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Maximal Oscillatory Singular Integrals with Kernels in $L\log L(\mathbf{S}^{n-1})$

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Abstract

In this paper, we study the L^p mapping properties of a certain class of maximal oscillatory singular integral operators. We establish the L^p boundedness of our operators provided that their kernels belong to the natural space $L\log^+ L(\mathbf{S}^{n-1})$. Our result substantially improves a previously known result. Moreover, the approach developed in this paper can be applied to handle more general maximal oscillatory singular integral operators.

Key Words: Oscillatory singular integrals, Rough kernels, Maximal functions.

1. Introduction and statement of Results

Let $\mathbf{R}^n$, $n \geq 2$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. For nonzero $y \in \mathbf{R}^n$, we shall let $y' = |y|^{-1} y$. Let $\Omega \in L^1(\mathbf{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbf{R}^n$ which satisfies the cancelation property

$$\int_{\mathbf{S}^{n-1}} \Omega(y')d\sigma(y') = 0. \tag{1.1}$$

For suitable mappings $\mathcal{P}(y) : \mathbf{R}^n \to \mathbf{R}^d$ and $\Phi : \mathbf{R}^n \to \mathbf{R}$, define the oscillatory singular integral operator $T_{\mathcal{P},\Phi,\Omega}$ and the maximal oscillatory singular integral operator $T^*_{\mathcal{P},\Phi,\Omega}$

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(initially for $\mathcal{C}_0^\infty$ functions on $\mathbf{R}^d$) by

$$T_{\mathcal{P},\Phi,\Omega}(f)(x) = \int_{\mathbf{R}^n} e^{i\Phi(y)} f(x-\mathcal{P}(y))\Omega(y')\,|y|^{-n}\,dy \tag{1.2}$$

$$T^*_{\mathcal{P},\Phi,\Omega}(f)(x) = \sup_{\varepsilon>0}\left|T^{\varepsilon}_{\mathcal{P},\Phi,\Omega}(f)(x)\right|, \tag{1.3}$$

where

$$T^{\varepsilon}_{\mathcal{P},\Phi,\Omega}(f)(x) = \int_{|y|>\varepsilon} e^{i\Phi(y)} f(x-\mathcal{P}(y))\Omega(y')\,|y|^{-n}\,dy.$$

It is clear that if $\Phi(y)=0$ and $\mathcal{P}(y)=y$, then the operators $T_{\mathcal{P},\Phi,\Omega}$ and $T^*_{\mathcal{P},\Phi,\Omega}$ are the classical Calderón-Zygmund singular integral operator and the maximal singular integral operator respectively. When $\Phi(y)=0$ and $\mathcal{P}(y)=y$, we shall simply let $T_\Omega = T_{\mathcal{P},\Phi,\Omega}$ and $T^*_\Omega = T^*_{\mathcal{P},\Phi,\Omega}$. In their fundamental work on singular integrals, Calderón and Zygmund established the L^p boundedness of the operators $\mathbf{T}_\Omega$ and T^*_Ω for $1<p<\infty$ under the condition that $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$, i.e.

$$\int_{\mathbf{S}^{n-1}} \left|\Omega(y^{'})\right| \log^+\left|\Omega(y^{'})\right| d\sigma(y^{'}) < \infty. \tag{1.4}$$

The condition in the form that $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$ turns out to be the most desirable size condition for the L^p boundedness of $\mathbf{T}_\Omega$ to hold. In fact, Calderón and Zygmund ([4], [5]) showed that $\mathbf{T}_\Omega$ may fail to be bounded on L^p for any p if the condition $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$ is replaced by any condition $\Omega \in L(\log^+ L)^{1-\varepsilon}(\mathbf{S}^{n-1})$, $\varepsilon>0$. It is worth pointing out that the space $L\log L(\mathbf{S}^{n-1})$ contains the space $L^q(\mathbf{S}^{n-1})$ (for any $q>1$) properly.

When $\Phi(y)=0$, the L^p boundedness properties of the operators (1.2)-(1.3) are well understood ([16], [18]; see also [2], [8], among others). However, for general mappings Φ and $\mathcal{P}$, the problem regarding the L^p boundedness of the corresponding operators $T_{\mathcal{P},\Phi,\Omega}$ and $T^*_{\mathcal{P},\Phi,\Omega}$ is still under investigation ([1], [3], [12], [13], [14], [15]).

