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Asymptotic Formulas for the Resonance Eigenvalues of the Schrödinger Operator

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Abstract

In this paper, we consider the Schrödinger operators defined by the differential expression

$$Lu = -\Delta u + q(x)u$$

in d-dimensional paralellepiped F, with the Dirichlet and the Neumann boundary conditions, where $q(x)$ is a real valued function of $L_2(F)$. We obtain the asymptotic formulas for the resonance eigenvalues of these operators.

First asymptotic formulas for the eigenvalues of the Schrödinger operator in parallelepiped with quasiperiodic boundary conditions are obtained in papers [8]–[11]. By some other methods, the asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases are obtained in [2], [3], [6], [7]. The asymptotic formulas for the eigenvalues of the Schrödinger operator with periodic boundary conditions are obtained in [4] and with Dirichlet boundary conditions in 2-dimensional case are obtained in [5].

Let $\Omega \equiv \{\sum_{i=1}^{d} m_i w_i; m_i \in Z, i = 1, 2, ..., d\}$ be a lattice in R^d with the reduced basis $w_1 = (a_1, 0, ..., 0)$, $w_2 = (0, a_2, 0, ..., 0)$,...,$w_d = (0, ..., 0, a_d)$,
$\Gamma \equiv \{\sum_{i=1}^{d} n_i \beta_i : n_i \in Z, i = 1, 2, ...d\}$ be the dual lattice of Ω, where
$\langle w_i, \beta_j \rangle = 2\pi\delta_{ij}$, $\langle ., . \rangle$ is inner product in R^d and $F \equiv [0, a_1] \times [0, a_2] \times ... \times [0, a_d]$.

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In this paper we consider the d-dimensional Schrödinger operators $L_D(q(x))$ and $L_N(q(x))$, defined by the differential expression

$$Lu = -\Delta u + q(x)u \tag{1}$$

in F, with the Dirichlet boundary condition

$$u|_{\partial F} = 0 \tag{2}$$

and the Neumann boundary condition

$$\frac{\partial u}{\partial n}|_{\partial F} = 0, \tag{3}$$

respectively, where ∂F denotes the boundary of the domain F, $x = (x_1, x_2, ..., x_d) \in R^d$, $d \geq 2$, Δ is the Laplace operator in R^d, and $\frac{\partial}{\partial n}$ denotes the differentiation along the outward normal n of ∂F.

We denote the eigenvalues and the normalized eigenfunctions of $L_D(q(x))$ by Λ_N and Ψ_N, respectively. The eigenvalues and the normalized eigenfunctions of $L_N(q(x))$ are denoted by Υ_N and Φ_N, respectively.

The eigenvalues of the operators $L_D(0)$ and $L_N(0)$ are $|\gamma|^2$, with the corresponding eigenfunctions

$$u_\gamma(x) = \sin\gamma^1 x_1 \sin\gamma^2 x_2 ... \sin\gamma^d x_d, \tag{4}$$

and

$$v_\gamma(x) = \cos\gamma^1 x_1 \cos\gamma^2 x_2 ... \cos\gamma^d x_d, \tag{5}$$

respectively, where $\gamma = (\gamma^1, \gamma^2, ..., \gamma^d) \in \frac{\Gamma}{2}$.

Since the orthogonal system $\{v_{\gamma'}(x)\}_{\gamma' \in \frac{\Gamma}{2}}$, is a basis in $L_2(F)$, the potential $q(x)$ in (1) can be written as

$$q(x) = \sum_{\gamma' \in \frac{\Gamma}{2}} q_{\gamma'} v_{\gamma'}(x), \tag{6}$$

where $q_{\gamma'}$ is the Fourier coefficient of $q(x)$ with respect to the basis $v_{\gamma'}(x), \gamma' \in \frac{\Gamma}{2}$. Without loss of generality we can take $q_0 = 0$.

In this paper, we assume that the Fourier coefficients of the potential $q(x)$ satisfy the condition

$$\sum_{\gamma' \in \frac{\Gamma}{2}} |q_{\gamma'}|^2 (1 + |\gamma'|^{2l}) < \infty, \tag{7}$$

where $l > \frac{(d-1)(d+20)}{2} + d + 1$. Therefore, one can write

$$q(x) = \sum_{\gamma' \in \Gamma(\rho^\alpha)} q_{\gamma'} v_{\gamma'}(x) + O(\rho^{-p\alpha}), \tag{8}$$

where $p = l - d$, $\Gamma(\rho^\alpha) = \{\gamma \in \frac{\Gamma}{2} : 0 < |\gamma| < \rho^\alpha\}$, $\alpha < \frac{1}{(d+20)}$ and ρ is a large parameter.

