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Simplex Codes Over the Ring $\sum_{n=0}^{s} u^n F_2$

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Abstract

In this paper, we introduce simplex linear codes over the ring $\sum_{n=0}^{n=s} u^n F_2$ of types α and β, where $u^{s+1} = 0$. And we determine their properties. These codes are an extension and generalization of simplex codes over the ring Z_{2^s}.

Key Words: Simplex codes, chain rings, Z_{p^s}-codes and $\sum_{n=0}^{n=s} u^n F_2$ -linear codes.

1. Introduction

Recently, there has been much interest in codes over finite rings, for example chain rings Z_{2^k}, where Z_{2^k} denotes the ring of integers modulo 2^k. In particular, codes over ring $F_2 + uF_2$ have been widely studied in [2] [4],[5], [9], [8], [10]. More recently in [3], Z_4-simplex codes and their Gray images, have been studied by Bhandari, Gupta and Lal. By following the same instruments, in [1] simplex codes over $F_2 + uF_2$ are studied.
In this paper we describe linear simplex codes and there properties over the chain ring $R = F_2 + uF_2 + u^2F_2 = F_2(u)/(u^3)$. These codes are extensions and generalizations of simplex codes over the ring Z_{2^k} which were studied by Bhandari, Gupta and Lal in [11].

1.1. The ring R

The ring R is introduced in [15], $R = F_2 + uF_2 + u^2F_2$ is a commutative chain ring of 8 elements which are $\{0, 1, u, u^2, v, v^2, uv, v^3\}$, where $u^3 = 0, v = 1 + u, v^2 = 1 + u^2, v^3 =$

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$1 + u + u^2, uv = u + u^2$. The elements of R are the polynomials over F_2 modulo the ideal (u^3) of $F_2[u]$, where F_2 is the binary field $\{0, 1\}$. Addition and multiplication operations over R are given in the following tables:

Table.

$+$	0	1	u	v	u^2	uv	v^2	v^3
0	0	1	u	v	u^2	uv	v^2	v^3
1	1	0	v	u	v^2	v^3	u^2	uv
u	u	v	0	1	uv	u^2	v^3	v^2
v	v	u	1	0	v^3	v^2	uv	u^2
u^2	u^2	v^2	uv	v^3	0	u	1	v
uv	uv	v^3	u^2	v^2	u	0	v	1
v^2	v^2	u^2	v^3	uv	1	v	0	u
v^3	v^3	uv	v^2	u^2	v	1	u	0

.	0	1	u	v	u^2	uv	v^2	v^3
0	0	0	0	0	0	0	0	0
1	0	1	u	v	u^2	uv	v^2	v^3
u	0	u	u^2	uv	0	u^2	u	uv
v	0	v	uv	v^2	u^2	u	v^3	1
u^2	0	u^2	0	u^2	0	0	u^2	u^2
uv	0	uv	u^2	u	0	u^2	uv	u
v^2	0	v^2	u	v^3	u^2	uv	1	v
v^3	0	v^2	uv	1	u^2	u	v	v^2

The ring R is a commutative chain ring with maximal ideal $uR = \{0, u, u^2, uv\}$. Since u is nilpotent with nilpotent index 3, we have

$$R \supset (uR) \supset (u^2R) \supset (u^3R) = 0. \tag{1.1}$$

Observe that $R/uR \cong F_2$, and

$$|u^iR| = 2|(u^{i+1}R)| = 2^{3-i}, \; i = 0, 1, 2. \tag{1.2}$$

This follows from the fact that $u^iR/u^{i+1}R$ is an R/uR vector space.
As R is a chain ring as in (1.1), every module M over R admits a decreasing filtration

$$M \supset uM \subset u^2M \supset u^3M = 0, \tag{1.3}$$

as well as a direct sum decomposition

$$M \cong (R/uR)^{l_1} \oplus (R/u^2R)^{l_2} \oplus (R/u^3R)^{l_3} \cong (u^2R)^{l_1} \oplus (uR)^{l_2} \oplus (R)^{l_3}. \tag{1.4}$$