It should be pointed out that the boundedness of the operators $T^*_{\mathcal{P},\Phi,\Omega}$ imply the boundedness of the corresponding operators $T_{\mathcal{P},\Phi,\Omega}$. In fact, establishing the a-priori bound $\left\|T^*_{\mathcal{P},\Phi,\Omega}f\right\|_p \leq C\left\|f\right\|_p$ with constant C independent of $f\in L^p$, implies that for any $f\in L^p$, $T^{\varepsilon}_{\mathcal{P},\Phi,\Omega}(f)$ converges (to $T_{\mathcal{P},\Phi,\Omega}(f)$) almost everywhere as $\varepsilon \to 0^+$. Hence, the boundedness of $T_{\mathcal{P},\Phi,\Omega}$ follows by an application of Fatou's lemma. For the significance of studying maximal operators of the form (1.3), we advice the reader to consult ([16], [17], [18], [19], among others).

In this paper, we focus our attention on studying the L^p mapping properties of a class of the maximal operators $T^*_{\mathcal{P},\Phi,\Omega}$. More specifically, in [10], Fan and Yang studied the operators $T^*_{\mathcal{P},\Phi,\Omega}$ under the conditions that $\mathcal{P}(y) = (P_1, \dots, P_d)$ where each P_j is a real valued polynomial and Φ is a homogeneous function that satisfies

$$\Phi(ty') = t^{\beta}\Phi(y') \text{ for } t > 0, \tag{1.5}$$

$$\Phi(y') \in L^{\infty}(\mathbf{S}^{n-1}), \text{ and } \int_{\mathbf{S}^{n-1}} \left|\Phi(y')\right|^{-\delta} d\sigma(y') < \infty, \tag{1.6}$$

for some $\delta > 0$ and for some $\beta \neq 0$. Fan and Yang proved the following theorem.

Theorem A ([10]). *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1) and that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Suppose also that $\mathcal{P}(y) = (P_1, \dots, P_d)$ is a polynomial mapping. If Φ is a homogeneous function that satisfies (1.5)-(1.6) with either the index $\beta \neq 0$ is not a positive integer or β is a positive integer larger than* $\max\{\deg(P_j) : 1 \leq j \leq d\}$, *then the operator $T^*_{\mathcal{P},\Phi,\Omega}$ is bounded on L^p for all $1 < p < \infty$. Moreover, the operator norm is independent of the coefficients of the polynomial mappings $\{P_j : 1 \leq j \leq d\}$.*

Since, by Calderón-Zygmund's result discussed above, the natural condition to impose on the function Ω is that $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$, the following question naturally arises.

Question. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1). Suppose also that $\mathcal{P}$, Φ,* and $T^*_{\mathcal{P},\Phi,\Omega}$ are as in Theorem A. *Does the result of Theorem A still hold if the condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$* is replaced by the weakest and more natural condition $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$?

In this paper, we shall answer this question in the affirmative. In fact, we have the following theorem.

Theorem B *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1) and that $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$. Suppose also that $\mathcal{P}(y) = (P_1, \dots, P_d)$ is a polynomial mapping. If Φ is a homogeneous function that satisfies (1.5)–(1.6) with either the index $\beta \neq 0$ is not a positive integer or β is a positive integer larger than* $\max\{\deg(P_j) : 1 \leq j \leq d\}$, *then the operator $T^*_{\mathcal{P},\Phi,\Omega}$ is bounded on L^p for all $1 < p < \infty$.*

Moreover, the operator norm is independent of the coefficients of the polynomial mappings $\{P_j : 1 \leq j \leq d\}$.

Throughout this paper the letter C will denote a constant that may vary at each occurrence, but it is independent of the essential variables. For a set A, we let χ_A denote the characteristic function of A.

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2. Some Lemmas

We shall begin by recalling the following result in [9]:

Lemma 2.1 ([9]). *Let* $\mathcal{P} = (P_1, ..., P_d)$ *be a polynomial mapping from* $\mathbf{R}^n$ *into* $\mathbf{R}^d$. *Suppose* $\Omega \in L^1(\mathbf{S}^{n-1})$ *and*

$$\mu_{\Omega,\mathcal{P}} f(x) = \sup_{j\in\mathbf{Z}} \int\limits_{2^j \leq |y| < 2^{(j+1)}} |f(x - \mathcal{P}(y))| \, |y|^{-n} \left|\Omega(y^{'})\right| dy.$$

Then for $1 < p \leq \infty$ *there exists a constant* $C_p > 0$ *independent of* Ω, *and the coefficients of* $P_1, ..., P_d$ *such that*

$$\|\mu_{\Omega,\mathcal{P}} f\|_p \leq C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_p$$

for every $f \in L^p(\mathbf{R}^d)$.

