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Characterizations of Augmented Graded Rings

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Abstract

In this paper, we introduce some characterizations for augmented graded rings in special cases.

Key Words: Graded rings, Augmented graded rings, Strongly graded rings.

Introduction

Let G be a group with identity e. A ring R is said to be a G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The G-graded ring R is denoted by (R, G). We denote by $supp(R, G)$ the support of G which is defined to be $\{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g. For $x \in R$, x can be written uniquely as $\sum_{g \in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \bigcup_{g \in G} R_g$.

In this paper, we give some charaterizations for the augmented graded rings for the case where $supp(R, G)$ is a subgroup of G. The general case is left open. One of the charaterizations has a connection with the second strong property.

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1. Preliminaries

In this section, we give some basic facts of graded rings. For more details, one can look in [3, 4, 5].

Lemma 1.1 *Let R be a G-graded ring and $x, y \in R, g \in G$. Then*
1. $(x+y)_g = x_g + y_g$.
2. $(xy)_g = \sum_{h \in G} x_h y_{h^{-1}g}$.

Proposition 1.2 *Let R be a G-graded ring. Then*
1. R_e is a subring of R and $1 \in R_e$.
2. R_g and R are left (resp. right) R_e-modules, for all $g \in G$.

Definition 1.3 *A G-graded ring R is said to be strongly graded if $R_g\ R_h = R_{gh}$ for all $g, h \in G$.*

Proposition 1.4 *Let R be a G-graded ring. Then (R, G) is strong iff $R_g R_{g^{-1}} = R_e$ for all $g \in G$.*

Corollary 1.5 *(R, G) is strong iff $1 \in R_g\ R_{g^{-1}}$ for all $g \in G$.*

Definition 1.6 *Let R be a G-graded ring. Then (R, G) is first strong if $R_g R_{g^{-1}} = R_e$ for all $g \in$ supp(R, G), or equivalently if $1 \in R_g R_{g^{-1}}$ for all $g \in$ supp(R, G).*

Proposition 1.7 *If (R, G) is first strong, then supp(R, G) is a subgroup of G.*

Definition 1.8 *Let R be a G-graded ring. Then (R, G) is said to be second strong if supp(R, G) is a monoid in G and $R_g R_h = R_{gh}$ for all $g, h \in$ supp(R, G).*

Remark 1.9 *Every first strongly graded ring is second strong but the converse is not true in general (see [5]).*

Definition 1.10 *A ring R is said to be an augmented G-graded ring if it satisfies the following conditions:*
1. $R = \bigoplus_{g \in G} R_g$ where R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$ (R is a G-graded ring).
2. If R_e is the identity component of the graduation then $R_e = \bigoplus_{g \in G} R_{e-g}$, where R_{e-g} is an additive subgroup of R_e and $R_{e-g} R_{e-h} \subset R_{e-gh}$ for all $g, h \in G$ (R_e is a G-graded ring).

3. For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h} r_g$. We assume $r_e = 1$.

4. If $g, h \in G$ and r_g, r_h are both non-zero, then $r_g\ r_h = r_{gh}$ and for all $x, y \in R_e$ we have $(xr_g)\ (yr_h) = xyr_{gh}$.

Remark 1.11 *It follows from the last definition that*
1. Condition 3 of the definition implies $R_h = R_e\ r_h$ for all $h \in G$.
2. R_g is a G-graded R_e-module with the usual multiplication on R and with the graduation $R_{g-h} = R_{e-h} r_g$ for all $h \in G$
3. $R_{g-h} R_{g'-h'} \subseteq R_{gg'-hh'}$ for all $g, g', h, h' \in G$. $R_{g-h} R_{g'-h'} = R_{e-h} r_g R_{e-h'} r_{g'}$
If $r_g, r_{g^{-1}}$ are both non-zero then $r_g\ R_e = R_e r_g = R_g$.