For more details about (1.1)–(1.4) see [13] and [17].
A linear code $\mathcal{C}$ of length n over the ring R is an R- submodule of R^n. An element of $\mathcal{C}$ is called a codeword of $\mathcal{C}$ and a generator matrix of $\mathcal{C}$ is a matrix whose rows generates $\mathcal{C}$. Following [15] we use the following terminology. The Hamming weight of a codeword

x in R^n is the number of non-zero components. The Lee weight a_r of an element r of the ring R is given by the following equations:

$$a_r = \begin{cases} 0 & \text{if } \ r = 0 \\ 1 & \text{if } \ r = 1, \ \text{or } v^2 \\ 2 & \text{if } \ r = u \text{ or } uv \\ 3 & \text{if } r = v \text{ or } v^3 \\ 4 & \text{if } r = u^2. \end{cases}$$

Then the Lee weight of an element $x = (x_1, x_2, ...x_n)$ of R^n is

$$wt_L(x) = \sum_{i=1}^{n} a_r. \tag{1.5}$$

This definition is analogous to the definition of the Lee weight of the elements of the ring Z_8, where $a_0 = 0,\ a_1 = a_7 = 1,\ a_2 = a_6 = 2,\ a_3 = a_5 = 3,\ a_4 = 4$.

Example 1.1 *Let* $x = (1, 0, 0, u, v, v^2, u^2, uv)$; *then* $wt_L(x) = 13$.

For $\mathbf{x} = (x_1, x_2, \cdots, x_n),\ \mathbf{y} = (y_1, y_2, \cdots, y_n) \in (R)^n,\ d_H(\mathbf{x}, \mathbf{y}) = |\{i : x_i \neq y_i\}|$ is called the Hamming distance between $\mathbf{x}$ and $\mathbf{y}$ and the minimum Hamming distance of $\mathcal{C}$ is denoted by d_H. The Lee distance between $\mathbf{x}$ and $\mathbf{y} \in (R)^n$ is denoted $d_L(\mathbf{x}, \mathbf{y}) = wt_L(\mathbf{x} - \mathbf{y})$. The minimum Lee distance d_L of a code $\mathcal{C}$ is defined analogously.

1.1.1. Generator matrices

For $k > 0, I_k$ denote the $k \times k$ identity matrix.

Definition 1.1 *(Generator Matrix) Let $\mathcal{C}$ be a code over R. A matrix G is called a generator matrix for $\mathcal{C}$ if the rows of G spans $\mathcal{C}$ and none of them can be written as a linear combination of the other rows of G. In [6] Sloane and Calderbank have defined the generator matrix for codes over Z_{p^s}. In [13] G. Norton And A. Salagean have defined the generator matrix over a finite chain rings. By the same theme we define the standard form of the generator matrix for code $\mathcal{C}$ over R as*

$$G = \begin{pmatrix} I_{k0} & A_{01} & A_{02} & A_{03} \\ \mathbf{0} & uI_{k1} & uA_{12} & uA_{13} \\ \mathbf{0} & \mathbf{0} & u^2 I_{k2} & u^2 A_{23} \end{pmatrix},$$

where A_{ij} are matrices over R and the columns are grouped into blocks of sizes k_i, where $0 \leq i < 3$. Let $k = \sum_{i=0}^{2}(3-i)k_i$. Then $|\mathcal{C}| = 2^k$. The code $\mathcal{C}$ is called free module if and only if $k_i = 0$ for all $i = 0, 1, 2$.

Remark 1.1 The presence of zero divisors in R creates a problem in finding the linear dependence of vectors in R^n. Consequently, defining the dimension of a module as a cardinality of its basis is not meaningful. Recently in [16] Vazirani, Saran and Sundar Rajan have introduced the notion of p-dimension for finitely generated modules over Z_{p^s}. As a consequence we define the 2-dimension for a code C over R in the following. A subset B of C is a 2-basis for the linear code C over R if B is 2-linearly independent and C is the 2-span of B. The number of vectors in any 2-basis for C is called 2-dimension of C, denoted 2-dim(C).

