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Valuations of Polynomials*

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Abstract

A tree is a connected (undirected) graph that contains no cycles. Trees play an important role in Computer Science. There are many applications in this field. Ordered binary decision diagrams are trees in the language of Boolean algebras. For the applications, it is important to measure the complexity of a tree or of a polynomial. The complexity of a polynomial over an arbitrary algebra can be regarded as a valuation. The concept of the valuations of terms was introduced by K. Denecke and S. L. Wismath in [5]. In [6], the author defined the depth of a polynomial which is an example of a complexity measure for polynomials. In this paper we study several other measures of the complexity of polynomials. In each of these measures, we have a mapping $v : P_\tau(X, \overline{A}) \longrightarrow I\!N$ from the set of all polynomials of type τ over $\overline{A}$ to the set of natural numbers (including 0) which assigns to each polynomial p a complexity number or a value $v(p)$. We will refer to such a function as a complexity or a cost function or a valuation and we study some properties of these valuations.

Key Words: Polynomials, valuations of polynomials, Order Condition, Algebraic Subpolynomial Condition, Subpolynomial Condition.

1. Introduction

We first introduce some definitions of polynomials of type τ over $\overline{A}$. Let $\overline{A}$ and X be disjoint sets and $\{f_i \mid i \in I\}$ be an indexed set of operation symbols where f_i is n_i-ary.

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We denote by X_n the n-element alphabet $X_n = \{x_1, \ldots, x_n\}$. The polynomials of type τ over $\overline{A}$, for short polynomials, are inductively defined by the following steps:

(i) each $x \in X$ is a polynomial, called a *variable*,

(ii) each $\overline{a} \in \overline{A}$ is a polynomial, called a *constant*,

(iii) if $p_1, \ldots, p_{n_i}$ are polynomials, then $f_i(p_1, \ldots, p_{n_i})$ is a polynomial.

The set $P_\tau(X_n, \overline{A})$ is called *the set of n-ary polynomials.* Let $P_\tau(X, \overline{A})$ be the set of all polynomials, i.e. $P_\tau(X, \overline{A}) := \bigcup\limits_{n=1}^{\infty} P_\tau(X_n, \overline{A})$. The set $W_\tau(X)$ of all terms of type τ is a subset of $P_\tau(X, \overline{A})$. The set $P_\tau(X, \overline{A})$ is the base set of the absolutely free algebra $\underline{P_\tau(X, \overline{A})} := (P_\tau(X, \overline{A}); (\overline{f}_i)_{i \in I})$, where the operations $\overline{f}_i$ are defined by $\overline{f}_i(p_1, \ldots, p_{n_i}) = f_i(p_1, \ldots, p_{n_i})$.

We begin with a list of examples of some complexity measures for polynomials.

The minimum depth of a polynomial p is the length of the shortest path from the root to a vertex in the tree, and is denoted by $mindepth(p)$ and defined by

(i) $mindepth(x) = 0$, if $x \in X$,

(ii) $mindepth(\overline{a}) = 0$, if $\overline{a} \in \overline{A}$,

(iii) $mindepth(f_i(p_1, \ldots, p_{n_i})) = min\{mindepth(p_1), \ldots, mindepth(p_{n_i})\} + 1$.

The maximum depth of a polynomial p is the length of the longest path from the root to a vertex in the tree, denoted by $maxdepth(p)$, and is defined by

(i) $maxdepth(x) = 0$, if $x \in X$,

(ii) $maxdepth(\overline{a}) = 0$, if $\overline{a} \in \overline{A}$,

(iii) $maxdepth(f_i(p_1, \ldots, p_{n_i})) = max\{maxdepth(p_1), \ldots, maxdepth(p_{n_i})\} + 1$.

The variable count of a polynomial p is the total number of occurrences of variables in p, denoted by $varcount(p)$, and is defined by

(i) $varcount(x) = 1$, if $x \in X$,

(ii) $varcount(\overline{a}) = 0$, if $\overline{a} \in \overline{A}$,

(iii) $varcount(f_i(p_1, \dots, p_{n_i})) = \sum_{j=1}^{n_i} varcount(p_j)$.

The constant count of a polynomial p is the total number of occurrences of constants in p, denoted by $constcount(p)$, and is defined by

(i) $constcount(x) = 0$, if $x \in X$,

(ii) $constcount(\overline{a}) = 1$, if $\overline{a} \in \overline{A}$,

(iii) $constcount(f_i(p_1, \dots, p_{n_i})) = \sum_{j=1}^{n_i} constcount(p_j)$.

