property is abelian, where the term *nice* can be replaced with locally finite, locally solvable, linear etc. The precise result is stated as Theorem 4.1(1). Infinite abelian groups with the weak wide commensurable property are easily seen to be either cyclic or isomorphic to a quasi-cyclic p-group for some prime p (Theorem 2.2). The second part of Theorem 4.1 contains a description of *non-abelian* groups that satisfy the weak wide commensurable property. Then we move on to describe the structure of infinite locally finite groups whose p-subgroups satisfy the weak wide commensurable property for all primes p (Theorem 4.3).

2. Definitions and Basic Properties

Definitions. Two subgroups $H, K \leq G$ are called *commensurable* if $[H : H \cap K]$ and $[K : H \cap K]$ are both finite. They are called *widely commensurable* if H and a conjugate of K in G are commensurable; i.e. there exists $g \in G$ such that $[H : H \cap K^g]$ and $[K^g : H \cap K^g]$ are both finite.

A group is said to satisfy the

- *weak* (*wide*) *commensurable property* if all of its non-identity, proper subgroups are (widely) commensurable.

- *strong* (*wide*) *commensurable property* if all of its non-identity subgroups are (widely) commensurable.

In short we will say a group is W(W)C or S(W)C respectively.

Recall that a group is called *quasi-finite* if each proper subgroup is finite. It is easy to see that quasi-finite groups coincide with groups whose proper subgroups are all widely commensurable, and with torsion WWC groups.

Examples. Clearly all finite groups satisfy all of the above properties. The additive group of integers is SC, the quasi-cyclic group is quasi-finite and non-SC. As mentioned in the Introduction, there are non-abelian WC groups constructed by Ol'shanskii (see [6, Theorem 31.4]). The group produced by Guba's method is an example of a WWC group which is not WC.

Non-examples. The direct sum of an infinite group and a non-trivial group is not WWC. The additive group of rationals is not WWC. In fact no infinite abelian group is WWC unless it is cyclic or quasi-cyclic (see Theorem 2.2 below).

Lemma 2.1 *A WWC group is either torsion or torsion-free. In particular, any infinite SWC group is torsion-free.*

Proof. Let G be a WWC group and contain a non-identity element x of finite order and an element y of infinite order. Then the index $[\langle y\rangle : \langle x\rangle^g \cap \langle y\rangle]$ is finite for some non-zero $g \in G$ and $\langle x\rangle^g \cap \langle y\rangle$ is finite by the choice of x. Hence $\langle y\rangle$ is finite, but this contradicts with the choice of y. $\square$

Theorem 2.2 *An infinite abelian group is WWC if and only if it is isomorphic to the additive group of integers* $\mathbf{Z}$ *or to the quasi-cyclic p-group* C_{p^∞} *for some prime p.*

Proof. Let G be an infinite abelian WWC group. First let's show that G is countable. If G is finitely generated, then G is countable. Otherwise take $x_1 \in G \setminus \{1\}$ and $x_{n+1} \in G \setminus \langle x_1, \dots, x_n\rangle$ for every $n \geq 2$. Then the index $[\cup_{n\geq 1}\langle x_1, \dots, x_n\rangle : \langle x_1\rangle]$ is infinite, hence $G = \cup_{n\geq 1}\langle x_1, \dots, x_n\rangle$ is countable.

Now to prove the theorem, first assume that G is torsion, hence proper subgroups of G are finite. By standard arguments, one can show that G is a divisible p-group for some prime p. Thus G is isomorphic to the quasi-cyclic p-group in this case.

Now by Lemma 2.1, we may assume that G is torsion-free. If G is finitely generated, then $G \cong \mathbf{Z}$. So let's assume that G is not finitely generated, but then every non-trivial proper subgroup of G is finitely generated and hence is isomorphic to $\mathbf{Z}$. Since G is countable, G is free abelian by ([7, 4.4.3]) and hence $G \cong \mathbf{Z}$. $\square$

The proof of the following is clear.

Lemma 2.3 *In any group, subgroups of finite index are all commensurable. In particular, if all non-trivial subgroups of a group are of finite index, then the ambient group is SC.*

Corollary 2.4 *The following are equivalent for an infinite group G.*
(a) *G is SWC.*
(b) *G is SC.*
(c) *G is isomorphic to the additive group of integers.*

Proof. (a)⇒(b): Let G be a SWC group. Then G and any non-trivial subgroup of G are widely commensurable. Thus all non-trivial subgroups in G are of finite index in G. Hence G is SC by Lemma 2.3.

(b)⇒(c): Let G be an infinite SC group. First assume that $Z(G) \neq 1$. Then $Z(G)$ has finite index in G and thus by Schur's Lemma (see [7, 10.1.4]), G' is finite. Hence $G' = 1$. Thus G is abelian torsion-free, so $G \cong \mathbf{Z}$ by Theorem 2.2.