Remark 1 *Notice that, if $q(x)$ is sufficiently smooth, ($q(x) \in W_2^l(F)$) and the support of* $\mathrm{grad} q(x) = (\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, ..., \frac{\partial q}{\partial x_d})$ *is contained in the interior of the domain F, then $q(x)$ satisfies the condition (7).*
There is also another class of functions $q(x)$, such that $q(x) \in W_2^l(F)$,

$$q(x) = \sum_{\gamma' \in \Gamma} q_{\gamma'} v_{\gamma'},$$

which is periodic with respect to Ω and thus also satisfies the condition (7).

As in the papers [11], [12], we divide the eigenvalues $|\gamma|^2$ for $|\gamma| \sim \rho$ of the Laplace operator into two groups, where $|\gamma| \sim \rho$ means that $c_1\rho < |\gamma| < c_2\rho$ and by c_i, $i = 1, 2, ...$, we denote the positive independent on ρ constants whose exact values are inessential. For this, we let $\alpha_k = 3^k\alpha$, $k = 1, 2, ..., d-1$ and introduce the following notations and definitions:

$$M = \sum_{\gamma' \in \frac{\Gamma}{2}} |q_{\gamma'}|, \tag{9}$$

$$V_b(\rho^{\alpha_1}) = \{x \in R^d : ||x|^2 - |x+b|^2| < \rho^{\alpha_1}\}, \quad E_1(\rho^{\alpha_1}, p) = \bigcup_{b \in \Gamma(p\rho^\alpha)} V_b(\rho^{\alpha_1}),$$

$$U(\rho^{\alpha_1}, p) = R^d \backslash E_1(\rho^{\alpha_1}, p), \quad E_k(\rho^{\alpha_k}, p) = \bigcup_{\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^\alpha)} (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})),$$

where the intersection $\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1, \gamma_2, ..., \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vectors in $\Gamma \bigcap \gamma_i R$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $|\gamma|^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$, for all $b \in \Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalue $|\gamma|^2$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

As noted in [12], the domain $V_b(\rho^{\alpha_1}) \setminus E_2$, called a single resonance domain, has asymptotically full measure on $V_b(\rho^{\alpha_1})$, that is

$$\frac{\mu((V_b(\rho^{\alpha_1}) \setminus E_2) \bigcap B(\rho))}{\mu(V_b(\rho^{\alpha_1}) \bigcap B(\rho))} \to 1, \ \ as \ \ \rho \to \infty,$$

where $B(\rho) = \{x \in R^d : |x| = \rho\}$, if

$$2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \quad \text{and} \quad \alpha_2 > 2\alpha_1, \tag{10}$$

hold. Since $\alpha < \frac{1}{d+20}$, the conditions in (10) hold.

In [1], we obtained the asymptotic formulas for the non-resonance eigenvalues of the d-dimensional Schrödinger operators $L_D(q(x))$ and $L_N(q(x))$ with the condition (7).

In continuation of the paper [1], in this paper we investigate the perturbation of the resonance eigenvalue $|\gamma|^2$, i.e., when $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, where δ is from $\{e_1, e_2, ..., e_d\}$ and $e_1 = (\frac{\pi}{a_1}, 0, ..., 0), e_2 = (0, \frac{\pi}{a_2}, 0, ..., 0), ..., e_d = (0, ..., 0, \frac{\pi}{a_d})$.

Now let $H_\delta = \{x \in R : \langle x, \delta \rangle = 0\}$ be the hyperplane which is orthogonal to δ. Then, we define the following sets:

$$\Omega_\delta = \{w \in \Omega : \langle w, \delta \rangle = 0\} = \Omega \bigcap H_\delta,$$

$$\Gamma_\delta = \{\gamma \in \frac{\Gamma}{2} : \langle \gamma, \delta \rangle = 0\} = \frac{\Gamma}{2} \bigcap H_\delta.$$

Clearly, for all $\gamma \in \frac{\Gamma}{2}$, we have the following decomposition

$$\gamma = j\delta + \beta, \qquad \beta \in \Gamma_\delta, \qquad j \in Z. \tag{11}$$

We write the decomposition (6) of $q(x)$ as

$$q(x) = \sum_{\gamma' \in \frac{\Gamma}{2}} q_{\gamma'} v_{\gamma'}(x) = q^\delta(x) + \sum_{\gamma \in \frac{\Gamma}{2} \setminus \delta R} q_\gamma v_\gamma(x), \tag{12}$$

where $q^\delta(x) \equiv Q(s) = \sum_{n\in Z} q_{n\delta} \cos n\langle x,\delta\rangle$, $q_{n\delta} = \int_F q(x)\cos n\langle x,\delta\rangle dx$, $s = \langle x,\delta\rangle$.