1.2. The Generalized Gray map

In [7] C. Carlet has defined a generalized gray map ϕ_{GL} form Z_{2^s} to Z_2^{2s-1} and has obtained the Z_{2^s} version of some binary codes and it is also shown that any Z_{2^s}-linear code is distance invariant under this map. In [11] it was shown that this map need not be linear. In [4] a linear isometry Gray map ϕ from the chain ring $F_2 + uF_2$ to F_2^2 was obtained and it was extended from

$$((F_2 + uF_2)^n, \text{ Lee distance}) \text{ to } (F_2^{2n}, \text{ Hamming distance}),$$

(see [4]).
In this paper we extend this result and define a generalized linear gray map ϕ_{GL} from $R = F_2 + uF_2 + u^2F_2$ to F_2^4 and we will extend it from

$$R^n \longrightarrow \text{to } F_2^{4n}$$

by applying ϕ_{GL} to each coordinate as follows:
For any element of R expressed as $x + uy + u^2z$, we let

$$\phi_{GL}(x + uy + u^2z) = (z, x+z, y+z, x+y+z), \text{where } x, y \text{ and } z \in F_2,$$

we extend this to vectors over R,

$$\Phi_{GL}(\mathbf{x} + u\mathbf{y} + u^2\mathbf{z}) = (\mathbf{z}, \mathbf{x}+\mathbf{z}, \mathbf{y}+\mathbf{z}, \mathbf{x}+\mathbf{y}+\mathbf{z}),$$

where $\mathbf{x},\ \mathbf{y},\ \mathbf{z} \in F_2^n$ and $(\mathbf{x}+u\mathbf{y}+u^2\mathbf{z}) \in R^n$. From the definition of this map we define the generalized Lee weight of any non-zero element $t \in R$ by

$$wt_{GL}(t) = wt_H(\phi_{GL}(t)) = \begin{cases} 2 & \text{if } t \neq u^2, \\ 4 & \text{if } t = u^2. \end{cases}$$

We also have the following matrix which gives the generalized Lee weight to each non-zero element of R

$$G_s = \begin{bmatrix} \phi_{GL}(1) \\ \phi_{GL}(u) \\ \phi_{GL}(v) \\ \phi_{Gl}(u^2) \\ \phi_{Gl}(v^2) \\ \phi_{GL}(uv) \\ \phi_{GL}(v^3) \end{bmatrix} = \begin{bmatrix} 0101 \\ 0011 \\ 0110 \\ 1111 \\ 1010 \\ 1100 \\ 1001 \end{bmatrix}.$$

The generalized gray map can be extended to $(R)^n$ by applying Φ_{GL} to its components. Note that this map is distance-preserving from
$((R)^n,$ Generalized Lee distance) to $((F_2)^{4n},$ Hamming distance).

Remark 1.2 From the definition of the generalized Gray map and the generalized Lee weights for the elements in the ring R, we extend the results that were given by Bonnecaze and Udaya in [4] to the ring $F_2 + uF_2$ and we have the following lemma.

Lemma 1.1 *If $\mathcal{C}$ is a linear code over R, so $\Phi_{GL}(\mathcal{C})$ is a linear binary code and the minimum Generalized Lee weight of $\mathcal{C}$ is the same as the minimum Hamming weight of $\Phi_{GL}(\mathcal{C})$.*

Proof. Let $t = x + uy + u^2z,\ \ t' = x' + uy' + u^2z' \in R$, then

$$t + t' = x + x' + u(y + y') + u^2(z + z')$$

and

$$\phi_{GL}(t+t') = (z+z', (x+x')+(z+z'), (y+y')+(z+z'), (x+x')+(y+y')+(z+z'))$$

$$= (z, x+z, y+z, x+y+z) + (z', x'+z', y'+z', x'+y'+z') = \phi_{GL}(t) + \phi_{GL}(t'),$$

So Φ_{GL} is a linear map.
Now let $\mathbf{c_i} = \mathbf{x_i} + \mathbf{uy_1} + \mathbf{u^2 z_i} \in \mathcal{C}$, where $i = 1, 2$.
And let $\mathbf{w_i} = \Phi_{\mathbf{GL}}(\mathbf{c_i})$, $i = 1, 2$.
Then

$$\Phi_{GL}(\mathbf{c_2} + \mathbf{c_2}) = (\mathbf{z_1} + \mathbf{z_2}, (\mathbf{x_1} + \mathbf{x_2}) + (\mathbf{z_1} + \mathbf{z_2}), (\mathbf{y_1} + \mathbf{y_2}) + (\mathbf{z_1} + \mathbf{z_2}), (\mathbf{x_1} + \mathbf{x_2}) + (\mathbf{y_1} + \mathbf{y_2}) + (\mathbf{z_1} + \mathbf{z_2}))$$

