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On Banach Lattice Algebras

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Abstract

In this study, without using the assumption $a^{-1} > 0$, it is shown that E is lattice - and algebra - isometric isomorphic to the reals $\mathbf{R}$ whenever E is a Banach lattice f-algebra with unit e, $\|e\| = 1$, in which for every $a > 0$ the inverse a^{-1} exists. Subsequently, an alternative proof to a result of Huijsmans is given for Banach lattice algebras.

Key Words: Algebra, inverse, lattice.

1. Introduction

Recall that the (real) vector lattice E is called a (real) lattice ordered algebra if E is also an associative algebra with the property that $a, b \in E_+$ implies $ab \in E_+$. We shall assume that E has a unit element $e > 0$. The lattice ordered algebra E is called an f-algebra whenever $a \wedge b = 0, c \in E_+$ implies $ac \wedge b = ca \wedge b = 0$. If the lattice ordered algebra E is Archimedean and uniformly complete we endow the complexification of E with the canonical absolute value; i.e., if $a = a_1 + ia_2$ with a_1 and a_2 real, then $|a| = sup\{(cos\Theta)a_1 + (sin\Theta)a_2 : 0 \leq \Theta \leq 2\pi\}$. The complexification is now called a complex lattice ordered algebra. For details on complex f-algebras we refer to [2].

Any lattice ordered algebra E which is at the same time a Banach lattice is called a Banach lattice algebra whenever $\|ab\| \leq \|a\| \|b\|$ holds for all $a, b \in E_+$. In addition, if E is an f-algebra then it is called Banach lattice f-algebra. Obviously, E is then

AMS Mathematics Subject Classification: 46B42 (06F25 16K40) a (real) Banach algebra. As above, it is assumed that E has a unit element $e > 0$. The complexification of E, $E_{\mathbf{C}}$, equipped with the canonical norm $\|a\|=\||a|\|$, is called a complex Banach lattice algebra and is also a Banach algebra. As customary, the spectrum of an element $a \in E$ is taken with respect to the complexification and is denoted by $Sp(a)$.

For the basic theory of vector lattices (Riesz spaces) and Banach lattices and for unexplained terminology we refer to [1], [8], [9], [10].

2. Main Results

Theorem 2.1. *Let E be a Banach lattice f-algebra with unit e, $\|e\| = 1$, in which for every $a > 0$ the inverse a^{-1} exists. Then E is lattice– and algebra–isometric isomorphic to $\mathbf{R}$.*

Proof. Let $a \in E$. Then there exist $\xi, \eta \in \mathbf{R}$ with $\xi + i\eta \in Sp(a)$,by theorem 13.7 in [3]. Since E is an f-algebra, $(\xi - a)^2 + \eta^2 \geq 0$ and $(\xi - a)^2 + \eta^2$ is not invertible, by theorem 13.8 in [3]. By hypothesis, $(\xi - a)^2 + \eta^2 = 0$ and so $(\xi - a)^2 = 0$. From theorem 142.5 in [10] , $a = \xi e$. Since $e > 0$, for each $a \in E$ there exists a unique $\xi \in \mathbf{R}$ such that $a = \xi e$ and also $|a| = |\xi|\, e$. The mapping $T : E \to \mathbf{R}$ defined by $T(a) = \xi$ is the desired lattice isomorphism. Since E and $\mathbf{R}$ are Archimedean f-algebras with unit element e and 1 respectively and $T : E \to \mathbf{R}$ is a lattice isomorphism which satisfies $T(e) = 1$, corollary 5.5 of [4] yields that T is also an algebra isomorphism. Furthermore, $\|T(a)\| = |\xi| = \|\xi e\| = \|a\|$. Therefore E is lattice– and algebra–isometric isomorphic to $\mathbf{R}$. □

Remark. Note that the proof is also obtained by Gelfand-Mazur theorem. Take $a + ib \neq 0$ $a, b \in E$. By assumption, $w = a^2 + b^2 > 0$ and so $w^{-1} \in E$. Then $(a+ib)(w^{-1}a-iw^{-1}b) = (w^{-1}a-iw^{-1}b)(a+ib) = e$ holds in $E_{\mathbf{C}}$, since E is commutative. From Gelfand-Mazur theorem, $E_{\mathbf{C}}$ is isomorphic to $\mathbf{C}$ [3]. Therefore, for each $a \in E$ there exists a unique $\xi \in \mathbf{R}$ such that $a = \xi e$. As above, E is lattice– and algebra–isometric isomorphic to $\mathbf{R}$.