Proposition 1.12 *Let R be an augmented G-graded ring such that supp(R, G) is a subgroup of G. Then (R, G) is first strong.*

Proof Let $g \in supp(R, G)$. Then $g^{-1} \in supp(R, G)$, i.e., $R_g \neq 0$ and $R_{g^{-1}} \neq 0$.
Since $R_g = R_e r_g$ and $R_{g^{-1}} = R_e r_{g^{-1}}$ we get $r_g \neq 0$, $r_{g^{-1}} \neq 0$ and hence
$r_g\ r_{g^{-1}} = r_{gg^{-1}} = r_e = 1$. Thus $1 \in R_g\ R_{g^{-1}}$, i. e., R is first strong. □

2. Characterizations of Augmented Graded Rings

In this section, we give characterizations for the augmented graded rings in the case where $supp(R, G)$ is a subgroup of G. The general case is still open.

Lemma 2.1 *Let $f : S \to G$ be a group isomorphism and R be a G-graded ring. Then R is S-graded ring with: $R_s = R_{f(s)}$ for all $s \in S$.*

Proof. Trivial.

Notation 2.2 *Suppose (R, G) is an augmented G-graded ring and $r_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h} r_g$. We let $F = \{r_g : g \in supp(R, G),\ r_g$ is fixed for each $g \in G\}$. It is easy to show $|F| = |supp(R, G)|$.* □

Lemma 2.3 *Let R be an augmented G-graded ring such that supp(R, G) is a subgroup of G. Then F is a multiplicative group with the multiplication of R restricted on F. Furthermore, F is isomorphic to supp(R, G).*

Proof. $F \neq \emptyset$ for $1 = r_e \in F$. Let $g, h \in supp(R, G)$. Then $r_g\ r_h = r_{gh} \in F$ because $gh \in supp(R, G)$. Hence, F is closed under multiplication.

Let $g \in supp(R,G)$ then $g^{-1} \in supp(R,G)$ and hence $r_g\ r_{g^{-1}} = r_{g^{-1}} r_g = r_e = 1$, i.e., r_g has an inverse in F. Namely, $r_g^{-1} = r_{g^{-1}}$. Since F inherits the associativity from R, F is a multiplicative group.

One can show that $f : supp(R,G) \to F$ given by $f(g) = r_g$ is a group isomorphism.□

Lemma 2.4 *Suppose R is an augmented G-graded ring and F given in Notation 2.2 is a multiplicative group. Then $supp(R,G)$ is subgroup of G and hence F is isomorphic to $supp(R,G)$.*

Proof. Suppose (R,G) is augmented and F is a multiplicative group.

Let $g,h \in supp(R,G)$. Then $r_g r_h = r_{gh} \in F$ and hence $gh \in supp(R,G)$. Thus $supp(R,G)$ is a monoid in G. Let $g \in supp(R,G)$. Then $r_g \in F$; So $r_g\ r_h = 1$ for some $r_h \in F$ and $h \in supp(R,G)$. So, $r_{gh} = r_g r_h = 1 = r_e$ and hence $gh = e$ and $h = g^{-1}$. Therefore, $g^{-1} \in supp(R,G)$. By Lemma 2.3, $supp(R,G)$ is isomorphic to F. □

Corollary 2.5 *Let R be an augmented G-graded ring. Then, $supp(R,G)$ is a subgroup of G iff F is a multiplicative group. Moreover, F is isomorphic to $supp(R,G)$.*

Proposition 2.6 *Let R be a G-graded ring such that $supp(R,G)$ is a subgroup of G. Then (R,G) is augmented iff the following conditions hold:*
1. R_e is a G-graded ring by any graduation.
2. For each $g \in supp(R,G)$ there exists $r_g \in R_g$ such that $R_g = R_e\ r_g$.
3. For each $g,h \in supp(R,G)$ we have $r_g r_h = r_{gh}$ and $x r_g = r_g x$ for each $x \in R_e$.

Proof. Suppose (R,G) is augmented then (1), (2) and (3) follow by Remark 1.11.

For the converse, suppose $R_e = \bigoplus_{h\in G} R_{e-h}$.

First, we show that $R_g = \bigoplus_{h\in G} R_{e-h} r_g$ for all $g \in G$. If $g \notin supp(R,G)$ we have $r_g = 0$. One can see that $R_g = \bigoplus_{h\in G} R_{e-h} r_g$. Suppose $g \in supp(R,G)$ and $x \in R_g = R_e r_g \in \sum_{h\in G} R_{e-h} r_g$. Then $x = s r_g$ and $s \in R_e$. Assume that $s = \sum_{i=1}^{n} y_{e-h_i}$ where $y_{e-h_i} \in R_{y_{e-h_i}}$ for $i = 1,2,\cdots,n$. Then, $x = \sum_{i=1}^{n} y_{e-h_i} r_g$. Hence, $R_g = \sum_{g\in G} R_{e-h} r_g$.