We consider the operators $L_D(q^\delta(x))$ and $L_N(q^\delta(x))$, defined by the differential expression

$$Lu = -\Delta u + q^\delta(x)u \tag{13}$$

with the Dirichlet boundary condition $u|_{\partial F} = 0$ and the Neumann boundary condition $\frac{\partial u}{\partial n}|_{\partial F} = 0$, respectively.

It can be easily verified by the method of separation of variables that the eigenvalues and the eigenfunctions of $L_D(q^\delta(x))$ are $\widetilde{\lambda}_{j,\beta} = \widetilde{\mu}_j + |\beta|^2$ and $\widetilde{\Theta}_{j,\beta} = \widetilde{\varphi}_j(s)u_\beta$, respectively, where $\beta \in \Gamma_\delta$, $\widetilde{\mu}_j$ is the eigenvalue and $\widetilde{\varphi}_j(s)$ is the corresponding eigenfunction of the operator $T_D^\delta(Q(s))$ defined by the differential expression

$$Ty(s) = -|\delta|^2 y''(s) + Q(s)y(s) \tag{14}$$

in $[0,\pi]$, with the Dirichlet boundary conditions $y(0) = y(\pi) = 0$.

Similarly, the eigenvalues and the eigenfunctions of $L_N(q^\delta(x))$ are $\lambda_{j,\beta} = \mu_j + |\beta|^2$ and $\Theta_{j,\beta} = \varphi_j(s)v_\beta$, respectively, where $\beta \in \Gamma_\delta$, and μ_j is the eigenvalue and $\varphi_j(s)$ is the corresponding eigenfunction of the Sturm-Liouville operator $T_N^\delta(Q(s))$, defined by the differential expression (14) in $[0,\pi]$, with the Neumann boundary conditions $y'(0) = y'(\pi) = 0$.

The eigenvalues of the operators $T_D^\delta(0)$ and $T_N^\delta(0)$ are $|n\delta|^2$ with the corresponding eigenfunctions $\sin ns$ and $\cos ns$, respectively. It is well known that the eigenvalue $\widetilde{\mu_j}$ of $T_D^\delta(Q(s))$ and the eigenvalue μ_j of $T_N^\delta(Q(s))$ such that $|\widetilde{\mu}_j - |j\delta|^2| < \sup Q(s), |\mu_j - |j\delta|^2| < \sup Q(s)$ together with the corresponding eigenfunction $\widetilde{\varphi}_j(s)$ of $T_D^\delta(Q(s))$ and the corresponding eigenfunction $\varphi_j(s)$ of $T_N^\delta(Q(s))$ satisfy the following relations:

$$\begin{aligned} \widetilde{\mu}_j &= |j\delta|^2 + O(\frac{1}{|j\delta|}), \qquad \widetilde{\varphi}_j(s) = \sin js + O(\frac{1}{|j\delta|}), \\ \mu_j &= |j\delta|^2 + O(\frac{1}{|j\delta|}), \qquad \varphi_j(s) = \cos js + O(\frac{1}{|j\delta|}). \end{aligned} \tag{15}$$

By the first equation in (15), the eigenvalue $|\gamma|^2 = |\beta|^2 + |j\delta|^2$ of $L_D(0)$ corresponds the eigenvalue $|\beta|^2 + \widetilde{\mu}_j$ of $L_D(q^\delta)$; and by the second equation in (15), the eigenvalue $|\gamma|^2 = |\beta|^2 + |j\delta|^2$ of $L_N(0)$ corresponds the eigenvalue $|\beta|^2 + \mu_j$ of $L_N(q^\delta)$. Now we prove that there is an eigenvalue Λ_N of $L_D(q)$ which is close to the eigenvalue $|\beta|^2 + \widetilde{\mu}_j$

of $L_D(q^\delta)$ and that there is an eigenvalue Υ_N of $L_N(q)$ which is closed to the eigenvalue $|\beta|^2+\mu_j$ of $L_N(q^\delta)$. For this we use the binding formula for $L_D(q)$ and $L_D(q^\delta)$

$$(\Lambda_N-\widetilde{\lambda}_{j,\beta})(\Psi_N,\widetilde{\Theta}_{j,\beta})=(\Psi_N,(q(x)-q^\delta(x))\widetilde{\Theta}_{j,\beta}), \tag{16}$$

and the binding formula for $L_N(q)$ and $L_N(q^\delta)$

$$(\Upsilon_N-\lambda_{j,\beta})(\Phi_N,\Theta_{j,\beta})=(\Phi_N,(q(x)-q^\delta(x))\Theta_{j,\beta}). \tag{17}$$