$$= (\mathbf{z_1}, \mathbf{x_1} + \mathbf{z_1}, \mathbf{y_1} + \mathbf{z_1}, \mathbf{x_1} + \mathbf{y_1} + \mathbf{z_1}) + (\mathbf{z_2}, \mathbf{x_2} + \mathbf{z_2}, \mathbf{y_2} + \mathbf{z_2}, \mathbf{x_2} + \mathbf{y_2} + \mathbf{z_2}) = \Phi_{\mathbf{GL}}(\mathbf{c_1}) + \Phi_{\mathbf{GL}}(\mathbf{c_2}) = \mathbf{w_1} + \mathbf{w_2}.$$

This implies that $\Phi_{GL}(\mathcal{C})$ is linear binary code over F_2.
Since the generalized Gray map Φ_{GL} is an isometry from

$$(R^n,\ GL) \text{ to } (F_2^{4n}, \text{Hamming distance}),$$

and from definition of Gray map,

$$wt_H(\Phi_{GL}(\mathbf{c})) = wt_{GL}(\mathbf{c}), \quad \mathbf{c} \in \mathcal{C},$$

$$d_H(\Phi_{GL}(\mathbf{c_1}), \Phi_{GL}(\mathbf{c_2})) = d_{GL}(\mathbf{c_1}, \mathbf{c_2}), \quad \mathbf{c_1}, \mathbf{c_2} \in \mathcal{C},$$

then the minimum generalized Lee weight of $\mathcal{C}$ is the same as the minimum Hamming weight of $\Phi_{GL}(\mathcal{C})$.
So the last assertion holds. □

Lemma 1.2 *Let $\mathcal{C}$ and $\mathcal{C}'$ be equivalent codes over R. Then $\phi(\mathcal{C})$ and $\phi(\mathcal{C}')$ are equivalent codes over F_2.*

A linear code over R of length n, 2-dimension k, minimum Hamming distance d_H, minimum Lee distance d_L and Generalized Lee distance d_{GL} is called an $[n, k, d_H, d_L, d_{GL}]$ code, or simply an $[n, k]$ code. The binary image under the Generalized Gray map $\Phi(\mathcal{C})$ of a code $\mathcal{C}$ over R is a linear code over F_2 of length $4n$, dimension k and minimum Hamming distance d_{GL}. Hence by the Griesmer bound for binary codes [12], we have

$$n \geq \lceil \frac{1}{4} \sum_{i=0}^{k-1} \lceil \frac{d_{GL}}{2^i} \rceil \rceil.$$

In [14] Rains has proved that for a linear code over Z_4, $d_H \geq \lceil \frac{d_L}{2} \rceil$. Also, the same result holds for codes over the ring $F_2 + uF_2$ [1]. The following corollary generalizes it for the ring R.

Corollary 1.3 *Let $\mathcal{C}$ be a linear code over R, then*

$$d_H \geq \lceil \frac{d_L}{4} \rceil, \text{ and } d_H \geq \lceil \frac{d_{GL}}{4} \rceil.$$

A linear code over $\mathcal{C}$ over R is said to be of type $\alpha(\beta)$ if

$$d_H = \lceil \frac{d_{GL}}{4} \rceil (d_H > \lceil \frac{d_{GL}}{4} \rceil).$$

Definition 1.2 *[11] For each $1 \leq i \leq n$, let $A_H(i)(A_L(i)$ or $A_{GL}(i))$ be the number of codewords of Hamming (Lee) or generalized Lee weight i in C. Then $\{A_H(0), A_H(1), ...A_H(n)\}, (\{A_L(0), A_L(1), ...A_L(n)\})$ or $(\{A_{GL}(0), A_{GL}(1), ...A_{GL}(n)\})$ is called the Hamming (Lee) or Generalized Lee weight distribution of C.*

2. R-Simplex Codes

In this section we will study the simplex codes of type α and β over R and also we study the properties of their images under the Generalized Gray map.
Let G_k be a $k \times 2^{3k}$ matrix over R defined inductively by

$$G_1 = [0, 1, u, v, u^2, uv, v^2, v^3],$$

$$G_k = \left[\begin{array}{c|c|c|c|c} 00...0 & 11...1 & \text{uu...u} & ... & v^3v^3...v^3 \\ \hline G_1 & G_1 & G_1 & ... & G_1 \end{array} \right]; k \geq 2. \tag{2.1}$$

Note that the columns of G_k consist of all distinct k-tuples over R. The code S_k^α generated by R has length 8^k and 2-dimension $3k$.