Let $x \in R_{e-\alpha} r_g \cap \sum_{h\in G-\{\alpha\}} R_{e-h} r_g$. Then $x = y_{e-\alpha} r_g = \sum_{h\in G-\{\alpha\}} y_{e-h} r_g$ and hence $\{y_{e-h} - \sum_{h\in G-\{\alpha\}} y_{e-h}\} r_g = 0$. Thus, $\{y_{e-\alpha} - \sum_{h\in G-\{\alpha\}} y_{e-h}\} r_g r_{g^{-1}} = 0$ or $\{y_{e-\alpha} -$

$\sum\limits_{h\in G-\{\alpha\}} y_{e-h}\} = 0$ where $r_g r_{g^{-1}} = r_e = 1$. Hence, $y_{e-\alpha} = 0$ and $y_{e-h} = 0$ for all $h \in G - \{\alpha\}$ because R_e is a G-graded ring . Therefore, $x = 0$ and $R_{e-\alpha}r_g \cap \sum\limits_{h\in G-\{\alpha\}} R_{e-h}r_g = 0$ for all $\alpha \in G$. Thus, we conclude that $R_g = \bigoplus\limits_{h\in G} R_{e-h}r_g$.

Second, we claim $(xr_g)(yr_h) = xy\ r_{gh}$ for all $x, y \in R_e$ and $g, h \in supp(R, G)$. Since $r_g r_h = r_{gh}$ for all $g, h \in supp(R, G)$ and $xr_g = r_g x$ for all $x \in R_e$, and $g \in supp(R, G)$, we have $(xr_g)(yr_h) = x(r_g y)r_h = x(yr_g)r_h = xy\ r_g r_h = xyr_{gh}$. □

Proposition 2.7 *Let R be a G-graded ring such that $H = supp(R, G)$ is a subgroup of G. Then (R, G) is augmented iff the following conditions hold*
1. R_e is G-graded ring by any graduation.
2. There exists a multiplicative group $F \subset h(R)$ such that F is isomorphic to $supp(R, G)$, $F \cap R_e = 1$, $R = R_e\ F$ and $ax = xa$ for each $x \in R_e$ and $a \in F$.

Proof. Suppose (R, G) is an augmented G-graded ring. Then condition (1) is clear. Let $R_g = \bigoplus\limits_{h\in G} R_{e-h}r_g$ for some $r_g \in R_g$ and $g \in G$. Fixing this r_g for each $g \in G$, taking $F = \{r_g\colon\ g \in H\}$ and using Lemma 2.3, we have F is a multiplicative group such that $F \subset h(R)$, F isomorphic to $supp(R, G)$, $F \cap R_e = \{1\}$ and $R = \bigoplus\limits_{g\in G} R_e r_g = \bigoplus\limits_{g\in H} R_e r_g = R_e F$.

By Remark 1.11, $r_g\ x = xr_g$ for all $x \in R_e$ and $g \in G$ (or $g \in H$).

Conversely, let $f : H \to F$ be a group isomorphism. We show (R, G) is augmented step by step.

Step1: If $g_1, g_2 \in H$ and $g_1 \neq g_2$ then $\sigma_1 \neq \sigma_2$ where $f(g_i) = R_{\sigma_i}$, for $i = 1, 2$. Otherwise, if $\sigma_1 = \sigma_2$ then $f(g_i) \in R_{\sigma_i}$, for $i = 1, 2$, and one can show that $f(g_i^{-1}) = R_{\sigma_i^{-1}}$. So, $f(g_1^{-1})f(g_2) \in R_{\sigma_1^{-1}} R_{\sigma_1} \subset R_e$ or $f(g_1^{-1}g_2) \in R_e \cap F = \{1\}$. Hence, $f(g_1^{-1}g_2) = 1$ and then $g_1^{-1}g_2 = e$, i.e., $g_1 = g_2$.