Now as in the non-resonance case, we decompose $(q(x)-q^\delta(x))\widetilde{\Theta}_{j,\beta}$ by $\widetilde{\Theta}_{j',\beta'}$ and $(q(x)-q^\delta(x))\Theta_{j,\beta}$ by $\Theta_{j',\beta'}$ then put these decompositions into (16) and (17), respectively. Let us find these decompositions. Writing (11) for every $\gamma_1\in\Gamma(\rho^\alpha)$ and using (8), we have

$$\gamma_1=n_1\delta+\beta_1,\qquad v_{\gamma_1}(x)=(\cos n_1 s)v_{\beta_1}(x),$$

$$q(x)-Q(s)=\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)v_{\beta_1}(x)+O(\rho^{-p\alpha}), \tag{18}$$

where $\beta_1\in\Gamma_\delta$, $d(\beta_1,n_1)=\int_F q(x)(\cos n_1 s)v_{\beta_1}(x)dx$ and
$\Gamma'(\rho^\alpha)=\{(\beta_1,n_1):\beta_1\in\Gamma_\delta\setminus\{0\},n_1\in Z,n_1\delta+\beta_1\in\Gamma(\rho^\alpha)\}$.

The fact that $\gamma=j\delta+\beta\in V_\delta(\rho^{\alpha_1})\setminus E_2$ implies

$$|j|<r_1,\qquad r_1\equiv\rho^{\alpha_1}|\delta|^2+1 \tag{19}$$

and $\beta\notin V_{e_k}(\rho^{\alpha_1})$, for all $e_k\neq\delta$, by which we have

$$|\beta^k|>\frac{1}{3}\rho^{\alpha_1},\qquad \forall k:e_k\neq\delta. \tag{20}$$

Clearly for $(\beta_1,n_1)\in\Gamma'(p\rho^\alpha)$, we have $|n_1\delta+\beta_1|<p\rho^\alpha$, and since β_1 is orthogonal to δ,

$$|\beta_1|<p\rho^\alpha,|n_1|<p\rho^\alpha,|n_1|<\frac{1}{2}r_1, \tag{21}$$

(see 19).

Now we prove that

$$\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)v_{\beta_1}(x)v_\beta(x)=\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)v_{\beta_1+\beta}(x), \tag{22}$$

for all $\beta \in \Gamma_{e_i}$ satisfying (20).

Clearly $v_\beta(x) = \frac{1}{|A_\beta|}\sum_{\alpha\in A_\beta} e^{i\langle\alpha,x\rangle}$, where $A_\beta = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in R^d : |\alpha_i| = |\beta^i|, i = 1, 2, ..., d\}$ and $|A_\beta|$ is the number of vectors in A_β. Using these, it is not difficult to verify that for all $\beta \in \Gamma_{e_i}$ satisfying (20) and for all a such that $(a, n_1) \in \Gamma'(\rho^\alpha)$, the following relations hold:

$$v_a(x)v_\beta(x) = \frac{1}{|A_a|}\frac{1}{|A_\beta|}\sum_{\gamma'\in A_a}\sum_{\alpha\in A_{\beta+\gamma'}} e^{i\langle\alpha,x\rangle} = \frac{1}{|A_a|}\sum_{\gamma'\in A_a} v_{\beta+\gamma'}, \tag{23}$$

since $|A_\beta| = |A_{\beta+\gamma'}| = 2^{d-1}$, because all components of β_i and $\beta_i + \gamma_i^{'}$ for all $i : e_i \neq \delta$ are different from zero and $\beta_k = 0$, $\beta_k + \gamma_k^{'} = 0$ for $k : e_k = \delta$. Really, the condition (20) implies $|\beta_i| > \frac{1}{3}\rho^{\alpha_1}$, $\forall i \neq k$. Also, if $(a, n_1) \in \Gamma'(\rho^\alpha)$, then for all $\gamma^{'} \in A_a$ we have $|\gamma_i^{'}| < \rho^\alpha$, $\forall i \neq k$. Therefore, $|\beta_i + \gamma_i^{'}| \geq ||\beta_k| - |\gamma_k^{'}|| > \frac{1}{4}\rho^{3\alpha}$.

The set A_a consists of the vectors $a^1, a^2, ..., a^s$, where $s = |A_a|$ and clearly,

$$A_{a^1} = A_{a^2} = ... = A_{a^s} = A_a, \qquad v_{a^1} = v_{a^2} = ... = v_{a^s} = v_a. \tag{24}$$

Hence in (23), the vector a can be replaced by $a^1, a^2, ..., a^s$. Summing the obtained s equality and using (24), we get

$$\sum_{k=1}^{s} v_{a^k}(x)v_\beta(x) = \sum_{\gamma'\in A_a} v_{\beta+\gamma'}(x) \Leftrightarrow \sum_{\gamma'\in A_a} v_{\gamma'}(x)v_\beta(x) = \sum_{\gamma'\in A_a} v_{\beta+\gamma'}(x).$$