Now assume that G has trivial center. Take $x, y \in G$ such that $\langle x, y\rangle$ is non-abelian. Set $H = \langle x, y\rangle$, now $[\langle x\rangle : \langle x\rangle \cap \langle y\rangle]$ is finite hence $x^n = y^m$ for some $n, m \in \mathbf{Z}^+$. Therefore $x^n \in Z(H)$ and $H \cong \mathbf{Z}$ by the arguments of the previous paragraph applied to H. The contradiction shows that G has a non-trivial center.

(c)⇒(a): Clear. $\square$

Corollary 2.5 *Let G be an infinite group such that $G \neq G'$. Then G is quasi-finite if and only if it is isomorphic to a quasi-cyclic p-group for some prime p.*

Proof. Let G be infinite quasi-finite and $G \neq G'$. Note that all proper subgroups of G are finite, in particular G is torsion, G' is finite and $C_G(x)$ are finite if $x \notin Z(G)$. Since G' is finite; for every $x \in G$, $C_G(x)$ is of finite index in G, hence G is abelian. Now by Theorem 2.2, $G \cong C_{p^\infty}$ for some prime p. The converse is clear. $\square$

3. WWC Groups

One of the main aims of this section is to determine WWC groups with an extra property which excludes quasi-finite groups. In all such results below, the classification gives either the infinite cyclic group or the quasi-cyclic p-groups for prime p. By Theorem 2.2 these are the only infinite abelian WWC groups. Hence most of the results are stated as: Any infinite WWC group with the property $\mathcal{P}$ is abelian.

A group is called an *FC-group* if every element has finitely many conjugates in the group.

Lemma 3.1 **(a)** *If an infinite locally finite group is WWC, then it is abelian.*
(b) *If an infinite FC-group is WWC, then it is abelian.*

Proof. **(a)** Let G be an infinite WWC locally finite group. Then G is quasi-finite. Any locally finite group contains an infinite abelian subgroup (see [4] or [2]) and hence G is abelian.

(b) Note that centralizers of elements are of finite index in an FC-group. Hence there is no infinite torsion WWC FC-group. On the other hand, torsion-free FC-groups are abelian ([7, 14.5.9]). □

Lemma 3.2 *Let G be an infinite WWC group.*
(a) *Infinite proper normal subgroups of G are all cyclic. In particular, an infinite subgroup of G is either cyclic or its conjugates generate G.*
(b) *If G is torsion-free and has trivial center, then G is simple.*
(c) *If G has a proper subgroup of finite index, then G is cyclic.*
(d) *If $G/Z(G)$ is finite or locally cyclic, then G is abelian.*

Proof. **(a)** Let G be an infinite WWC group and N an infinite proper normal subgroup of G. If L is a non-trivial subgroup of N, then for some $g \in G$, $[N^g : N^g \cap L] = [N : N \cap L] = [N : L]$ is finite. Hence N is SC by Lemma 2.3 and hence infinite cyclic by Corollary 2.4.

(b) Let G be a torsion-free WWC group with a trivial center. To derive a contradiction, assume that N is an infinite proper normal subgroup in G. By (a), N is cyclic, say x generates N. Choose $y \in G$ such that $[x, y] \neq 1$. Since N is normal in G, $yxy^{-1} = x^{-1}$. Also $\langle x \rangle$ and $\langle y \rangle$ are wide commensurable hence $[\langle y \rangle : \langle y \rangle \cap \langle x \rangle^g]$ is finite for some $g \in G$. Therefore $y^m = (x^g)^l = x^{kl}$ for some $k, l, m \in \mathbf{Z}$. Now take the kl-th powers of $x^{-1} = yxy^{-1}$ to get $x^{-kl} = (yxy^{-1})^{kl} = y^m = x^{kl}$. Since G is torsion-free, we get $k = 0$ or $l = 0$. But none of these cases is possible.

(c) Let G be an infinite WWC group with a proper subgroup of finite index, then clearly G has a proper normal subgroup N of finite index. If G is torsion, then it is easy to see that G is finite. If G is torsion-free, then by part (b), $Z(G) \neq 1$. Thus N and $Z(G)$ are commensurable and hence $Z(G)$ is of finite index in G. Being torsion-free FC, G is abelian. Hence G is cyclic by Theorem 2.2.

(d) This is obvious. □

The next lemma is about torsion-free groups in general, and it will help us to analyse torsion-free WWC groups.