The following lemma will be useful in handling the needed oscillatory integrals:

Lemma 2.2 (van der Corput [18]). *Suppose* ϕ *is real-valued and smooth in* (a, b), *and that* $\left|\phi^{(k)}(t)\right| \geq 1$ *for all* $t \in (a, b)$. *Then the inequality*

$$\left|\int_a^b e^{-i\lambda\phi(t)} \psi(t) dt\right| \leq C_k \, |\lambda|^{-\frac{1}{k}}$$

holds when:

(i) $k \geq 2$, *or*

(ii) $k = 1$ *and* $\phi^{'}$ *is monotonic.*

The bound C_k *is independent of* a, b, ϕ, *and* λ.

We now prove the following lemma.

Lemma 2.3 *Let $\mathcal{P} = (P_1, ..., P_d)$ be a polynomial mapping from $\mathbf{R}^n$ into $\mathbf{R}^d$. Suppose $m \in \mathbf{N}$, $\Omega \in L^1(\mathbf{S}^{n-1})$, and $\Phi \in L^\infty(\mathbf{S}^{n-1})$ is a homogeneous function of degree $\beta \neq 0$. Let*

$$\begin{aligned} D^{(1,m)} &= \{y \in \mathbf{R}^n : |y| > 2^{2m}\}, \\ D^{(2,m)} &= \{y \in \mathbf{R}^n : |y| < 2^{2m}\}, \end{aligned}$$

and

$$D^{(0,m)} = \{y \in \mathbf{R}^n : 1 \leq |y| < 2^{2m}\}.$$

Let $M^{(0)}_{\mathcal{P},\Phi,\Omega}$, $M^{(1)}_{\mathcal{P},\Phi,\Omega}$, and $M^{(2)}_{\mathcal{P},\Phi,\Omega}$ be the operators given by

$$M^{(0)}_{\mathcal{P},\Phi,\Omega}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} e^{i\Phi(y)} f(x - \mathcal{P}(y)) \, |y|^{-n} \, \Omega(y^{'}) \chi_{D^{(0,m)}} \right| dy$$

and

$$M^{(i)}_{\mathcal{P},\Phi,\Omega}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} (e^{i\Phi(y)} - 1) f(x - \mathcal{P}(y)) \, |y|^{-n} \, \Omega(y^{'}) \chi_{D^{(i,m)}} \right| dy,$$

for $i = 1, 2$. Then for all $1 < p < \infty$ there exists a constant $C_p > 0$ independent of Ω and m such that

$$\left\| M^{(i)}_{\mathcal{P},\Phi,\Omega}(f) \right\|_p \leq m \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \, C_p \, \|f\|_p \tag{2.1}$$

for $i = 0, 1, 2$ with $\beta < 0$ for $i = 1$ and $\beta > 0$ for $i = 2$.

Proof. We start by proving (2.1) for $i = 0$. Notice that

$$\begin{aligned} M^{(0)}_{\mathcal{P},\Phi,\Omega}(f)(x) &\leq \int_{1 \leq |y| < 2^{2m}} \left| \Omega(y^{'}) \right| |y|^{-n} \, |f(x - \mathcal{P}(y))| \, dy \\ &= \sum_{l=0}^{2m-1} \{ \int_{2^l \leq |y| < 2^{l+1}} \left| \Omega(y^{'}) \right| |y|^{-n} \, |f(x - \mathcal{P}(y))| \, dy \} \\ &\leq \sum_{l=0}^{2m-1} \mu_{\Omega,\mathcal{P}} f(x) = 2m \mu_{\Omega,\mathcal{P}} f(x), \end{aligned} \tag{2.2}$$

where $\mu_{\Omega,\mathcal{P}}f$ is the operator given in Lemma 2.1. Hence (2.1) for $i=0$ follows by (2.2) and Lemma 2.1.

Now, we prove (2.1) for $i=1$. First, observe that

$$\begin{aligned} M^{(1)}_{\mathcal{P},\Phi,\Omega}(f)(x) &\le \int_{|y|>2^{2m}} |\Phi(y)| \left|\Omega(y^{'})\right| |y|^{-n} |f(x-\mathcal{P}(y))|\, dy \\ &\le \int_{|y|>2^{2m}} |\Phi(y')| \left|\Omega(y^{'})\right| |y|^{-n+\beta} |f(x-\mathcal{P}(y))|\, dy. \end{aligned} \tag{2.3}$$