Step2: We show $R_\sigma \cap F \neq \emptyset$ for each $\sigma \in H$.

Let $K = \{\sigma \in H : R_\sigma \cap F = \emptyset\}$. Then $R = R_e F = \sum\limits_{g\in H} R_e f(g)$ and $R_e f(g) \subset R_{\sigma_g}$ where $f(g) \in R_{\sigma_g}$. Also, if $g_1 \neq g_2$ then $\sigma_{g_1} \neq \sigma_{g_2}$ and hence $R = \bigoplus\limits_{g\in H} R_e f(g)$. Let $g \in H$ and $m \in R_{\sigma_g}$. Then, $m = \sum\limits_{i=1}^{n} x_i f(g_i)$ where $x_i \in R_e$ and $g_i \in H$ for all $i = 1, \cdots, n$. Thus,

$n = 1$ and $g_1 = g$ with $f(g) \in R_{\sigma_g}$, i.e., $m = x_1 f(g) \in R_e f(g)$. Therefore, $R_{\sigma_g} = R_e f(g)$ and hence $R_{\sigma_g} \cap F \neq \emptyset$. So, $\sigma_g \in H - K$ for all $g \in H$. Since $R = \bigoplus_{g \in H} R_e f(g) = \bigoplus_{g \in H} R_{\sigma_g} \subset \bigoplus_{\sigma \in H-K} R_\sigma \subset \bigoplus_{\sigma \in H} R_\sigma = R$ we have $R = \bigoplus_{\sigma \in H} R_\sigma = \bigoplus_{\sigma \in H-K} R_\sigma$ and hence $\bigoplus_{\sigma \in K} R_\sigma = 0$. Since $K \subset H$, $K = \emptyset$. Therefore, $R_\sigma \cap F \neq \emptyset$ for all $\sigma \in H$.

Step3: Define $\zeta : H \to H$ by $\zeta(g) = \sigma_g$ where $f(g) \in R_{\sigma_g}$. Our aim now is to show that ζ is a group isomorphism.

Cearly ζ is well-defined and monomorphism.

Let $\sigma \in H$. By Step2, $R_\sigma \cap F \neq \emptyset$. Thus there exists $a \in F \cap R_\sigma$. Moreover, $a \in F$ and f is onto imply $a = f(g) \in R_\sigma$ and hence $\sigma = \zeta(g)$ where $g \in H$, i.e., ζ is onto.

By Lemma 2.1, R is an H-graded ring with $R_{(h)} = R_{\zeta(h)}$ for all $h \in H$. Hence, R is G-graded with $R_{(g)} = R_{\zeta(g)}$ if $g \in H$ and $R(g) = 0$ if $g \notin H$.

Let $R = \bigoplus_{g \in G} R_{(g)}$. Then, by Proposition 2.6, $R = \bigoplus_{g \in G} R_{(g)}$ is an augmented G-graded ring. □

Remark 2.8 *Let G be an abelian multiplicative group and $\{H_\alpha : \alpha \in \Delta\}$ be a family of subgroups of G. We write $G = \bigotimes_{\alpha \in \Delta} H_\alpha$ if for each $g \in G$, $g = \prod_{\alpha \in \Delta} g_\alpha$ where $g_\alpha \in H_\alpha$ and $g_\alpha = e$ for all $\alpha \in \Delta$ except finitely many and if $H_\beta \cap \left(\bigcap_{\alpha \in \Delta - \{\beta\}} H_\alpha \right) = e$ where e is the identity of G, for all $\beta \in \Delta$. Indeed, this is the internal direct product of the multiplicative subgroups of G.*

If $g \in G$. Then g has a unique decomposition of the form $g = \prod_{\alpha \in \Delta} g_\alpha$.

Proposition 2.9 *Let R be a G-graded ring such that $supp(R, G)$ is an abelian subgroup of G. Suppose $supp(R, G) = \bigotimes_{\alpha \in \Delta} < g_\alpha >$, where $g_\alpha \in supp(R, G)$ and $< g_\alpha >$ is the cyclic group generated by g_α for all $\alpha \in \Delta$, and $xy = yx$ for each $x, y \in h(R) - R_e$. Then (R, G) is augmented iff the following conditions hold*

1. R_e is G-graded ring with any graduation.

2. (R, G) is second strong.