Lemma 3.3 *If a cyclic subgroup is of infinite index in a torsion-free group, then every cyclic subgroup is of infinite index.*

Proof. Assume H is torsion-free and there exists $1 \neq x \in H$ such that $[H : \langle x \rangle]$ is infinite. Pick an arbitrary $y \in H$ and write $H = \sqcup_{i \in I} c_i \langle y \rangle$ where I is an index set. Then for every $n \in \mathbf{Z}$, there exists $i_n \in I$ and $t_n \in \mathbf{Z}$ such that $x^n = c_{i_n} y^{t_n}$. Assume x^n and x^m are in the same coset of $\langle y \rangle$, then $x^m = c_{i_n} y^{t_m}$. Hence $c_{i_n} = x^n y^{-t_n} = x^m y^{-t_m}$ and $x^{n-m} = y^{t_n - t_m} \in \langle x \rangle \cap \langle y \rangle$. If $\langle x \rangle \cap \langle y \rangle = \langle x^k \rangle = \langle y^l \rangle$ for some non-zero $k, l \in \mathbf{Z}$, then $\langle y \rangle$ is of infinite index in H and we are done. If $\langle x \rangle \cap \langle y \rangle = 1$, then we get $n = m$. Therefore x^n, x^m are in the same coset of $\langle y \rangle$ iff $n = m$. Thus $\langle y \rangle$ is of infinite index in H. □

The next theorem is about torsion-free WWC groups. Recall that there are non-abelian torsion-free WWC groups (see the Introduction).

Theorem 3.4 *If G is a torsion-free non-abelian WWC group, then proper subgroups of G are cyclic and G is 2-generated.*

Proof. Let H be a proper non-trivial subgroup of G. Assume there exists $1 \neq x \in H$ such that $[H : \langle x \rangle]$ is infinite. Since G is WWC, for some $g \in G$, $[\langle x \rangle^g : H \cap \langle x \rangle^g]$ is finite. Hence $y := (x^g)^n \in H$ for some $n \in \mathbf{Z}$, but then $\langle y \rangle$ has finite index in H which contradicts with Lemma 3.3. Therefore $\langle x \rangle$ is of finite index in H for all $x \in H$. In particular being a torsion-free FC-group, H is abelian and non-trivial subgroups of H are of finite index in H. Thus H is cyclic.

Since G is not abelian, there exist $a, b \in G$ that do not commute, hence $G = \langle a, b \rangle$ is 2-generated. □

Next we consider infinite non-perfect WWC groups.

Theorem 3.5 *Any infinite non-perfect WWC group is abelian.*

Proof. Let G be infinite WWC and $G \neq G'$. First assume that G is torsion, then G is quasi-finite and hence abelian by Corollary 2.5. Now assume that G is torsion-free. If $G' \neq 1$, then $Z(G) \neq 1$ by Lemma 3.2. After an application of Corollary 2.5 and Lemma 3.2 to $G/Z(G)$, the result follows. □

A group is called *locally graded* if every finitely generated subgroup has a proper subgroup of finite index. Now a result on locally graded WWC groups is in order.

Theorem 3.6 *Any infinite locally graded WWC group is abelian.*

Proof. We have already observed that torsion WWC groups are quasi-finite. Since the group under consideration is also locally graded, it is locally finite hence Theorem 3.1 applies. Now we may assume that G is a locally graded torsion-free WWC group. By Theorem 3.4, we may assume that G is 2-generated, thus there is an infinite proper normal subgroup of finite index in G and the result follows from Lemma 3.2 (c). □

Theorem 3.7 *Any infinite locally solvable WWC group is abelian.*

Proof. Recall that a torsion WWC group is quasi-finite. A quasi-finite group satisfies min (i.e. each non-empty collection of subgroups has a minimal element). A locally solvable group that satisfies min has an ascending normal series with abelian factors (see [7, 12.5.3]). Hence G is infinite locally finite and the result follows from Lemma 3.1. We may assume that G is a locally solvable torsion-free WWC group. If G has a trivial center, then G is simple by Lemma 3.2, but there is no infinite simple locally solvable group (see [7, 12.5.2]). Hence G has non-trivial center, in this case the result follows from Lemma 3.2 (d) applied to $G/Z(G)$. □

4. Main Theorems

The first part of the following theorem is basically a summary of the results of the previous section.

Theorem 4.1 **(1)** *Let G be an infinite group which is locally finite or FC or non-perfect or locally graded or locally solvable or residually finite or linear. If G is a WWC group, then G is abelian; more precisely G is isomorphic to the additive group of integers or to a quasi-cyclic p-group for some prime p.*
(2) *Let G be an infinite non-abelian WWC group. Then either*
(2.1) *G is a 2-generated torsion-free group whose proper subgroups are all cyclic (and either G is simple or $G/Z(G)$ is quasi-finite),*
(2.2) *or G is a finitely generated quasi-finite group.*

Proof. **(1)** We need to consider the last two cases only. A group is called residually finite if for every non-identity element in the group there is a normal subgroup of finite index which does not contain the given element. Now the statement on residually finite WWC groups is an easy consequence of Lemma 3.2 (c).