Thus, by (2.3) and the assumption that $\Phi \in L^{\infty}(\mathbf{S}^{n-1})$, we have

$$\begin{aligned} M^{(1)}_{\mathcal{P},\Phi,\Omega}(f)(x) &\le \|\Phi\|_{\infty} \int_{|y|>2^{2m}} \left|\Omega(y^{'})\right| |y|^{-n+\beta} |f(x-\mathcal{P}(y))|\, dy \\ &= \|\Phi\|_{\infty} \sum_{j=2}^{\infty} \int_{2^{mj}<|y|<2^{m(j+1)}} \left|\Omega(y^{'})\right| |y|^{-n+\beta} |f(x-\mathcal{P}(y))|\, dy \\ &\le \|\Phi\|_{\infty} \sum_{j=2}^{\infty} \{2^{m\beta j} \int_{2^{mj}<|y|<2^{m(j+1)}} \left|\Omega(y^{'})\right| |y|^{-n} |f(x-\mathcal{P}(y))|\, dy\} \\ &\le \|\Phi\|_{\infty}\, m\{\sum_{j=2}^{\infty} 2^{m\beta j}\} \mu_{\Omega,\mathcal{P}}f(x). \end{aligned}$$

Therefore, Since $\beta < 0$, we immediately obtain

$$M^{(1)}_{\mathcal{P},\Phi,\Omega}(f)(x) \le \|\Phi\|_{\infty} \frac{2^{\beta} m}{1-2^{\beta}} \mu_{\Omega,\mathcal{P}}f(x). \tag{2.4}$$

Hence, (2.1) for $i=1$ follows from (2.4) and Lemma 2.1. Similarly, one can obtain (2.1) for $i=2$. We omit the details. This ends the proof. □

The following lemma will play an important role in the proof of our result.

Lemma 2.4 *Suppose that $\Omega \in L^{\infty}(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies $\|\Omega\|_{L^1} \le 1$ and $\|\Omega\|_{L^{\infty}} \le 2^m$ for some $m \ge 1$. Suppose also that $\mathcal{P}(y) = (P_1, \ldots, P_d)$ is a polynomial mapping and Φ is a homogeneous function that satisfies (1.5)-(1.6) with either the index $\beta \ne 0$ is not a positive integer or β is a positive integer larger than* $\max\{\deg(P_j) : 1 \le j \le d\}$. *Let $\psi_{k,m}$ be a smooth function on $\mathbf{R}$ that* *satisfies* $0 \leq \psi_{k,m} \leq 1$, *supp*$(\psi_{k,m}) \subseteq [2^{-m(k+1)}, 2^{-m(k-1)}]$, *and* $\left|\frac{d\psi_{k,m}}{du}(u)\right| \leq Cu^{-1}$ *with constant* C *independent of* m *and* k*. Then*

$$J_k(\Phi,\xi,\Omega) = \left|\int_{\mathbf{S}^{n-1}} \Omega(y') \int_0^\infty e^{i\{t^\beta \Phi(y') - \mathcal{P}(ty')\cdot\xi\}} \frac{\psi_{k,m}(t)dtd\sigma}{t}\right| \leq mC2^{\beta\alpha(k+1)}$$

for some constants $0 < \alpha < 1$ *and* $C > 0$ *which are independent of* m, k*, and the coefficients of* $P_1, ..., P_d$.

Proof. By the properties of $\psi_{k,m}$, and the fact that $\|\Omega\|_{L^1} \leq 1$, we have

$$J_k(\Phi,\xi,\Omega) \leq 2m\ln 2. \tag{2.5}$$

On the other hand, since $\|\Omega\|_{L^\infty} \leq 2^m$, we have

$$J_k(\Phi,\xi,\Omega) \leq 2^m \int_{\mathbf{S}^{n-1}} \left|\int_0^\infty e^{i\{t^\beta \Phi(y') - \mathcal{P}(ty')\cdot\xi\}} \frac{\psi_{k,m}(t)}{t} dt\right| d\sigma(y'). \tag{2.6}$$

Next, let

$$I_k(\Phi,\xi) = \left|\int_0^\infty e^{i\{t^\beta \Phi(y') - \mathcal{P}(ty')\cdot\xi\}} \frac{\psi_{k,m}(t)}{t} dt\right|. \tag{2.7}$$

Then by the support property of $\psi_{k,m}$, (2.7) reduces to

$$I_k(\Phi,\xi) = \left|\int_1^{2^{2m}} e^{i\{(a_{k,m}t)^\beta \Phi(y') - \mathcal{P}(a_{k,m}ty')\cdot\xi\}} \frac{\psi_{k,m}(a_{k,m}t)}{t} dt\right|, \tag{2.8}$$

where we set $a_{k,m} = 2^{-m(k+1)}$.