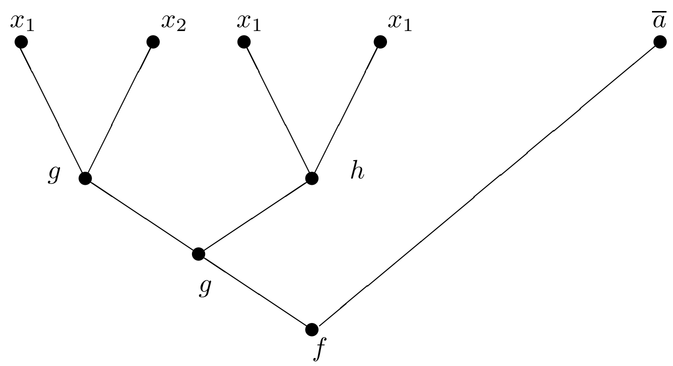
The operation symbol count of a polynomial p is the total number of occurrences of operation symbols in p, denoted by $opcount(p)$, and is defined by

(i) $opcount(x) = 0$, if $x \in X$,

(ii) $opcount(\overline{a}) = 0$, if $\overline{a} \in \overline{A}$,

(iii) $opcount(f_i(p_1, \dots, p_{n_i})) = \sum_{j=1}^{n_i} opcount(p_j) + 1$.

As an example we consider the tree type $\tau = (2, 2, 2)$ with binary operation symbols f, g, h. For a polynomial $p = f(g(g(x_1, x_2), h(x_1, x_1)), \overline{a})$, as shown below in Figure,



Figure

we have $mindepth(p) = 1, maxdepth(p) = 3, varcount(p) = 4, constcount(p) = 1$, and $opcount(p) = 4$.

These complexity functions assign the same complexity values (either 0 or 1) to all variables, all constants and all the operation symbols. For example, each constant contributes a value of 1 in *constcount*, and a value 0 in *varcount*. So, it is reasonable to have a cost function which variously weighs different operations.

Let $w : \{f_i \mid i \in I\} \longrightarrow I\!\!N$ be a function which assigns to each operation symbol f_i a weight $w(f_i)$. Then the complexity measure v_w on $P_\tau(X, \overline{A})$ is inductively defined as follows:

(i) v_w assigns some constant values to each variable and to each constant;

(ii) $v_w(f_i(p_1, \ldots, p_{n_i})) = w(f_i) + \sum_{j=1}^{n_i} v_w(p_j)$.

2. Generalized Complexity Functions

Next, we consider the basic properties of our examples of the complexity measures on $P_\tau(X, \overline{A})$. In the previous section we have used several features of the algebra $I\!\!N$ including the addition operation and the order relation $\leq$ on $I\!\!N$. We used additional n_i-ary operations on $I\!\!N$ for each $i \in I$. So, we regard the set $I\!\!N$ as an algebra of the fixed type τ, denoted by $I\!\!N_\tau$.

Definition 2.1 Let a and b be fixed elements of $I\!\!N$ and let $I\!\!N_\tau = (I\!\!N; (f_i^{I\!\!N})_{i \in I})$ be an algebra of type τ with the base set $I\!\!N$. Let v be the constant mapping $v : X \cup \overline{A} \longrightarrow I\!\!N$ defined by

(i) $v(x) = a$, for all $x \in X$,

(ii) $v(\overline{a}) = b$, for all $\overline{a} \in \overline{A}$.

Then v has a unique extension, denoted by $\overline{v}$, to the set $P_\tau(X, \overline{A})$ which is a homomorphism from the free algebra $\underline{P_\tau(X, \overline{A})}$ into $I\!\!N_\tau$, i.e. $\overline{v} : \underline{P_\tau(X, \overline{A})} \longrightarrow I\!\!N_\tau$, where $\overline{v}(f_i(p_1, \ldots, p_{n_i})) = f_i^{I\!\!N}(\overline{v}(p_1), \ldots, \overline{v}(p_{n_i}))$. Such an extension $\overline{v}$ is called *a valuation of polynomials of type τ over $\overline{A}$ into $I\!\!N_\tau$* if the following conditions are satisfied:

(iii) $\overline{v}(p) \geq \overline{v}(x)$, for all $x \in X$ and for all $p \in P_\tau(X, \overline{A})$,

(iv) $\overline{v}(p) \geq \overline{v}(\overline{a})$, for all $\overline{a} \in \overline{A}$ and for all $p \in P_\tau(X, \overline{A})$.

It follows immediately from the conditions (iii) and (iv) that when these conditions are met it must be the case that the values a and b assigned to all variables and all constants, respectively, must be equal. Thus the definition of a valuation requires that $a = b$.