Remark 2.1 If A_{k-1} denotes an array of codewords in S_{k-1}^α and if $\mathbf{i}$=$(i, i, i, ..., i)$, then an array of all codewords of S_k^α is given by

$$\begin{bmatrix} A_{k-1} & A_{k-1} & A_{k-1} & \dots & A_{k-1} \\ A_{k-1} & \mathbf{1} + A_{k-1} & \mathbf{u} + A_{k-1} & \dots & \mathbf{v^3} + A_{k-1} \\ A_{k-1} & \mathbf{u} + A_{k-1} & \mathbf{u^2} + A_{k-1} & \dots & \mathbf{v^2} + A_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k-1} & \mathbf{v^3} + A_{k-1} & \mathbf{v^2} + A_{k-1} & \dots & \mathbf{1} + A_{k-1} \end{bmatrix}.$$

Remark 2.2 If $R_1, R_2, ...R_k$ denote the rows of the matrix G_k then $w_H(R_i) = 2^{3k} - 2^{3(k-1)}$, $w_H(u^2R_i) = 2^{3k-1}$,$w_L(R_i) = 2^{3(k+1)-2}$, and $w_{GL}(R_i) = 2^{3(k+1)-2}$.

For each $m,\ 0 \leq m \leq 3$, let $S_0 = \{0\}, S_1 = \{0, u^2\}, S_2 = \{0, u, u^2, uv\}$,$S_3 = R$. Note that S_2 is the set of all zero divisors of R. A codeword $\mathbf{c} = (c_1, c_2, c_3, ...c_n) \in S_k^\alpha$ is said to be of type m if all of its components belong to the set S_m. From the observation of G_k, we have that each element of R occurs equally in every row of G_k, for this we have the following lemma.

Lemma 2.1 *Let* $\mathbf{c} \in S_k^\alpha$ *be a type* m *codeword. Then all the components of* $\mathbf{c}$ *will occur equally often* 2^{3k-m} *times.*

Proof. By remark (2.1) , since for any $x \in S_{k-1}^\alpha$ we have the following codewords of S_k^α :

$y_1 = \Big(\text{ x } \Big| \text{ x } \Big| \text{ x } \Big| \ \dots \ \Big| \text{ x } \Big)$, $\quad y_2 = \Big(\text{ x } \Big| \ \mathbf{1}\text{+x } \Big| \ \mathbf{u}\text{+ x } \Big| \ \dots \ \Big| \ \mathbf{v^3}\text{+x } \Big)$,

$y_3 = \Big(\text{ x } \Big| \ \mathbf{u}\text{+x } \Big| \ \mathbf{u^2}\text{+ x } \Big| \ \dots \ \Big| \ \mathbf{uv}\text{+ x } \Big)$,....,and

$y_8 = \Big(\text{ x } \Big| \ \mathbf{v^3}\text{+x } \Big| \ \mathbf{uv}\text{+ x } \Big| \ \dots \ \Big| \ \mathbf{1}\text{+x } \Big)$.

The result holds by induction on k and by remark (2.1). □

To determine weight distribution of S_k^α we need to determine the number of codewords of type m in S_k^α for $1 \leq m \leq 3$. Following [11], let C_m be the matrix defined by

$$C_1 = \begin{bmatrix} u^2R_1 \\ u^2R_2 \\ \vdots \\ u^2R_k \end{bmatrix}, \quad C_2 = \begin{bmatrix} uR_1 \\ u^2R_1 \\ uR_2 \\ u^2R_2 \\ \vdots \\ uR_k \\ u^2R_k \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1R_1 \\ uR_1 \\ u^2R_1 \\ 1R_2 \\ uR_2 \\ u^2R_2 \\ \vdots \\ 1R_k \\ uR_k \\ u^2R_k \end{bmatrix},$$

where R_i is the i^{th} row of the matrix G_k. The sub-codes $\mathcal{D}^{(m)}$ of $\mathcal{C}$ generated by the 2-linear combinations of the rows of C_m will have 2^{mk} codewords. Note that the codewords generated by the matrix C_1 have components either 0 or u^2 and C_3 yields the whole code S_k^α. Thus, for all $m,\ 1 \le m \le 3$, a codeword of type m will occur $2^{mk} - 2^{(m-1)k}$ times in S_k^α. This proves the following lemma.