Now, notice that

$$\frac{\left|\frac{d^{l+1}}{dt^{l+1}}((a_{k,m}t)^\beta \Phi(y') - \mathcal{P}(a_{k,m}ty')\cdot\xi)\right|}{-\beta(1-\beta)...(l-\beta)(a_{k,m})^\beta \left|2^{2m(\beta-l-1)}\Phi(y')\right|} \geq 1$$

for all $1 \leq t \leq 2^{2m}$, where l is the degree of $\mathcal{P}$. Thus by Lemma 2.2, we have

$$\left|\int_1^u e^{i\{t^\beta a_{k,m}\Phi(y') - \mathcal{P}(a_{k,m}ty')\cdot\xi\}} dt\right| \leq 2^{2m\frac{(l+1-\beta)}{l+1}} C \left|(a_{k,m})^\beta \Phi(y')\right|^{-\frac{1}{l+1}}, \tag{2.9}$$

for all $1 < u \leq 2^{2m}$ where C is a constant independent of m. Therefore, by (2.8), (2.9), and integration by parts, we obtain

$$I_k(\Phi, \xi) \leq 2^{2m\frac{(l+1-\beta)}{l+1}} C \left| (a_{k,m})^{\beta} \Phi(y') \right|^{-\frac{1}{l+1}} C(k, m), \tag{2.10}$$

where

$$C(k, m) = \frac{\psi_{k,m}(a_{k,m} 2^{2m})}{2^{2m}} + \int_{1}^{2^{2m}} \left| \frac{(a_{k,m} t \psi'_{k,m}(a_{k,m} t) - \psi_{k,m}(a_{k,m} t)}{t^2} \right|. \tag{2.11}$$

By the properties of $\psi_{k,m}$, we immediately obtain

$$C(k, m) \leq \frac{1}{2^{2m}} - \frac{2}{2^{2m}} + 2 = 1 - \frac{1}{2^{2m}} \leq 1. \tag{2.12}$$

Thus, by (2.10) and (2.12), we get

$$I_k(\Phi, \xi) \leq 2^{2m\frac{(l+1-\beta)}{l+1}} C \left| (a_{k,m})^{\beta} \Phi(y') \right|^{-\frac{1}{l+1}}; \tag{2.13}$$

which when interpolated with the trivial estimate $I_k(\Phi, \xi) \leq 2m \ln 2$, imply that

$$I_k(\Phi, \xi) \leq mC \left| (a_{k,m})^{\beta} \Phi(y') \right|^{-\frac{\delta}{l+1}}. \tag{2.14}$$

By (2.14), (2.6), and (1.6), we obtain

$$J_k(\Phi, \xi, \Omega) \leq mC 2^m \left| (a_{k,m})^{\beta} \right|^{-\frac{\delta}{l+1}}. \tag{2.15}$$

Now, by an interpolation between (2.5) and (2.15), we get the desired result. This completes the proof. □

Lemma 2.5 *Let $k \geq 0, m \geq 1$, and $\delta < 0$. Suppose that $\{\sigma_{m,k-j} : j \leq 1\}$ is a sequence of Borel measures on $\mathbf{R}^n$ such that*

(i) $\sup_{\xi \in \mathbf{R}^n} |\hat{\sigma}_{m,k-j}(\xi)| \leq mC 2^{\delta(k-j)}$;

(ii) The corresponding maximal function

$$M_{m,k}(f)(x) = \sup_{j<1} |\sigma_{m,k-j} * f(x)|$$

satisfies

$$\|M_{m,k}(f)(x)\|_p \leq mC \|f\|_p \tag{2.16}$$

for all $1 < p < \infty$.

Then, for $1 < p < \infty$ *there exist positive constants* α_p *and* C *which are independent of* k *and* m *such that*

$$\|M_{m,k}(f)(x)\|_p \leq mC2^{\delta\alpha_p k} \|f\|_p .$$

Proof. We start by observing that

$$M_{m,k}(f)(x) \leq \sum_{j=-\infty}^{1} |\sigma_{m,k-j} * f(x)| .$$

Therefore, by (i) and Plancherel's theorem, we have

$$\begin{aligned}
\|M_{m,k}(f)\|_2 &\leq \sum_{j=-\infty}^{1} \|\sigma_{m,k-j} * f\|_2 \leq \|f\|_2 \sum_{j=-\infty}^{1} \|\sigma_{m,k-j}\|_\infty \\
&\leq mC2^{\delta k}(\sum_{j=-\infty}^{1} 2^{-\delta j}) \|f\|_2 \leq mC2^{\delta k} \|f\|_2 . \qquad (2.17)
\end{aligned}$$

Hence, by interpolation between (2.15) and (2.17), we get the desired result. This completes the proof.

We end this section with the following lemma.