The algebra $\mathbb{N}_\tau$ is called *the valuation algebra of the valuation* $\overline{v}$.

The definition uses the order relation $\leq$ on $\mathbb{N}$. More generally, we could use a valuation structure $\mathcal{A}_\tau = (A; (f_i^{\mathcal{A}})_{i \in I}, \leq)$, where $\leq$ is a partial order on the set A. From this definition and the examples in section 1, we have the following results.

For *mindepth*, the operations $f_i^{\mathbb{N}}$ are defined by $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) := min\{a_1, \ldots, a_{n_i}\} + 1$. Then $mindepth(p) = 0$ if $p = x \in X$ or $p = \overline{a} \in \overline{A}$, and the conditions (iii), (iv) are also satisfied.

For *maxdepth*, the operations $f_i^{\mathbb{N}}$ are defined by $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) := max\{a_1, \ldots, a_{n_i}\} + 1$. Then $maxdepth(p) = 0$ if $p = x \in X$ or $p = \overline{a} \in \overline{A}$, and the conditions (iii), (iv) are also satisfied.

For *varcount*, the operations $f_i^{\mathbb{N}}$ are defined by $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) := \sum_{j=1}^{n_i} a_j$, with $varcount(x) = 1$ for $x \in X$ and $varcount(\overline{a}) = 0$ for $\overline{a} \in \overline{A}$, and the condition (iv) is also satisfied. But the condition (iii) is not satisfied since $varcount(\overline{a}) < varcount(x)$.

For *constcount*, the operations $f_i^{\mathbb{N}}$ are defined by $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) := \sum_{j=1}^{n_i} a_j$, with $constcount(x) = 0$ for $x \in X$ and $constcount(\overline{a}) = 1$ for $\overline{a} \in \overline{A}$, and the condition (iii) is also satisfied. But the condition (iv) is not satisfied since $constcount(x) < constcount(\overline{a})$.

So *varcount* and *constcount* are not valuations of polynomials of type τ into $\mathbb{N}_\tau$.

For *opcount*, the operations $f_i^{\mathbb{N}}$ are defined by $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) := \sum_{j=1}^{n_i} a_j + 1$. Then $opcount(p) = 0$ if $p = x \in X$ or $p = \overline{a} \in \overline{A}$, and the conditions (iii), (iv) are also satisfied.

For the weighted complexity function v_w in section 1, we assign $v(x) = a$ for some the fixed value a, $v(\overline{a}) = b$ for some the fixed value b, and take $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) := w(f_i) + \sum_{j=1}^{n_i} a_j$.

Next, we consider the property of the operations $f_i^{\mathbb{N}}$ of the valuation algebra defined by K. Denecke and S. L. Wismath in [5].

The operations $f_i^{\mathbb{N}}$ of the valuation algebra are monotone, so the following Order Condition (OC) is satisfied:
(OC) If $a_j \leq b_j$ for $1 \leq j \leq n_i$ and f_i is an n_i-ary operation symbol of type τ, then for the corresponding $f_i^{\mathbb{N}}$ we have $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \leq f_i^{\mathbb{N}}(b_1, \ldots, b_{n_i})$.

Proposition 2.2 *For any valuation $\overline{v}$ of polynomials of type τ over $\overline{A}$ into $\mathbb{N}_\tau$ and any polynomials $p, p_1, \ldots, p_n$, we have*

$$\overline{v}(p(p_1, \ldots, p_n)) \geq \overline{v}(p).$$

Here, $p(p_1, \ldots, p_n)$ denotes the superposition of the polynomial p with the polynomials $p_1, \ldots, p_n$.

Proof. We will prove the claim by induction on the complexity *opcount*(p) of the polynomial p. If *opcount*$(p) = 0$, then $p = x_j$ for some $j \in \{1, \ldots, n\}$ or $p = \overline{a}$ for some $\overline{a} \in \overline{A}$. If $p = x_j$ for some $j \in \{1, \ldots, n\}$, then $\overline{v}(x_j(p_1, \ldots, p_n)) = \overline{v}(p_j) \geq \overline{v}(x_j)$. If $p = \overline{a}$, then $\overline{v}(\overline{a}(p_1, \ldots, p_n)) = \overline{v}(\overline{a}) \geq \overline{v}(\overline{a})$. If $p = f_i(p'_1, \ldots, p'_{n_i})$ and assume that $\overline{v}(p'_j(p_1, \ldots, p_n)) \geq \overline{v}(p'_j)$ for $1 \leq j \leq n_i$, then