Lemma 2.2 *Let $0 < m \le 3$. Then the number of codewords of type m in S_k^α is $2^{(m-1)k}(2^k-1)$.*

Theorem 2.3 *The hamming, Lee and generalized Lee weight distributions of S_k^α are:*

1. $A_H(0) = 1, A_H(2^{3k-m}(2^m-1)) = 2^{(m-1)k}(2^k-1)$ *for* $1 \le m \le 3$,
2. $A_L(0) = 1, A_L(2^{3(k+1)-2}) = 2^{3k}-1$ *and*
3. $A_{GL}(0) = 1, A_{GL}(2^{3(k+1)-2}) = 2^{3k}-1$.

Proof. Let $\mathbf{c} \in S_k^\alpha$ be a codeword of type $m \neq 0$.The by Lemma 2.1, $wt_H(\mathbf{c}) = 2^{3k} - 2^{3k-m}$ and hence by Lemma 2.2, $A_H(2^{3k} - 2^{3k-m}) = 2^{(m-1)k}(2^k-1)$. For $m = 0$, $A_H(0) = 1$. Also, by Lemma 2.1 $wt_L(\mathbf{c}) = 2^{3k-m}(\sum_{t=0}^{(3^m-1)} wt_L(t.u^{3-m})) = 2^{3(k+1)-2}$ which is independent of m. Thus all type $m \neq 0$ codewords will have same Lee weight Similar argument holds for generalized Lee weight. □

Note:-S_k^α is an equidistant code with respect to Lee and generalized Lee distances and it is of type α.
As the length of S_k^α is large, we can puncture some columns from G_k to yield good codes over R.
Let G_k^α be the $k \times 2^{2(k-1)}(2^k-1)$ matrix defined inductively by

$$G_2^\beta = \left[\begin{array}{c|c|c|c|c} 111\ldots1 & 0 & u & u^2 & uv \\ \hline 0,1,u,v,u^2,uv,v^2,v^3 & 1 & 1 & 1 & 1 \end{array}\right],$$

and for $k > 2$,

$$G_k^\beta = \left[\begin{array}{c|c|c|c|c} 111\ldots1 & 00\ldots0 & u,u\ldots u & u^2,u^2\ldots,u^2 & uv,uv,\ldots,uv \\ \hline G_{k-1} & G_{k-1}^\beta & G_{k-1}^\beta & G_{k-1}^\beta & G_{k-1}^\beta \end{array}\right],$$

where G_{k-1} is the matrix of S_{k-1}^α. By induction, it easy to verify that no two columns of G_k^β are multiple of each other. Let S_k^β be the linear code over R generated by G_k^β. Note that S_k^β is $[2^{2(k-1)}(2^k-1), 3k]$ code.

Remark 2.3 If $A_{k-1}(B_{k-1})$ denotes an array of codewords in $S^{\alpha}_{k-1}(S^{\beta}_{k-1})$ and if $\mathbf{i} = (i, i, ..., i)$ then an array of all codewords of S^{β}_{k} is given by

$$\begin{bmatrix} A_{k-1} & B_{k-1} & B_{k-1} & B_{k-1} & B_{k-1} \\ \mathbf{1}+A_{k-1} & B_{k-1} & \mathbf{u}+B_{k-1} & \mathbf{u^2}+B_{k-1} & \mathbf{uv}+B_{k-1} \\ \mathbf{u}+A_{k-1} & B_{k-1} & \mathbf{u^2}+B_{k-1} & B_{k-1} & \mathbf{u^2}+B_{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{v^3}+A_{k-1} & B_{k-1} & \mathbf{uv}+B_{k-1} & \mathbf{u^2}+B_{k-1} & \mathbf{u}+B_{k-1} \end{bmatrix}.$$

Remark 2.4 Each row of G^{β}_{k} has Hamming weight $2^{(3-1)(k-1)-3}[(2^k-1)(2^3-1)+1]$, and generalized Lee weight $2^{3k-k-1}(2^k-1)$. The Lee weight of the first row will be $2^{3(k-1)} + 2^{3k-2} - 2^{3k-k-1}$.