Thus, we have

$$\sum_{\gamma'\in A_a} d(\gamma', n_1)(\cos n_1 s)v_{\gamma'}(x)v_\beta(x) = \sum_{\gamma'\in A_a} d(\gamma', n_1)(\cos n_1 s)v_{\gamma'+\beta}(x), \tag{25}$$

for all $n_1 \in \mathbb{Z}$, since $d(\gamma', n_1)\cos n_1 s = d(a, n_1)\cos n_1 s$, for all $\gamma' \in A_a$, $n_1 \in \mathbb{Z}$. Clearly, there exist vectors $\beta_1, \beta_2, ..., \beta_m \in \Gamma_{e_i}$ such that

$$\Gamma'(\rho^\alpha) \subseteq (\bigcup_{j=1}^{m} A_{\beta_j}) \times \{n_1 \in \mathbb{Z} : |n_1| < \frac{1}{2}r_1\}. \tag{26}$$

In (25), replacing a by β_j, for $j = 1, 2, ..., m$, summing all obtained m equations and using

(26), we get (22). Similarly, one can prove

$$\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)v_{\beta_1}(x)u_\beta(x) = \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)u_{\beta_1+\beta}(x), \tag{27}$$

for all $\beta \in \Gamma_{e_i}$ satisfying (20).

Now multiplying the both sides of the equation (18) by $\widetilde{\Theta}_{j',\beta'}$, where β' satisfies (20) and using (27), we obtain

$$\begin{aligned}(q(x)-Q(s))\widetilde{\Theta}_{j',\beta'} &= \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)v_{\beta_1}\widetilde{\Theta}_{j',\beta'} + O(\rho^{-p\alpha}) \\ &= \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)\widetilde{\varphi}_{j'}(s)u_{\beta_1+\beta'} + O(\rho^{-p\alpha}). \end{aligned} \tag{28}$$

Similarly, multiplying the both sides of the equation (18) by $\Theta_{j',\beta'}$ and using (22), we get

$$(q(x)-Q(s))\Theta_{j',\beta'} = \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)(\cos n_1 s)\varphi_{j'}(s)v_{\beta_1+\beta'} + O(\rho^{-p\alpha}), \tag{29}$$

To decompose the right hand sides of (16) and (17) by $\widetilde{\Theta}_{j',\beta'}$ and $\Theta_{j',\beta'}$, respectively, we use the following lemmas:

Lemma 1 *Let r be a number no less than r_1, i.e. $r \geq r_1$, and j, m be integers satisfying $|j|+1 < r, |m| \geq 2r$. Then,*

$$(\varphi_j(s), \cos ms) = O(\frac{1}{|m\delta|^{l-1}}), \tag{30}$$

$$\varphi_j(s) = \sum_{|m|<2r} (\varphi_j(s), \cos ms)\cos ms + O(\frac{1}{\rho^{(l-2)\alpha}}) \tag{31}$$

and

$$|(\widetilde{\varphi}_j(s), \sin ms)| = O(\frac{1}{|m\delta|^{l-1}}), \tag{32}$$

$$\widetilde{\varphi}_j(s) = \sum_{|m|<2r} (\widetilde{\varphi}_j(s), \sin ms)\sin ms + O(\frac{1}{\rho^{(l-2)\alpha}}). \tag{33}$$

Proof. First we prove (30) and (31) using the following binding formula for $T_N^\delta(Q(s))$ and $T_N^\delta(0)$

$$(\mu_j - |m\delta|^2)(\varphi_j(s), \cos ms) = (\varphi_j(s)Q(s), \cos ms). \tag{34}$$

Using the obvious decomposition (see (7)) for $Q(s)$,

$$Q(s) = \sum_{|l_1\delta|<\frac{\lfloor m\delta\rfloor}{2l}} q_{l_1\delta}\cos l_1 s + O(|m\delta|^{-(l-1)}). \tag{35}$$

into (34), we get

$$\begin{aligned}(\mu_j - |m\delta|^2)(\varphi_j(s), \cos ms) &= (\varphi_j(s) \sum_{|l_1\delta|<\frac{\lfloor m\delta\rfloor}{2l}} q_{l_1\delta}\cos l_1 s, \cos ms) + O(|m\delta|^{-(l-1)}) \\ &= \sum_{|l_1\delta|<\frac{\lfloor m\delta\rfloor}{2l}} q_{l_1\delta}(\varphi_j(s), \cos l_1 s.\cos ms) + O(|m\delta|^{-(l-1)}) \\ &= \sum_{|l_1\delta|<\frac{\lfloor m\delta\rfloor}{2l}} q_{l_1\delta}(\varphi_j(s), \frac{1}{2}[\cos(m+l_1)s + \cos(m-l_1)s]) + O(|m\delta|^{-(l-1)}) \\ &= \sum_{|l_1\delta|<\frac{\lfloor m\delta\rfloor}{2l}} q_{l_1\delta}(\varphi_j(s), \cos(m-l_1)s) + O(|m\delta|^{-(l-1)}).\end{aligned}$$