$$\begin{aligned}
\overline{v}(p(p_1, \ldots, p_n)) &= \overline{v}(f_i(p'_1, \ldots, p'_{n_i})(p_1, \ldots, p_n)) \\
&= \overline{v}(f_i(p'_1(p_1, \ldots, p_n), \ldots, p'_{n_i}(p_1, \ldots, p_n))) \\
&= f_i^{\mathbb{N}}(\overline{v}(p'_1(p_1, \ldots, p_n)), \ldots, \overline{v}(p'_{n_i}(p_1, \ldots, p_n))) \\
&\geq f_i^{\mathbb{N}}(\overline{v}(p'_1), \ldots, \overline{v}(p'_{n_i})) \\
&= \overline{v}(f_i(p'_1, \ldots, p'_{n_i})) \\
&= \overline{v}(p).
\end{aligned}$$

□

Since any valuation $\overline{v}$ is a homomorphism from $\underline{P_\tau(X, \overline{A})}$ to $\mathbb{N}_\tau$, the image $\overline{v}(P_\tau(X, \overline{A}))$ is a subalgebra of $\mathbb{N}_\tau$ and $ker(\overline{v})$ is a congruence on $P_\tau(X, \overline{A})$, although not necessary a fully invariant congruence. The next proposition shows when $ker(\overline{v})$ is a fully invariant congruence.

Proposition 2.3 *Let $\overline{v}$ be a valuation of polynomials of type τ over $\overline{A}$ into $\mathbb{N}_\tau$. The kernel of $\overline{v}$ is a fully invariant congruence iff $\overline{v}$ is a constant function on $P_\tau(X, \overline{A})$.*

Proof. Let $ker(\overline{v})$ be a fully invariant congruence. By definition, any two variables x and y assigned the same value are in the same block of $ker(\overline{v})$. Since $ker(\overline{v})$ is a fully invariant congruence, for every endomorphism φ on $P_\tau(X,\overline{A})$, we have $(\varphi(x),\varphi(y)) \in ker(\overline{v})$. But for any two polynomials p_1 and p_2 there is an endomorphism φ which maps x and y to p_1 and p_2, respectively. This shows that every pair (p_1,p_2) of polynomials is in $ker(\overline{v})$. Thus $\overline{v}$ is a constant function on $P_\tau(X,\overline{A})$. Conversely, let $\overline{v}$ be a constant function. Then all polynomials have the same value. So $ker(\overline{v}) = P_\tau(X,\overline{A}) \times P_\tau(X,\overline{A})$ is a fully invariant congruence. □

A valuation of polynomials of type τ over $\overline{A}$ into $I\!N_\tau$ lets us to assign to any polynomial of a given type a complexity value, since polynomial identities are pairs of polynomials, and polynomial varieties of type τ are classes defined by sets of polynomial identities. (For more details on polynomial identities and polynomial varieties, see [1].) So we can measure the complexity of polynomial identities and polynomial varieties of type τ by using our valuation functions. Let $\overline{v}$ be a valuation of polynomials of type τ over $\overline{A}$ into $I\!N_\tau$, and let S be some fixed non-empty subset of $I\!N$. For any set Σ of polynomial identities, we define

$$N_S^E(\Sigma) = \{p_1 \approx p_2 \in \Sigma \mid p_1,p_2 \in P_\tau(X,\overline{A}) \text{ and } \overline{v}(p_1),\overline{v}(p_2) \in S\}.$$

We denote the set of all polynomial equations $p_1 \approx p_2$ of type τ over $\overline{A}$ by $P_\tau(X,\overline{A})^2$ and denote its power set by $\mathcal{P}(P_\tau(X,\overline{A})^2)$. Then we have a mapping $N_S^E : \mathcal{P}(P_\tau(X,\overline{A})^2) \longrightarrow \mathcal{P}(P_\tau(X,\overline{A})^2)$. If Σ is all of $P_\tau(X,\overline{A})^2$, then $N_S^E(P_\tau(X,\overline{A})^2)$ is the set of all polynomial identities when both polynomials have valuation in S. We have the following proposition.