Lemma 2.6 *Suppose that* $h \in L^\infty(\mathbf{R}^+)$ *and* $\mathcal{P} = (P_1, ..., P_d)$ *is a polynomial mapping from* $\mathbf{R}^n$ *into* $\mathbf{R}^d$*. Suppose also that* $\Omega \in L^\infty(\mathbf{S}^{n-1})$ *is a homogeneous function of degree zero on* $\mathbf{R}^n$*that satisfies (1.1) with* $\|\Omega\|_{L^1} \leq 1$ *and* $\|\Omega\|_{L^\infty} \leq 2^m$ *for some* $m \geq 1$*. Then the operator*

$$S^*_{\mathcal{P},\Omega,h}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} f(x - \mathcal{P}(y))\Omega(y')h(|y|) |y|^{-n} dy \right|$$

satisfies

$$\left\|S^*_{\mathcal{P},\Omega,h}(f)\right\|_p \leq m \|h\|_\infty C_p \|f\|_p$$

for all $1 < p < \infty$ *with constant* C_p *independent of* m, h, Ω, *and the coefficients of* $P_1, ..., P_d$.

It should be pointed out that Lemma 2.6 was proved in (see [8], Theorem 1.2 therein) under the assumption that Ω is in the Hardy space $H^1(\mathbf{S}^{n-1})$. But, in our case, it is essential to determine the dependence of the L^p bounds on the parameter m. However, the latter can be obtained by following similar argument as in the proof of Theorem 1.1 in [2]. We omit the details.

3. Proof of Main Result

Proof of Theorem B. Assume that $\Omega \in L\log L(\mathbf{S}^{n-1})$ and satisfies (1.1). Let Φ, β, and $\mathcal{P}(y) = (P_1, \dots, P_d)$ be as in the statement of Theorem B. We start by decomposing the function Ω as follows.

For $m \in \mathbf{N}$, let $\mathbf{E}_m$ be the set of points $y' \in \mathbf{S}^{n-1}$ which satisfy $2^m \leq |\Omega\ (y')| < 2^{m+1}$. Also, we let $\mathbf{E}_0$ be the set of all those points $y' \in \mathbf{S}^{n-1}$which satisfy $|\Omega\ (y')| < 2$. For $m \in \mathbf{N} \cup \{0\}$, set $b_m = \Omega\chi_{\mathbf{E}_m}$ and $\theta_m = \|b_m\|_1$. Set

$$\mathbf{D} = \left\{m \in \mathbf{N} : \theta_m \geq 2^{-3m}\right\}.$$

For $m \in \mathbf{D}$, define the function A_m on $\mathbf{S}^{n-1}$ by

$$A_m(y') = (\theta_m)^{-1}\{b_m(y') - \int_{\mathbf{S}^{n-1}} b_m(y')d\sigma(y')\}.$$

We also define G on $\mathbf{S}^{n-1}$ by

$$G(y') = b_0(y') + \sum_{m \notin \mathbf{D}} b_m(y') - \int_{\mathbf{S}^{n-1}} b_0(y')d\sigma(y') - \sum_{m \notin \mathbf{D}} \int_{\mathbf{S}^{n-1}} b_m(y')d\sigma(y').$$

Then, it is straightforward to show that the following hold:

$$\begin{aligned}
\int_{\mathbf{S}^{n-1}} A_m(y')d\sigma(y') &= 0, \text{ and } \int_{\mathbf{S}^{n-1}} G(y')d\sigma(y') = 0; && (3.1)\\
\|A_m\|_1 &\leq C,\ \|A_m\|_\infty \leq C2^{4(m+1)}, && (3.2)\\
\Omega(y') &= G(y') + \sum_{m\in\mathbf{D}} \theta_m A_m(y'); && (3.3)\\
G &\in L^2(\mathbf{S}^{n-1}); && (3.4)\\
\sum_{m\in\mathbf{D}} m\theta_m &\leq C\|\Omega\|_{L(\log L)(\mathbf{S}^{n-1})}. && (3.5)
\end{aligned}$$

Thus by (3.3), we have

$$T^*_{\mathcal{P},\Phi,\Omega}f(x) \leq T^*_{\mathcal{P},\Phi,G}f(x) + \sum_{m\in\mathbf{D}} \theta_m T^*_{\mathcal{P},\Phi,A_m}f(x). \tag{3.6}$$

Since $G \in L^2(\mathbf{S}^{n-1})$, it follows from Theorem A that

$$\left\|T^*_{\mathcal{P},\Phi,G}f\right\|_p \leq C\|f\|_p \tag{3.7}$$

for all $1 < p < \infty$. Therefore by (3.6), (3.7), and (3.5), it suffices to show that

$$\left\|T^*_{\mathcal{P},\Phi,A_m}f\right\|_p \leq mC\|f\|_p \tag{3.8}$$

for all $1 < p < \infty$ and $m \in \mathbf{D}$ with constant C independent of m.