3. R_{g_α} is isomorphic to R_e as a left and right R_e-module for all $\alpha \in \Delta$. In the case $g_\alpha = e$ for some $\alpha \in \Delta$ we suppose R_e isomorphic to itself by the identity isomorphism.

Proof. Suppose (R,G) is augmented graded ring. Since *supp*(R,G) is subgroup of G it follows by Proposition 1.12, (R,G) is second strong. By Remark 1.11, $xr_g = r_g\, x$ for all $x \in R_e$ and $g \in G$ where $r_g \in R_g$ with $R_g = \bigoplus_{h\in G} R_{e-h}r_g$. Also, $R_g = R_e\, r_g$ for all $g \in G$ and $r_g \neq 0$ iff $g \in$ *supp*(R,G).

Define $f : R_e \to R_g = R_eR_g$ by $f(x) = xr_g$ for each $x \in R_e$. Then clearly f is well-defined and f is R_e-module isomorphism for all $g \in$ *supp*(R,G) and hence for g_α where $\alpha \in \Delta$.

For the converse, assume that conditions (1), (2) and (3) hold. Since *supp*(R,G) is a subgroup of G and (R,G) is second strong then (R,G) is first strong. Let f_α : $R_e \to R_{g_\alpha}$ be an R_e-module isomorphism. For each $x \in R_e$, $f_\alpha(x) = xf_\alpha(1) = f_\alpha(1)x$. Let $r_{g_\alpha} = f_\alpha(1)$ for each $\alpha \in \Delta$. Then for each $x_{g_\alpha} \in R_{g_\alpha}$ there exists $x \in R_e$ such that $x_{g_\alpha} = f_\alpha(x) = xr_\alpha = r_\alpha x$, and x is unique because f_α is 1-1. Hence $R_{g_\alpha} = R_e r_{g_\alpha} = r_{g_\alpha} R_e$ for all $\alpha \in \Delta$.

Since R is first strong, $R_{g_\alpha} R_{g_\alpha^{-1}} = R_{g_\alpha^{-1}} R_{g_\alpha} = R_e$ for all $\alpha \in \Delta$. So, $(r_{g_\alpha} R_e)R_{g_\alpha^{-1}} = R_{g_\alpha^{-1}}(R_e r_{g_\alpha})$ or $r_{g_\alpha}(R_e R_{g_\alpha^{-1}}) = (R_{g_\alpha^{-1}} R_e)r_{g_\alpha}$ and hence we obtain $r_{g_\alpha} R_{g_\alpha^{-1}} = R_{g_\alpha^{-1}} r_{g_\alpha} = R_e$. Therefore there exist $x, y \in R_{g_\alpha^{-1}}$ such that $1 = xr_{g_\alpha} = r_{g_\alpha} y$. Clearly, $xr_{g_\alpha}, r_{g_\alpha} y \in R_e$. Thus $f_\alpha(xr_{g_\alpha}) = f_\alpha(r_{g_\alpha} y)$ and hence $r_{g_\alpha}(xr_{g_\alpha}) = (r_{g_\alpha} y)r_{g_\alpha}$. Multiplying both sides by x from the left to get $(xr_{g_\alpha})(xr_{g_\alpha}) = (xr_{g_\alpha})(yr_{g_\alpha})$ which gives $xr_{g_\alpha} = yr_{g_\alpha}$. Multiplying both sides from the right by y to get $x(r_{g_\alpha} y) = y(r_{g_\alpha} y)$ and so $x = y$. Thus, $xr_g = r_{g_\alpha} x = 1$, i.e., r_{g_α} is a unit in R, for each $\alpha \in \Delta$. Since $R_{g_\alpha} R_{g_\alpha^{-1}} = R_e$, $r_{g_\alpha} R_{g_\alpha^{-1}} = R_e$ and hence $R_{g_\alpha^{-1}} = r_{g_\alpha}^{-1} R_e$.