Proposition 2.4 *Let S be a fixed non-empty subset of $I\!N$. Then the mapping N_S^E is a kernel operator on $\mathcal{P}(P_\tau(X,\overline{A})^2)$ which preserves intersections and unions.*

Proof. It is clear that $N_S^E(\Sigma) = \Sigma \cap N_S^E(P_\tau(X,\overline{A})^2) \subseteq \Sigma$. Thus $N_S^E(\Sigma)$ is intensive. If $\Sigma_1 \subseteq \Sigma_2$, then $N_S^E(\Sigma_1) = \Sigma_1 \cap N_S^E(P_\tau(X,\overline{A})^2) \subseteq \Sigma_2 \cap N_S^E(P_\tau(X,\overline{A})^2) = N_S^E(\Sigma_2)$. So N_S^E is isotone. For idempotence, consider

$$N_S^E(N_S^E(\Sigma)) = (\Sigma \cap N_S^E(P_\tau(X,\overline{A})^2) \cap N_S^E(P_\tau(X,\overline{A})^2) = N_S^E(\Sigma).$$

Next, consider

$$\begin{aligned} N_S^E(\bigcap\{\Sigma_i \mid i \in I\}) &= \bigcap\{\Sigma_i \mid i \in I\} \cap N_S^E(P_\tau(X,\overline{A})^2) \\ &= \bigcap\{\Sigma_i \cap N_S^E(P_\tau(X,\overline{A})^2) \mid i \in I\} \\ &= \bigcap\{N_S^E(\Sigma_i) \mid i \in I\}, \end{aligned}$$

and

$$\begin{aligned} N_S^E(\bigcup\{\Sigma_i \mid i \in I\}) &= \bigcup\{\Sigma_i \mid i \in I\} \cap N_S^E(P_\tau(X,\overline{A})^2) \\ &= \bigcup\{\Sigma_i \cap N_S^E(P_\tau(X,\overline{A})^2) \mid i \in I\} \\ &= \bigcup\{N_S^E(\Sigma_i) \mid i \in I\}. \end{aligned}$$

This finishes the proof. □

In general, the set of fixed points of a kernel operator defined on a lattice forms a sublattice of the given lattice. Our operator N_S^E is defined on a complete lattice, the power set $\mathcal{P}(P_\tau(X,\overline{A})^2)$; and since the generalized meet and join operation of this lattice agree with $\bigcap$ and $\bigcup$, so we have the following corollary.

Corollary 2.5 *The set* $\{\Sigma \in \mathcal{P}(P_\tau(X,\overline{A})^2) \mid N_S^E(\Sigma) = \Sigma\}$ *forms a complete lattice of the power set lattice* $\mathcal{P}(P_\tau(X,\overline{A})^2)$. □

Next, we consider the interconnections between the valuations of polynomials of type τ over $\overline{A}$ and polynomial equational theory. We denote the class of all algebras $\underline{B}$ of type τ such that $\underline{B}$ contains a subalgebra $\underline{A}$ which has the same cardinality as $\overline{A}$ by $K_{\overline{A}}$. Algebras from $K_{\overline{A}}$ are called $\overline{A}$- *algebras*. For any subclass $K \subseteq K_{\overline{A}}$ and for any set $\Sigma \subseteq P_\tau(X,\overline{A})^2$, we define

$$PIdK := \{p_1 \approx p_2 \mid p_1, p_2 \in P_\tau(X,\overline{A}) \text{ and for all } \underline{B} \in K \subseteq K_{\overline{A}}(\underline{B} \text{ satisfies } p_1 \approx p_2 \text{ as a polynomial identity})\}$$

and

$$PMod\Sigma := \{\underline{B} \mid \underline{B} \text{ is an } \overline{A}\text{-algebra and for all } p_1 \approx p_2 \in \Sigma(\underline{B} \text{ satisfies } p_1 \approx p_2 \text{ as a polynomial identity})\}.$$

It is shown in [1] that the pair of mappings (PMod, PId) forms a Galois connection and PMod$P$Id and PIdPMod are closure operators. The subclasses $K \subseteq K_{\overline{A}}$ with PMod$P$Id $K = K$ form a complete lattice $P\mathcal{L}_{\overline{A}}(\tau)$ and the subsets $\Sigma \subseteq P_\tau(X, \overline{A})^2$ with PIdPMod $\Sigma = \Sigma$ form a complete lattice $P\mathcal{E}_{\overline{A}}(\tau)$. Both lattices are dually isomorphic. The classes from $P\mathcal{L}_{\overline{A}}(\tau)$ are called *polynomial varieties of type* τ and the elements of $P\mathcal{E}_{\overline{A}}(\tau)$ are called *polynomial equational theories of type* τ. Polynomial equational theories of type τ can be described as sets of polynomial identities closed under a closure operator E which describes the algebraic derivation concept; that is, under finitary applications of the usual five derivation rules for polynomial identities, which are reflexivity, symmetry, transitivity, the replacement rule and the substitution rule. We consider sets of the form $N_S^E(\Sigma)$ where $\Sigma = P\text{Id}\ K$ is closed, and in particular in whether such sets are closed under these five derivation rules. At first, to ensure reflexivity we add all pairs (p, p) of polynomials from the diagonal $\Delta_{P_\tau(X,\overline{A})}$. Thus we will consider the operator DN_S^E, mapping any set Σ of polynomial identities to $N_S^E(\Sigma) \cup \Delta_{P_\tau(X,\overline{A})}$. It is clearly that any set $DN_S^E(\Sigma)$ is closed under the first three derivation rules, that is, such sets form equivalence relations. In the next section we examine the conditions on our valuations which are needed to ensure closure under the replacement rule and the substitution rule.