And, again using (34), we get

$$(\mu_j - |m\delta|^2)(\varphi_j(s), \cos ms) = \sum_{|l_1\delta|<\frac{\lfloor m\delta\rfloor}{2l}} q_{l_1\delta}\frac{(\varphi_j(s)Q(s), \cos(m-l_1)s)}{\mu_j - |(m-l_1)\delta|^2} + O(|m\delta|^{-(l-1)}).$$

Putting (35) into the last equation, we obtain

$$\begin{aligned}(\mu_j - |m\delta|^2)(\varphi_j(s), \cos ms) \quad = \quad & \sum_{\substack{|l_1\delta|<\frac{\lfloor m\delta\rfloor}{2l}, \\ |l_2\delta|<\frac{\lfloor m\delta\rfloor}{2l}}} q_{l_1\delta}q_{l_2\delta}\frac{(\varphi_j(s), \cos(m-l_1-l_2)s)}{\mu_j - |(m-l_1)\delta|^2} \\ & + O(|m\delta|^{-(l-1)}).\end{aligned}$$

In this way, iterating $k = [\frac{l}{2}]$ times and dividing both sides of the obtained equation by

$\mu_j - |m\delta|^2$, we have

$$(\varphi_j(s), \cos ms) = \sum_{\substack{|l_1\delta|<\frac{|m\delta|}{2l},\dots,\\ |l_k\delta|<\frac{|m\delta|}{2l}}} q_{l_1\delta}...q_{l_k\delta}\frac{(\varphi_j(s), \cos(m-l_1-...-l_k)s)}{\prod_{t=0}^{k-1}(\mu_j - |(m-l_1-...-l_t)\delta|^2} + O(|m\delta|^{-(l-1)}), \tag{36}$$

where the integers $m, l_1, ..., l_k$ satisfy the conditions
$|l_i| < \frac{|m|}{2l}, i = 1, 2, ..., k, \quad |j|+1 < \frac{|m|}{2}$ (see assumption of the lemma). These conditions imply that $||m - l_1 - ... - l_t| - |j|| > \frac{|m|}{5}$. This together with (15) give

$$\frac{1}{|\mu_j - |(m-l_1-...-l_t)\delta|^2|} = \frac{1}{||j\delta|^2 + O(\frac{1}{|j\delta|}) - |(m-l_1-...-l_t)\delta|^2|} = O(|m\delta|^{-2}), \tag{37}$$

for $t = 0, 1, ..., k-1$. Hence by (36),(37) and (9), we have $|(\varphi_j(s), \cos ms)| = O(|m\delta|^{-(l-1)})$. (30) is proved.

To prove (31), for j satisfying $|j| + 1 < r$, we write the Fourier series of $\varphi_j(s)$ with respect to the basis $\{\cos ms : m \in Z\}$, i.e.,

$$\begin{aligned}\varphi_j(s) &= \sum_{m\in Z}(\varphi_j(s), \cos ms)\cos ms \\ &= \sum_{|m|<2r}(\varphi_j(s), \cos ms)\cos ms + \sum_{m\geq 2r}(\varphi_j(s), \cos ms)\cos ms.\end{aligned}$$

By (30), for $|m| \geq 2r$ and $|j| + 1 < r$, we have $(\varphi_j(s), \cos ms) = O(|m\delta|^{-(l-1)})$. Using this relation, we get

$$\varphi_j(s) = \sum_{|m|<2r}(\varphi_j(s), \cos ms)\cos ms + O(|m\delta|^{-(l-2)}),$$

since $|m\delta| > \rho^\alpha$, (31) is proved.

In the same way, instead of (34), using the the following binding formula for $T_D^\delta(Q(s))$ and $T_D^\delta(0)$

$$(\widetilde{\mu}_j - |m\delta|^2)(\widetilde{\varphi}_j(s), \sin ms) = (\widetilde{\varphi}_j(s)Q(s), \sin ms), \tag{38}$$

(32) and (33) can be easily proved. □

Lemma 2 *Let r be a number no less than r_1, i.e. $r \geq r_1$, and j be integer satisfying $|j|+1<r$. Then*

$$(\cos n_1 s)\varphi_j(s) = \sum_{|j_1|<6r} a(n_1, j, j+j_1)\varphi_{j+j_1}(s) + O(r^{-(l-3)}), \tag{39}$$

$$(\cos n_1 s)\widetilde{\varphi}_j(s) = \sum_{|j_1|<6r} \widetilde{a}(n_1, j, j+j_1)\widetilde{\varphi}_{j+j_1}(s) + O(r^{-(l-3)}), \tag{40}$$

for $(n_1, \beta_1) \in \Gamma'(p_1\rho^\alpha)$, where $a(n_1, j, j+j_1) = ((\cos n_1 s)\varphi_j(s), \varphi_{j+j_1}(s))$ and $\widetilde{a}(n_1, j, j+j_1) = ((\cos n_1 s)\widetilde{\varphi}_j(s), \widetilde{\varphi}_{j+j_1}(s))$.