First, let us show that (3.8) and (3.7) will imply the theorem. Given $1 < p < \infty$. Then by (3.6), (3.7), and (3.8), we have

$$\begin{aligned}
\left\|T^*_{\mathcal{P},\Phi,\Omega}f\right\|_p &\leq \left\|T^*_{\mathcal{P},\Phi,G}f\right\|_p + \sum_{m\in\mathbf{D}} \theta_m \left\|T^*_{\mathcal{P},\Phi,A_m}f\right\|_p\\
&\leq C\{1 + \sum_{m\in\mathbf{D}} m\theta_m\}\|f\|_p \leq C\|f\|_p,
\end{aligned}$$

where the last inequality follows by (3.5).

Now, we turn to the proof of (3.8). By an elementary procedure, choose a collection of $\mathcal{C}^\infty$ functions $\{\psi_{k,m}\}_{k\in\mathbf{Z}}$ on $(0,\infty)$ with the properties:

$$\begin{aligned}
supp(\psi_{k,m}) &\subseteq [2^{-m(k+1)}, 2^{-m(k-1)}], 0 \leq \psi_{k,m} \leq 1, \sum_{k\in\mathbf{Z}} \psi_{k,m}(u) = 1,\\
\left|\frac{d^s\psi_{k,m}}{du^s}(u)\right| &\leq C_s u^{-s}
\end{aligned}$$

with constants C_s independent of m (see [2] for more details).

Now, as in [10], we have two cases. The fist case is when $\beta < 0$ and the second case is when β is a positive integer larger than $\max\{\deg(P_j) : 1 \leq j \leq d\}$. We shall only prove the case for $\beta < 0$. The proof for the other case follows by minor modifications.

Assume that $\beta < 0$. Let

$$\begin{aligned} \eta(y) &= \sum_{k=-\infty}^{-1} \psi_{k,m}(|y|); \\ K_{m,\infty}(y) &= A_m(y')\eta(y); \\ K_{m,0}(y) &= \sum_{k=0}^{\infty} A_m(y')\psi_{k,m}(|y|). \end{aligned}$$

Then, it is clear that

$$supp(K_{m,\infty}) \subset \{y \in \mathbf{R}^n : |y| \geq 1\}; \tag{3.9}$$

$$K_{m,\infty}(y) = A_m(y') \text{ for all } |y| > 2^{2m}; \tag{3.10}$$

$$supp(K_{m,0}) \subset \{y \in \mathbf{R}^n : |y| \leq 2^m\}. \tag{3.11}$$

Therefore, we have

$$T^*_{\mathcal{P},\Phi,A_m} f(x) \leq T^*_{\mathcal{P},\Phi,K_{m,\infty}}(f)(x) + T^*_{\mathcal{P},\Phi,K_{m,0}}(f)(x). \tag{3.12}$$

Now, by (3.9) and (3.10), we can decompose the factor $e^{i\Phi(y)} |y|^{-n} K_{m,\infty}(y)$ as follows:

$$\begin{aligned} e^{i\Phi(y)} |y|^{-n} K_{m,\infty}(y) &= |y|^{-n} A_m(y')\chi_{\{|y|>2^{2m}\}} + \\ &\quad (e^{i\Phi(y)} - 1) |y|^{-n} A_m(y')\chi_{\{|y|>2^{2m}\}} + \\ &\quad e^{i\Phi(y)} |y|^{-n} K_{m,\infty}(y)\chi_{\{1\leq|y|<2^{2m}\}}. \end{aligned} \tag{3.13}$$

This immediately implies that

$$T^*_{\mathcal{P},\Phi,K_{m,\infty}}(f)(x) \leq S^*_{\mathcal{P},A_m,h_m}(f)(x) + M^{(1)}_{\mathcal{P},\Phi,A_m}(f)(x) + M^{(0)}_{\mathcal{P},\Phi,\Omega}(f)(x),$$

where $h_m = \chi_{\{|y|>2^{2m}\}}$, $M^{(0)}_{\mathcal{P},\Phi,\Omega}, M^{(1)}_{\mathcal{P},\Phi,A_m}$, and $S^*_{\mathcal{P},A_m,h_m}$ are the operators given in Lemma 2.3 and Lemma 2.6. Thus, by Lemma 2.3 and Lemma 2.6, we obtain

$$\left\| T^*_{\mathcal{P},\Phi,K_{m,\infty}}(f) \right\|_p \leq mC \left\| f \right\|_p \tag{3.14}$$

for all $1 < p < \infty$.