3. The Subpolynomial Condition

In this section we consider another property of valuations defined by K. Denecke and S. L. Wismath in [5]. We describe this property algebraically, as a condition on the valuation algebra $\mathbb{N}_\tau$, or as a condition on a valuation $\overline{v}$. We will say that $\mathbb{N}_\tau$ satisfies the Algebraic Subpolynomial Condition, (ASPC), if

$$\forall i \in I\, \forall j \in \{1, \ldots, n_i\}\, \forall a_1, \ldots, a_{n_i} \in \mathbb{N}\, (f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \geq a_j).$$

A valuation $\overline{v}$ is said to satisfy the Subpolynomial Condition (SPC) if any subpolynomial p' of a polynomial p has $\overline{v}(p') \leq \overline{v}(p)$.

The following proposition shows that these two properties are connected.

Proposition 3.1 *Let $\overline{v}$ be a valuation of polynomials of type τ over $\overline{A}$ into the algebra $\mathbb{N}_\tau$. If $\mathbb{N}_\tau$ satisfies (ASPC), then v satisfies (SPC).*

Proof. Let $\overline{v}$ be a valuation of polynomials of type τ over $\overline{A}$ into the algebra $\mathbb{N}_\tau$, where the algebra $\mathbb{N}_\tau$ satisfies (ASPC). We prove by induction on the complexity of the polynomial p that any subpolynomial of p has value less than or equal to the value of p. If $p = x \in X$ is a variable, then any subpolynomial of x is equal to x and has the same value as x. If $p = \overline{a} \in \overline{A}$ is a constant, then any subpolynomial of $\overline{a}$ is equal to $\overline{a}$ and has the same value as $\overline{a}$. If $p = f_i(p_1, \ldots, p_{n_i})$ and p' is a subpolynomial of p different from p, then p' is a subpolynomial of p_j for some $j \in \{1, \ldots, n_i\}$. In this case we have $\overline{v}(p) = \overline{v}(f_i(p_1, \ldots, p_{n_i}) = f_i^{\mathbb{N}}(\overline{v}(p_1), \ldots, \overline{v}(p_{n_i})) \geq \overline{v}(p_j)$, by the assumption that $\mathbb{N}_\tau$ satisfies (ASPC). Thus by the induction hypothesis we conclude that $\overline{v}(p_j) \geq \overline{v}(p')$, and by transitivity we get $\overline{v}(p) \geq \overline{v}(p')$. $\square$

Lemma 3.2 *Let $\overline{v}$ be a valuation of polynomials of type τ over $\overline{A}$ into $\mathbb{N}_\tau$. If $\overline{v}$ satisfies (SPC), then for every $i \in I$ and all polynomials $p_1, \ldots, p_{n_i}$ we have*

$$\overline{v}(f_i(p_1, \ldots, p_{n_i})) \geq max\{\overline{v}(p_j) \mid 1 \leq j \leq n_i\}.$$

Proof. Since each polynomial p_j is a subpolynomial of $f_i(p_1, \ldots, p_{n_i})$, then by (SPC) we get $\overline{v}(f_i(p_1, \ldots, p_{n_i})) \geq \overline{v}(p_j), 1 \leq j \leq n_i$. Thus $\overline{v}(f_i(p_1, \ldots, p_{n_i})) \geq max\{\overline{v}(p_j) \mid 1 \leq j \leq n_i\}$. $\square$

Now we will consider that when $DN_S^E(P\text{Id } K)$ is a polynomial equational theory, for K a polynomial variety of type τ. We have seen that these sets are always at least equivalence relations. Let $S_k = \{m \in \mathbb{N} \mid m \geq k\}$, with the corresponding sets $N_{S_k}^E(P\text{Id } K)$ and $DN_{S_k}^E(P\text{Id } K)$.