Remark 2.5 Let $j \in R$ and let $\mathbf{c}$ be a codeword in the code C we denote $w_j(\mathbf{c}) = |\{k : c_k = j\}|$.

Let U, Z be the set of units and zero divisors of R, respectively. The following proposition in the determination of the weight distribution of S^{α}_{k}.

Proposition 2.4 *Let $1 \leq j \leq k$ and let R_j be the j^{th} row of G^{β}_{k}. Then $\sum_{i\in U}\omega_i = 2^{3(k-1)}$, and each zero divisor in R occurs $2^{(3-1)(k-2)}(2^{k-1}-1)$ times in R_j.*

Proof. The proof follows directly from above using the definition of R_j. □

Proposition 2.5 *Let $\mathbf{c} \in S^{\beta}_{k}$. If one of the coordinates of $\mathbf{c}$ is a unit then $\sum_{i\in U}\omega_i = 2^{3(k-1)}$, and each zero divisor in R occurs $2^{(3-1)(k-2)}(2^{k-1}-1)$ times in $\mathbf{c}$.*

Proof. The proof is follows by induction from remark (2.3). □

Let $\mathcal{C}$ be a linear code over R. We can define the reduction code $\mathcal{C}^{(1)}$ and the torsion code $\mathcal{C}^{(2)}$ of $\mathcal{C}$ as follows. $\mathcal{C}^{(1)} = \{\mathbf{x} \in F_2^n | \exists \mathbf{y},\ \mathbf{z} \in F_2^n | \mathbf{x} + \mathbf{y}u + \mathbf{z}u^2 \in \mathcal{C}\}$ and $\mathcal{C}^{(2)} = \{\mathbf{x} \in F_2^n | u^2\mathbf{x} \in \mathcal{C}\}$. If $\mathcal{C}$ is a free module then $\mathcal{C}^{(1)} = \mathcal{C}^{(2)}$. Hence the reduction and the torsion codes of $S^{\alpha}_{k}(S^{\beta}_{k})$ are equal.

Proposition 2.6 *The torsion code of $S^{\alpha}_{k}(S^{\beta}_{k})$ is equivalent to $2^{(3-1)k}$ copies of the extended binary simplex code ($2^{(3-1)(k-1)}$ copies of the binary simplex code).*

Proof. The proof is by induction on k. □

Theorem 2.7 *The Hamming and Generalized Lee weight distribution of S_k^β are*

1. $A_H(0) = 1, A_H(2^{2(k-1)}[2^{k-m}\{2^m - 1\} + \{2^{1-m} - 1\}]) = 2^{(m-1)k}(2^k - 1)$, *for each* m; $1 \leq m \leq 3$, *and*

2. $A_{GL}(0) = 1, A_{GL}(2^{3k} - 1) = 2^k - 1, A_{GL}(2^{3k-k-1}(2^k - 1)) = 2^k(2^{2k} - 1)$.

Proof. By induction on k. By Theorem (2.3) and Remark (2.3) it easy to see that the possible nonzero Hamming (Generalized Lee) weights of S_k^β are $\{2^{2(k-1)}(2^{k-m}(2^m - 1) + (2^{1-m} - 1)) : 1 \leq m \leq 3\}(\{2^{3k-1}, 2^{3k-k-1}(2^k - 1)\})$. By lemma (2.2), Hamming weight of type m will occur $2^{(m-1)k}(2^k - 1)$ times. Moreover, generalized Lee weight 2^{3k-1} will occur $2^k - 1$ times. Thus the other weight will occur $2^{3k} - 2^k$ times. □

2.1. Gray Image Families

Let $\mathcal{C}$ be an $[n, k, d_H, d_{GL}]$ linear code over R and let Φ_{GL} be the generalized Gray map defined in section (1.2). Then $\Phi_{GL}(\mathcal{C})$ is a binary code having 2^k codewords of length $4n$, and Hamming distance d_{GL}. Also, $\Phi_{GL}(\mathcal{C})$ is always a linear binary code.