Similarly, $R_{g_\alpha^{-1}} = R_e r_{g_\alpha}^{-1}$ for each $\alpha \in \Delta$. We define $r_{g_\alpha^{-1}} = r_{g_\alpha}^{-1}$; $\alpha \in \Delta$. Thus $R_e = R_{g_\alpha} R_{g_\alpha^{-1}} = R_e r_{g_\alpha} r_{g_\alpha^{-1}}$. We let $r_e = r_{g_\alpha} r_{g_\alpha^{-1}} = 1$.

If $\alpha \in \Delta$ and $n \in N$ then $R_{g_\alpha^n} = R_{g_\alpha} \cdots R_{g_\alpha}$ (n-times) and hence $R_{g_\alpha^n} = (R_e r_{g_\alpha}) \cdot \cdot (R_e r_{g_\alpha})$ (n-times) which gives $R_{g_\alpha^n} = R_e r_{g_\alpha} \cdots r_{g_\alpha}$ where r_{g_α} is product with itself n-times. We define $r_{g_\alpha^n} = r_{g_\alpha}^n$.

If $n \in Z - (N \cup \{0\})$, i.e., $n < 0$ we have $R_{g_\alpha^n} = R_{(g_\alpha^{-1})^n} = R_{g_\alpha^{-1}} \cdots R_{g_\alpha^{-1}}$ ($|n|$-times) and hence $R_{g_\alpha^n} = (R_e r_{g^{-1}}) \cdots (R_e r_{g^{-1}})$ ($|n|$-times) and so $R_{g_\alpha^n} = R_e r_{g_\alpha^{-1}} \cdots r_{g_\alpha^{-1}} = R_e r^{|n|}_{g_\alpha^{-1}}$. We define $r_{g_\alpha^n} = r^{|n|}_{g_\alpha^{-1}} = (r_{g_\alpha}^{-1})^{|n|} = (r_{g_\alpha})^{-|n|} = r_{g_\alpha}^n$ for all $\alpha \in \Delta$.

Therefore, for any $\alpha \in \Delta$ and $n \in Z$ we define $r_{g_\alpha^n} = r_{g_\alpha}^n$ and hence $R_{g_\alpha^n} = R_e r_{g_\alpha}^n$. Similarly, we can show that $R_{g_\alpha^n} = r_{g_\alpha^n} R_e$, for all $n \in Z$ and $\alpha \in \Delta$.

Now, let $h \in supp(R,G)$. By Remark 2.8, h can be written uniquely as $h = \prod_{\alpha\in\Delta} g_\alpha^{n_\alpha}$ where $n_\alpha \in Z$ and $g_\alpha^{n_\alpha} = e$ for all α except finitely many.

Without loss of generality, suppose $h = g_{\alpha_1}^{n_1} \cdots g_{\alpha_m}^{n_m}$. Then $R_h = R_{g_{\alpha_1}^{n_1}} \cdots R_{g_{\alpha_m}^{n_m}} = (R_e r_{g_{\alpha_1}^{n_1}}) \cdots (R_e r_{g_{\alpha_m}^{n_m}}) = R_e r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}} = r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}} R_e$. We define $r_h = r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}} = r_{g_{\alpha_1}}^{n_1} \cdots r_{g_{\alpha_m}}^{n_m}$. Since $g_\alpha^{n_\alpha} = e$ for each $\alpha \notin \{1, \cdots, m\}$ we have $r_{g_\alpha^{n_\alpha}} = r_e = 1$. So, it is possible to write $r_h = \prod_{\alpha\in\Delta} r_{g_\alpha^{n_\alpha}}$. Clearly, $R_h = R_e r_h$ and similarly $R_h = r_h R_e$.

Let $g, h \in supp(R,G)$. Then $g = \prod_{\alpha\in\Delta} g_\alpha^{n_\alpha}$ and $h = \prod_{\alpha\in\Delta} g_\alpha^{m_\alpha}$ and hence $gh = \prod_{\alpha\in\Delta} g_\alpha^{n_\alpha+m_\alpha} = \prod_{\alpha\in\Delta} g_\alpha^{m_\alpha+n_\alpha} = hg$.