Proposition 3.3 *Let $\overline{v}$ be a valuation of polynomials of type τ over $\overline{A}$ into $\mathbb{N}_\tau$, let K be a polynomial variety of type τ over $\overline{A}$, and let $k \geq 0$. If $\overline{v}$ satisfies (SPC), then $DN_{S_k}^E(PId\ K)$ is closed under the replacement derivation rule.*

Proof. Let f_i be an n_i-ary operation symbol of type τ, and let $p_1 \approx p_1', \ldots, p_{n_i} \approx p_{n_i}'$ be elements of $DN_{S_k}^E(P\text{Id } K)$. We will show that $f_i(p_1, \ldots, p_{n_i}) \approx f_i(p_1', \ldots, p_{n_i}')$ is also in $DN_{S_k}^E(P\text{Id } K)$. If $p_j = p_j'$ holds for all $1 \leq j \leq n_i$, then $f_i(p_1, \ldots, p_{n_i}) = f_i(p_1', \ldots, p_{n_i}')$. Otherwise, we have at least one value $1 \leq m \leq n_i$ for which $p_m \neq p_m'$, and both $\overline{v}(p_m), \overline{v}(p_m') \geq k$. By the condition (SPC) which follows from (ASPC) we have both $\overline{v}(f_i(p_1, \ldots, p_{n_i})) \geq k$ and $\overline{v}(f_i(p_1', \ldots, p_{n_i}')) \geq k$. Since for all $1 \leq j \leq n_i$, we have $p_j \approx p_j' \in P\text{Id } K$, so $f_i(p_1, \ldots, p_{n_i}) \approx f_i(p_1', \ldots, p_{n_i}') \in P\text{Id } K$. Thus

$f_i(p_1,\ldots,p_{n_i}) \approx f_i(p'_1,\ldots,p'_{n_i})$ is in $DN^E_{S_k}(P\text{Id } K)$. $\square$

Proposition 3.4 *Let $\overline{v}$ be any valuation of polynomials of type τ over $\overline{A}$ into $I\!N_\tau$, let K be a polynomial variety of type τ over $\overline{A}$, and let $k \geq 0$. If $\overline{v}$ satisfies (OC), then $DN^E_{S_k}(PId\, V)$ is closed under the substitution derivation rule.*

Proof. Let $p_1 \approx p_2$ be an element of the set $DN^E_{S_k}(P\text{Id } K)$ with $p_1 \neq p_2$, so that $p_1 \approx p_2 \in P\text{Id } K$ and $\overline{v}(p_1), \overline{v}(p_2) \geq k$. We will show that for any substitution $\varphi : X \cup \overline{A} \longrightarrow P_\tau(X, \overline{A})$, the polynomial identity $\hat{\varphi}(p_1) \approx \hat{\varphi}(p_2)$ is also in $DN^E_{S_k}(P\text{Id } V)$, where $\hat{\varphi}$ is the canonical endomorphism of $\underline{P_\tau(X, \overline{A})}$ extending φ from $X \cup \overline{A}$ to $P_\tau(X, \overline{A})$. Since $P\text{Id } K$ is a polynomial equational theory and closed under such substitutions, we have $\hat{\varphi}(p_1) \approx \hat{\varphi}(p_2) \in P\text{Id } K$. Next, we will prove that $\overline{v}(\hat{\varphi}(p_1)), \overline{v}(\hat{\varphi}(p_2)) \geq k$. It will suffice to prove that, for any polynomial p we have $\overline{v}(\hat{\varphi}(p)) \geq \overline{v}(p)$. We prove this by induction on p. First, if $p = x_j$ is a variable, then $\overline{v}(\hat{\varphi}(p)) \geq \overline{v}(x_j) = \overline{v}(p)$, by the definition of a valuation. If $p = \overline{a}$ is a constant, then $\overline{v}(\hat{\varphi}(p)) \geq \overline{v}(\overline{a}) = \overline{v}(p)$, by the definition of a valuation. Inductively, if $p = f_i(p_1,\ldots,p_{n_i})$ for some polynomials $p_1,\ldots,p_{n_i}$ for which this claim holds, then

$$
\begin{array}{rcl}
\overline{v}(\hat{\varphi}(p)) & = & \overline{v}(\hat{\varphi}(f_i(p_1,\ldots,p_{n_i}))) \\
 & = & \overline{v}(f_i(\hat{\varphi}(p_1),\ldots,\hat{\varphi}(p_{n_i}))) \\
 & \geq & \overline{v}(f_i(p_1,\ldots,p_{n_i})) \\
 & = & \overline{v}(p).
\end{array}
$$