Proof. First we prove (39). Consider the Fourier series of $(\cos n_1 s)\varphi_j(s)$ with respect to the basis $\{\varphi_{j+j_1}(s) : j_1 \in Z\}$

$$\begin{aligned}(\cos n_1 s)\varphi_j(s) &= \sum_{j_1\in Z} ((\cos n_1 s)\varphi_j(s), \varphi_{j+j_1}(s))\varphi_{j+j_1}(s) \\ &= \sum_{|j_1|<6r} a(n_1, j, j+j_1)\varphi_{j+j_1}(s) + \sum_{|j_1|\geq 6r} a(n_1, j, j+j_1)\varphi_{j+j_1}(s).\end{aligned}$$

To prove (39), we must prove $\sum_{|j_1|\geq 6r} |a(n_1, j, j+j_1)| = O(r^{-(l-3)})$ or, equivalently,

$$|a(n_1, j, j+j_1)| = O(r^{-(l-2)}), \qquad \forall j_1 : |j_1| \geq 6r. \tag{41}$$

Decomposing $\varphi_j(s)$ by $\cos ms$, we have $\varphi_j(s) = \sum_{m\in Z}(\varphi_j(s), \cos ms)\cos ms$ and multiplying this decomposition by $\cos n_1 s$, we obtain

$$\begin{aligned}(\cos n_1 s)\varphi_j(s) &= \sum_{m\in Z} (\varphi_j(s), \cos ms)(\cos ms)(\cos n_1 s), \\ &= \sum_{m\in Z} (\varphi_j(s), \cos ms)\frac{1}{2}[\cos(n_1+m)s + \cos(n_1-m)s] \\ &= \sum_{m\in Z} (\varphi_j(s), \cos ms)\cos(n_1+m)s. \end{aligned} \tag{42}$$

Using (42) and the decomposition $\varphi_{j+j_1}(s) = \sum_{k\in Z}(\varphi_{j+j_1}(s), \cos ks)\cos ks$, we get

$$\begin{aligned} a(n_1, j, j+j_1) &= ((\cos n_1 s)\varphi_j(s), \varphi_{j+j_1}(s)) \\ &= (\sum_{m\in Z}(\varphi_j(s), \cos ms)\cos(n_1+m)s, \sum_{k\in Z}(\varphi_{j+j_1}(s), \cos ks)\cos ks) \\ &= \sum_{m,k\in Z}(\varphi_j(s), \cos ms)\overline{(\varphi_{j+j_1}(s), \cos ks)}(\cos(n_1+m)s, \cos ks) \\ &= \sum_{k\in Z}(\varphi_j(s), \cos(k-n_1)s)\overline{(\varphi_{j+j_1}(s), \cos ks)}. \end{aligned} \tag{43}$$

Consider the following two cases:

Case 1: $|k| > \frac{1}{2}|j_1| \geq 3r$. Since $|n_1| + 1 < r$ (see 21), $|k - n_1| > 2r$. Hence by (31)

$$\sum_{|k|>\frac{1}{2}|j_1|} |(\varphi_j(s), \cos(k-n_1)s)| = \sum_{|k-n_1|>2r} O(\frac{1}{|(k-n_1)\delta|^{l-1}}) = O(r^{-(l-2)}). \tag{44}$$

Case 2: $|k| \leq \frac{1}{2}|j_1|$. By assumptions $|j| < r$ and $|j_1| \geq 6r$, we have $|j_1 + j| > 5r$. For any integers $l_1, ..., l_t$ satisfying $|l_i| < \frac{|j_1|}{3l}, i = 1, 2, ..., t$, where $t = [\frac{l}{2}]$, we have $|j_1 + j| - |k - l_1 - ... - l_t| > \frac{1}{6}|j_1|$. This together with (15) gives

$$\frac{1}{|\mu_j - |(k - l_1 - ... - l_i)\delta|^2|} = O(|j_1\delta|^{-2}), \tag{45}$$

for $i = 0, 1, ..., t$. Arguing as the proof of (31), we get

$$\sum_{|k|\leq\frac{1}{2}|j_1|} |(\varphi_{j_1+j}(s), \cos ks)| = O(r^{-(l-2)}). \tag{46}$$

Using (44) and (46), we have

$$\begin{aligned} |a(n_1, j, j+j_1)| \leq & \sum_{|k|\leq\frac{1}{2}|j_1|} |(\varphi_j(s), \cos(k-n_1)s)||\overline{(\varphi_{j+j_1}(s), \cos ks)}| \\ & + \sum_{|k|>\frac{1}{2}|j_1|} |(\varphi_j(s), \cos(k-n_1)s)||\overline{(\varphi_{j+j_1}(s), \cos ks)}| = O(r^{-(l-2)}). \end{aligned}$$

(41), hence (39) is proved.