But $r_{gh} = \prod_{\alpha\in\Delta} r_{g_\alpha^{n_\alpha+m_\alpha}} = \prod_{\alpha\in\Delta} r_{g_\alpha}^{n_\alpha+m_\alpha} = \prod_{\alpha\in\Delta} r_{g_\alpha}^{m_\alpha+n_\alpha} = r_{hg}$ implies $r_{gh} = r_{hg}$ for all $g, h \in supp(R,G)$. Moreover, if $g, h \in supp(R,G)$ such that $g = \prod_{\alpha\in\Delta} g_\alpha^{n_\alpha}$ and $h = \prod_{\alpha\in\Delta} g_\alpha^{m_\alpha}$ then $r_g r_h = \left(\prod_{\alpha\in\Delta} r_{g_\alpha}^{n_\alpha}\right)\left(\prod_{\alpha\in\Delta} r_{h_\alpha}^{m_\alpha}\right) = \prod_{\alpha\in\Delta} r_{g_\alpha}^{n_\alpha} r_{g_\alpha}^{m_\alpha} = \prod_{\alpha\in\Delta} r_{g_\alpha}^{n_\alpha+m_\alpha} = r_{gh}$ because the homogeneous elements commute, i.e., $r_g r_h = r_{gh}$ for all $g, h \in supp(R,G)$.
$(*)$ If $g \notin supp(R,G)$ we let $r_g = 0$. Then we have the following

1. R_e is G-graded with any graduation.
2. For each $g \in G$ there exists $r_g \in R_g$ such that $R_g = R_e\, r_g$.
3. For all $g, h \in supp(R,G)$ and noticing $(*)$ we have $r_g r_h = r_{gh}$.

If $x \in R_e$ we have $x r_{g_\alpha} = r_{g_\alpha} x$ for all $\alpha \in \Delta$ and hence $r_{g_\alpha}^{-1} x = x r_{g_\alpha}^{-1}$, i. e., $r_{g_\alpha^{-1}} x = x r_{g_\alpha^{-1}}; \alpha \in \Delta$. If $n \in Z$, $x r_{g_\alpha^n} = x r_{g_\alpha}^n = r_{g_\alpha}^n x = r_{g_\alpha^n} x$ follows by associativity of R, for $\alpha \in \Delta$.

Let $h \in supp(R,G)$. Then $h = \prod_{\alpha\in\Delta} g_\alpha^{n_\alpha}$ and $r_h = \prod_{\alpha\in\Delta} r_{g_\alpha}^{n_\alpha}$. Without loss of generality, suppose $h = g_{\alpha_1}^{n_1} \cdots g_{\alpha_m}^{n_m}$ and $g_\alpha^{n_\alpha} = e$ for all $\alpha \notin \{\alpha_1, \cdots, \alpha_m\}$. Then $x r_h = x r_{g_{\alpha_1}}^{n_1} \cdots r_{g_{\alpha_m}}^{n_m} = r_{g_{\alpha_1}}^{n_1} x r_{g_{\alpha_2}}^{n_2} \cdots r_{g_{\alpha_m}}^{n_m} = r_{g_{\alpha_1}}^{n_1} r_{g_{\alpha_2}}^{n_2} \cdots x r_{g_{\alpha_m}}^{n_m} = r_{g_{\alpha_1}}^{n_1} r_{g_{\alpha_2}}^{n_2} \cdots r_{g_{\alpha_m}}^{n_m} x = r_h x$. Therefore, $x r_h = r_h x$ for all $x \in R_e$ and $h \in supp(R,G)$. If $h \notin supp(R,G)$ then $r_h = 0$ and clearly $x r_h = r_h x$.

Therefore, by Proposition 2.6, (R,G) is augmented graded ring. □

Corollary 2.10 *Suppose (R,G) is commutative ring such that $supp(R,G)$ is an abelian subgroup of G. Suppose $supp(R,G) = \bigotimes_{\alpha\in\Delta} < g_\alpha >$, where $g_\alpha \in supp(R,G)$ and $< g_\alpha >$*

is the cyclic group generated by g_α *for all* $\alpha \in \Delta$*.Then* (R, G) *is augmented ring iff the following conditions hold*

1. R_e *is* G*-graded ring with any graduation.*

2. (R, G) *is second strong.*

3. R_{g_α} *is isomorphic to* R_e *as an* R_e*-module for all* $\alpha \in \Delta$*.*

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