This finishes the proof. $\square$

Corollary 3.5 *If $\overline{v}$ is a valuation which satisfies (OC) and (SPC), then the set $DN^E_{S_k}(PIdK)$ is a polynomial equational theory for any polynomial variety K and any integer $k \geq 0$. In particular, the set $DN^E_{S_k}(\tau)$ is a polynomial equational theory.*

$\square$

The next proposition shows that the set $P_k := \{m \in I\!N \mid m \geq k\} \cap im(\overline{v})$ forms a subalgebra of the image algebra $\overline{v}(P_\tau(X, \overline{A}))$ and has the ideal property, i.e. if f_i is an n_i-ary operation symbol of type τ, we have $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \in P_k$ when any one of $a_1, \ldots, a_{n_i}$ is in P_k.

Proposition 3.6 *Let $\overline{v}$ be a valuation of polynomials of type τ over $\overline{A}$ satisfying (SPC), and let $k \geq 0$. The set $P_k := \{m \in \mathbb{N} \mid m \geq k\} \cap im(\overline{v})$, where $im(\overline{v})$ is the image of $\overline{v}$, forms a subalgebra of the image algebra $\overline{v}(P_\tau(X, \overline{A}))$ and have an ideal property.*

Proof. Let f_i be an n_i-ary operation symbol for $i \in I$. Assume that $a_1, \ldots, a_{n_i}$ are elements of P_k. Then $a_j \geq k$ for every $j \in \{1, \ldots, n_i\}$ and for every $j \in \{1, \ldots, n_i\}$ there is a polynomial p_j such that $a_j = \overline{v}(p_j)$. The condition (SPC) gives $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \geq a_j \geq k$, so that $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \in \{m \in \mathbb{N} \mid m \geq k\}$. Moreover, $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) = f_i^{\mathbb{N}}(\overline{v}(p_1), \ldots, \overline{v}(p_{n_i})) = \overline{v}(f_i(p_1, \ldots, p_{n_i}))$. This shows that $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i})$ is in the image algebra. Therefore $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \in P_k$. A similar argument shows that it is enough, under (SPC), to have even one input $a_j \geq k$. $\square$

The ideal property is strong enough to guarantee closure of $N_S^E(P\text{Id } K)$ under the replacement rule.

Proposition 3.7 *Let $\overline{v}$ be a valuation of polynomials of type τ and let K be a polynomial variety of type τ over $\overline{A}$. If the set S is an ideal of $\mathbb{N}_\tau$, then $DN_S^E(P\text{Id } K)$ is closed under the replacement rule.*

Proof. Let f_i be an n_i-ary operation symbol of type τ, and let $p_1 \approx p'_1, \ldots, p_{n_i} \approx p'_{n_i}$ be elements of $DN_S^E(P\text{Id } K)$. We will show that $f_i(p_1, \ldots, p_{n_i}) \approx f_i(p'_1, \ldots, p'_{n_i})$ is also in $DN_S^E(P\text{Id } K)$. If $p_j = p'_j$ holds for all $1 \leq j \leq n_i$, then $f_i(p_1, \ldots, p_{n_i}) = f_i(p'_1, \ldots, p'_{n_i})$. Otherwise, we have at least one value $1 \leq m \leq n_i$ for which $p_m \neq p'_m$, and both $\overline{v}(p_m), \overline{v}(p'_m) \in S$. By the ideal property of S, we have $f_i^{\mathbb{N}}(\overline{v}(p_1), \ldots, \overline{v}(p_{n_i}))$, $f_i^{\mathbb{N}}(\overline{v}(p'_1), \ldots, \overline{v}(p'_{n_i})) \in S$. Since for all $1 \leq j \leq n_i$ we have $p_j \approx p'_j \in P\text{Id } K$, so $f_i(p_1, \ldots, p_{n_i}) \approx f_i(p'_1, \ldots, p'_{n_i}) \in P\text{Id } K$. This means that $f_i(p_1, \ldots, p_{n_i}) \approx f_i(p'_1, \ldots, p'_{n_i})$ is in $DN_S^E(P\text{Id } K)$. $\square$

Next, suppose that $p \approx p'$ and $\overline{v}(p), \overline{v}(p') \in S$. Let $\varphi : X \cup \overline{A} \longrightarrow P_\tau(X, \overline{A})$ be a substitution of type τ. We proved in Proposition 3.4 that for any valuation $\overline{v}$ satisfying

(OC), we have $\overline{v}(\hat{\varphi}(p)) \geq \overline{v}(p)$ and $\overline{v}(\hat{\varphi}(p')) \geq \overline{v}(p')$, so it is sufficient for S to be closed under the substitution rule.

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