Let E be an Archimedean lattice ordered algebra with unit element $e > 0$. The principal ideal and band generated by e in E are denoted by I_e and B_e, respectively. It is shown in [7] that B_e is an Archimedean f-algebra with unit e and is a full subalgebra of E. The proof of this result is easier for Banach lattice algebras. It is stated next.

Theorem 2.2. *Let E be a Banach lattice algebra with unit element $e > 0$. Then I_e is full subalgebra of E, that is, each $a \in I_e$ invertible in E has its inverse in I_e.*

Proof. It is shown in [5] that $E = I_e \oplus I_e^d$ and $I_e = B_e$. A simple argument shows that I_e is an Archimedean f-algebra with unit e. Assume that $a \in I_e$ is invertible in E. Then there exist $u \in I_e, v \in I_e^d$ such that $a^{-1} = u + v$. Therefore, $au + av = e$ holds. Since $av = e - au, av \in I_e$. Furthermore, $|av| \leq |a|\,|v|$ holds in E. We obtain that $av \in I_e^d$ and so $av = 0$. This implies that $v = 0$, i.e., $a^{-1} \in I_e$. □

Let E be a Banach lattice. Recall that the e-uniform norm of an element $a \in I_e$ is defined by $\|a\|_e = inf(\lambda > 0 : |a| \leq \lambda e)$. It is well known that $(I_e, \|.\|_e)$ is a Banach lattice [1].

Corollary 2.3. *Let E be a Banach lattice algebra with unit element $e > 0$ in which for every $a > 0$ the inverse a^{-1} exists. Then $(I_e, \|.\|_e)$ is lattice– and algebra–isometric isomorphic to* **R**.

Proof. By hypothesis and theorem 2.2, $(I_e, \|.\|_e)$ is a Banach lattice f-algebra with unit e, $\|e\|_e = 1$, in which for every $a > 0$ the inverse a^{-1} exists. From theorem 2.1, $(I_e, \|.\|_e)$ is lattice– and algebra– isometric isomorphic to **R** □

Theorem 2.4. *Let E be an Archimedean lattice ordered algebra with unit element $e > 0$ in which for every $w > e$ has a positive inverse. Then $I_e^d = \{0\}$. If, in addition, E is a Banach lattice algebra then $E = I_e$.*

Proof. Take $a \in I_e^d$. The inequality $e \leq e + |a|$ yields $0 < (e + |a|)^{-1} \leq e$ and so $0 \leq (e+|a|)^{-1}\,|a|\,(e+|a|)^{-1} \leq |a|$. On the other hand, $|a| \leq e+|a|$ yields $|a| \leq (e+|a|)^2$ and so $0 \leq (e + |a|)^{-1}\,|a|\,(e + |a|)^{-1} \leq e$. Therefore $(e + |a|)^{-1}\,|a|\,(e + |a|)^{-1} = 0$ and so $a = 0$. Hence $I_e^d = \{0\}$. Let now E be a Banach lattice algebra. Since $E = I_e \oplus I_e^d$, we obtain that $E = I_e$ [5]. The proof of the theorem is now complete. □

Following result is first obtained by C. B. Huijsmans in [6] for Archimedean lattice ordered algebras.

Corollary 2.5. *Let E be a Banach lattice algebra with unit element $e > 0$ in which every positive element has a positive inverse. Then E is lattice– and algebra– isometric isomorphic to* **R** *with respect to e-uniform norm.*

Proof. It immediately follows from corollary 2.3 and theorem 2.4. □

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