Now let G be an infinite linear WWC group. If G is torsion linear, then G is locally finite by a result of Schur [5, Theorem 1.L.1]. Hence we are done in this case. If G is torsion-free, then G is either abelian or 2-generated by Theorem 3.4. By a result of Mal'cev ([8, Theorem 4.2]), finitely generated linear groups are residually finite. Hence G is abelian by the above paragraph and hence by Theorem 2.2 is the infinite cyclic group. (Note that the infinite cyclic group embeds in the center of a maximal unipotent subgroup of the general linear group of dimension at least 2 over a field of characteristic zero.)

(2.1) If G is an infinite torsion-free non-abelian WWC group, then by Theorem 3.3, G is 2-generated, say $G = \langle a, b\rangle$ and all proper subgroups of G are cyclic.

If $\langle a\rangle \cap \langle b\rangle = 1$, then G is simple. Moreover any two elements of G that do not belong to a cyclic subgroup of G generate G, and the maximal groups that they belong to are disjoint and widely commensurable. In this group, if two maximal subgroups intersect non-trivially then they coincide and all maximal subgroups are conjugate. (As mentioned in the Introduction, such a group can be constructed by using the method in [1].)

If $\langle a\rangle \cap \langle b\rangle \neq 1$, then $Z(G) \neq 1$ and hence $G/Z(G)$ is quasi-finite. (Recall that there is such a group constructed by Ol'shanskii [6, Theorem 31.4].)

(2.2) Torsion WWC groups coincide with quasi-finite groups. If such a group is not finitely generated then it is locally finite hence the result follows. $\square$

Corollary 4.2 *An infinite non-abelian WC group is either quasi-finite or center-by-quasi-finite.*

4.1. WWC p-subgroups

The following describes the structure of infinite locally finite groups whose p-subgroups are all WWC.

Theorem 4.3 *Let G be an infinite locally finite group. All p-subgroups of G are WWC for every prime p if and only if G can be expressed as a product of a normal abelian subgroup A and a residually finite group F satisfying the following:*
a. $A \cap F = 1$.
b. $A = \mathrm{Dr}_{i \in I} \mathbf{C}_{p_i^\infty}$ *where I is an index set and p_i for $i \in I$ are distinct primes.*
c. *Sylow subgroups of F are finite and Sylow p_i-subgroups of F are trivial for every $i \in I$.*
d. *There exists a locally solvable subgroup S of finite index in F.*
e. *For any finite set of primes $\pi = \{q_1, \ldots, q_k\}$ dividing the orders of elements in S, S can be written as a product of (finite) Sylow q_i-subgroups of S and a Hall π'-subgroup of S.*

Proof. First note that Sylow p-subgroups of G are either finite or isomorphic to a quasi-cyclic p-group by Lemma 3.1; therefore in any case G satisfies min-p (i.e. p-subgroups satisfy the minimal condition) for every prime p. Clearly every section of G also satsifies min-p for every prime p. By a result of Hartley and Shute [3], an infinite simple locally finite group satisfying min-q is a group of Lie type over an infinite locally finite field of characteristic not q. Hence the characteristic is not prime and clearly it is not zero. Therefore G does not involve any infinite simple subgroups.

Thus G has a divisible abelian normal subgroup A such that G/A is residually finite and has finite Sylow subgroups by [5, Corollary 3.18]. Clearly A is a direct product of quasi-cyclic p-groups for distinct primes p (see the Non-examples in Section 2). This proves part (b).

Assume that A has a direct factor A_p isomorphic to $\mathbf{C}_{p^\infty}$ and G has a Sylow p-subgroup P such that PA/A is non-trivial to derive a contradiction. Since PA/A is finite by the above paragraph, we can pick an element $x \in P \setminus A$ such that a finite power of x lies in A. Now the weak wide commensurable property applied to A_p and $\langle x \rangle$ gives the desired contradiction.

Since the orders of elements in A and G/A are relatively prime, there exists a subgroup F in G such that $A \cap F = 1$ and $G = AF$ by a well-known generalization of Schur-Zassenhaus Theorem. By the arguments of the second paragraph of this proof, F is residually finite and has finite Sylow subgroups. Also note that F is almost locally solvable by Feit–Thompson's Odd Order Theorem. This completes the proofs of parts (a), (c) and (d).

The proof of part (e) is clear. $\square$

Corollary 4.4 *There is no infinite simple locally finite group with WWC p-subgroups for every prime p.*

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