Next, by (3.11), we have

$$T^*_{\mathcal{P},\Phi,K_{m,0}}(f)(x) = \sup_{0<\varepsilon<2^m} \left| \sum_{k=0}^{\infty} \int_{|y|>\varepsilon} e^{i\Phi(y)} |y|^{-n} A_m(y')\psi_{k,m}(|y|) f(x-\mathcal{P}(y))dy \right|. \quad (3.15)$$

Now, $0 < \varepsilon < 2^m$, choose $j \leq 1$ such that $2^{m(j-1)} \leq \varepsilon < 2^{mj}$. Therefore,

$$\left| \sum_{k=0}^{\infty} \int_{|y|>\varepsilon} e^{i\Phi(y)} |y|^{-n} A_m(y')\psi_{k,m}(|y|) f(x-\mathcal{P}(y))dy \right| \leq I_1(f)(x) + I_2(f)(x), \quad (3.16)$$

where

$$\begin{aligned} I_1(f)(x) &= \left| \sum_{k=0}^{\infty} \int_{2^{mj}\leq|y|<2^m} e^{i\Phi(y)} |y|^{-n} A_m(y')\psi_{k,m}(|y|) f(x-\mathcal{P}(y))dy \right|; \\ I_2(f)(x) &= \left| \sum_{k=0}^{\infty} \int_{\varepsilon<|y|<2^{mj}} e^{i\Phi(y)} |y|^{-n} A_m(y')\psi_{k,m}(|y|) f(x-\mathcal{P}(y))dy \right|. \end{aligned}$$

It is clear that

$$\begin{aligned} I_2(f)(x) &\leq \sum_{k=\max\{0,-1-j\}}^{k=2-j} \int_{2^{m(j-1)}\leq|y|<2^{mj}} |y|^{-n} |A_m(y')| \, |f(x-\mathcal{P}(y))| \, dy \\ &\leq 3m\mu_{A_m,\mathcal{P}} f(x), \end{aligned} \quad (3.17)$$

where $\mu_{A_m,\mathcal{P}} f(x)$ is the operator given in Lemma 2.1 with Ω is replaced by A_m

On the other hand, by the support property of $\psi_{k,m}$ we have

$$\begin{aligned}
I_1(f)(x) &= \left|\sum_{k=0}^{1-j}\int_{2^{mj}\le|y|<2^m} e^{i\Phi(y)}\frac{A_m(y')}{|y|^n}\psi_{k,m}(|y|)f(x-\mathcal{P}(y))dy\right| \\
&\le \left|\sum_{k=0}^{1-j}\int_{\mathbf{R}^n} e^{i\Phi(y)}\frac{A_m(y')}{|y|^n}\psi_{k,m}(|y|)f(x-\mathcal{P}(y))dy\right| + \\
&\quad \left|\sum_{k=-j}^{1-j}\int_{|y|<2^{mj}} e^{i\Phi(y)}\frac{A_m(y')}{|y|^n}\psi_{k,m}(|y|)f(x-\mathcal{P}(y))dy\right| \\
&\le \left|\sum_{k=0}^{1-j}\int_{\mathbf{R}^n} e^{i\Phi(y)}\frac{A_m(y')}{|y|^n}\psi_{k,m}(|y|)f(x-\mathcal{P}(y))dy\right| + 2m\mu_{A_m,\mathcal{P}}f(x).
\end{aligned} \tag{3.18}$$

Therefore by (3.15)–(3.18), we

$$T^*_{\mathcal{P},\Phi,K_{m,0}}(f)(x) \le G(f)(x) + 5m\mu_{A_m,\mathcal{P}}f(x), \tag{3.19}$$

where

$$G(f)(x) = \sup_{j<1}\left|\sum_{k=0}^{1-j}\int_{\mathbf{R}^n} e^{i\Phi(y)}\,|y|^{-n}\,A_m(y')\psi_{k,m}(|y|)f(x-\mathcal{P}(y))dy\right|.$$

Let $\sigma_{m,k}$ be the measure defined by

$$\int f d\sigma_{m,k} = \int e^{i\Phi(y)}\,|y|^{-n}\,A_m(y')\psi_{k,m}(|y|)f(\mathcal{P}(y))dy. \tag{3.20}$$

Then

$$G(f)(x) = \sup_{j<1}\left|\sum_{k=0}^{1-j}\sigma_{m,k} * f(x)\right| \le \sum_{k=0}^{\infty} M_{m,k}(f)(x), \tag{3.21}$$

where $M_{m,k}$ is the operator given in Lemma 2.5.

Thus, by (3.21), Lemma 2.1, Lemma 2.4, and Lemma 2.5, we have

$$\|G(f)\|_p \le mC\,\|f\|_p \tag{3.22}$$

for all $1 < p < \infty$; which when combined with (3.19) and Lemma 2.1, we obtain

$$\left\| T^{*}_{\mathcal{P},\Phi,K_{m,0}}(f) \right\|_{p} \leq mC \left\| f \right\|_{p} \tag{3.23}$$

for all $1 < p < \infty$. Hence, (3.8) follows by (3.12), (3.13), and (3.23). This completes the proof. □

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