Remark 2.6 (i) Let $\overline{S}\ _k^\alpha$ be the punctured code of S_k^α obtained by deleting the zero coordinate, then $\Phi_{GL}(\overline{S}\ _k^\alpha)$ is a binary code of length $2^2(2^{3k-1})$ and minimum Hamming distance $2^{3(k+1)-2}$.

(ii) $\Phi_{GL}(S_k^\beta)$ is a binary code of length $2^{2k}(2^k - 1)$ and minimum Hamming distance $2^{3k-k-1}(2^k - 1)$.

2.2. Conclusion

In this paper we have studied R- simplex codes and some of their properties. Other properties of these codes will reported in future study. One can also extend these ideas to a more general rings like $\sum_{n=0}^{s} u^n F_2$ and to $\sum_{n=0}^{s} u^n F_p$, where p is a prime integer and $u^{s+1} = 0$.

References

[1] AL-Ashker M. *Simplex codes over* $F_2 + uF_2$, The Arabian Journal for Science and engineering, To appear.

[2] Bachoc C. *Application of coding Theory to the construction of modular lattices*, J. Combin. Theory Ser. A 78 (1997) 92-119.

[3] Bhandrri M. C., Gupta M. K. and Lal A. K. *On* Z_4 *simplex codes and their gray images*, *AACC-13 Lecture notes in Computer Sciences* Vol.**1719** (1999) pp. 170–180.

[4] Bonnecaze A., Udaya P. *Cyclic codes and self-dual codes over* F_2+uF_2, *IEEE Trans. Inform. Theory*, vol **45**. No. 4, may-(1999) pp. 1250-1254.

[5] Bonnecaze A., Udaya P. *Decoding of Cyclic codes over* $F_2 + uF_2$, *IEEE Trans. Inform. Theory*, vol **45**. No. 6, may-(1999) pp. 2148-2156.

[6] Calderbank A., and Sloane N.J.A. *Modular and p-adic codes*, Designs codes and Cryptography, 6: 21-35 (1995).

[7] Carlet C. Z_{2^s}*-linear codes, IEEE Trans. Inform. Theory*, vol. IT-44(4): pp. 1543–1547, 1998.

[8] Dougherty ST.,Gaborit P., Harda M. and Solé P. *Type II codes over* $F_2 + uF_2$, *IEEE Trans. Inform. Theory* **45**(1999) pp. 32–45.

[9] El-Atrash M. and Al-Ashker M. *Linear codes over the ring* $F_2 + uF_2$, Islamic University Journal for natural Science, Vol. 11, No. 2, pp. 53-68, June, (2003).

[10] S.Sadek, EL-Atrash M. and Naji A. *Codes of Constant Lee or Euclidean weight over the ring* $F_2 + uF_2$,Al-Aqsa University Journal, To appear.

[11] Gupta M. *On some linear codes over* Z_{2^s}, Ph.D. Thesis, Department of Mathematics, IIT. Kanpur, India, (July 2000), pp. 1-98.

[12] Macwillams F. J.and Sloane N. J. A. *Theory of Error- Correcting codes* North Holland, Amsterdam, (1998).

[13] Norton G. and S$\tilde{a}$l$\tilde{a}$gean A. *On the structure of linear and cyclic codes over finite chain rings* , Application Algebra in enginearing communiction and computing **10**, (2000), pp. 489–506.

[14] Rain's E. *Optimal self- dual codes over* Z_4, Discrete Mathematics, 203 (1999), pp. 215–228

[15] Sadek S., EL-Atrash M. and Naji A. *Linear Codes over* $F_2 + uF_2 + u^2F_2$ *of Constant Lee weight*, The second conference of the Islamic University on Mathematical Science-Gaza, 27-28 Aug. 2002.

[16] Vazirani V.V., Sran H. And Rajan B. S. *In efficient algorithm for constructing minimal trellises for codes over finite a belian groups.* IEEE. Trans. Infor. Theory, 42 (6): 1839–1854, 1996.

[17] Wood Jay A. *The Structure of linear codes of constant weight* , Trans. Amer. Math. Soc. 354, pp. 1007-1026, (2002).

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