Similarly, to prove (40), instead of (41), we must prove

$$|\widetilde{a}(n_1, j, j + j_1)| = O(r^{-(l-2)}), \qquad \forall j_1 : |j_1| \geq 6r \tag{47}$$

which can be proved in the same way as (41). Lemma is proved. □

Now substituting (39) into (29) and (40) into (28), we get

$$(q(x) - Q(s))\Theta_{j',\beta'} = \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} A(j', \beta', j' + j_1, \beta' + \beta_1)\Theta_{j'+j_1,\beta'+\beta_1} + O(\rho^{-p\alpha}), \tag{48}$$

and

$$(q(x) - Q(s))\widetilde{\Theta}_{j',\beta'} = \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} \widetilde{A}(j', \beta', j' + j_1, \beta' + \beta_1)\widetilde{\Theta}_{j'+j_1,\beta'+\beta_1} + O(\rho^{-p\alpha}), \tag{49}$$

respectively, for every j' satisfying $|j'| + 1 < r$, where

$$Q(\rho^\alpha, 6r) = \{(j, \beta) : |j\delta| < 6r, 0 < |\beta| < \rho^\alpha\},$$

$$A(j', \beta', j' + j_1, \beta' + \beta_1) = \sum_{n_1:(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1, n_1)a(n_1, j', j' + j_1),$$

and

$$\widetilde{A}(j', \beta', j' + j_1, \beta' + \beta_1) = \sum_{n_1:(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1, n_1)\widetilde{a}(n_1, j', j' + j_1).$$

We need to prove that

$$\sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} |A(j', \beta', j' + j_1, \beta' + \beta_1)| < c_1 \tag{50}$$

and

$$\sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} |\widetilde{A}(j', \beta', j' + j_1, \beta' + \beta_1)| < c_2. \tag{51}$$

First we prove (50). By definition of $A(j',\beta',j'+j_1,\beta'+\beta_1)$, $d(\beta_1,n_1)$, (9) and (43), we have

$$\sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} |A(j',\beta',j'+j_1,\beta'+\beta_1)| \le \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} |d(\beta_1,n_1)| \sum_{|j_1|\le 6r} |a(n_1,j',j'+j_1)|$$
$$\le M \sum_{k\in Z} |(\varphi_j(s),\cos(k-n_1)s)| \sum_{|j_1|\le 6r} |(\varphi_{j+j_1}(s),\cos ks)|.$$

Hence (50) follows from the inequalities $\sum_{k\in Z}|(\varphi_j(s),\cos(k-n_1)s)|<c_3$ and $\sum_{|j_1|\le 6r}|(\varphi_{j+j_1}(s),\cos ks)|<c_4$, which can be easily obtained by (34). (51) can be proved similarly.

The decomposition (48) together with the binding formula (17) for $L_N(q)$ and $L_N(q^\delta)$ give

$$(\Upsilon_N-\lambda_{j',\beta'})(\Phi_N,\Theta_{j',\beta'}) = (\Phi_N,(q(x)-Q(s))\Theta_{j',\beta'})$$
$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} A(j',\beta',j'+j_1,\beta'+\beta_1)(\Phi_N,\Theta_{j'+j_1,\beta'+\beta_1}) + O(\rho^{-p\alpha}) \qquad (52)$$

and the decomposition (49) together with the binding formula (16) for $L_D(q)$ and $L_D(q^\delta)$ give

$$(\Lambda_N-\widetilde{\lambda}_{j',\beta'})(\Psi_N,\widetilde{\Theta}_{j',\beta'}) = (\Psi_N,(q(x)-Q(s))\widetilde{\Theta}_{j',\beta'})$$
$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} \widetilde{A}(j',\beta',j'+j_1,\beta'+\beta_1)(\Psi_N,\widetilde{\Theta}_{j'+j_1,\beta'+\beta_1}) + O(\rho^{-p\alpha}). \qquad (53)$$

If the conditions (iterability conditions for the triple (N,j',β'))

$$|\Upsilon_N-\lambda_{j',\beta'}|>c_7 \quad \text{and} \quad |\Lambda_N-\widetilde{\lambda}_{j',\beta'}|>c_8 \qquad (54)$$

hold, then the formulas (52) and (53) can be written in the following forms:

$$(\Phi_N,\Theta_{j',\beta'}) = \frac{(\Phi_N,(q(x)-Q(s))\Theta_{j',\beta'})}{\Upsilon_N-\lambda_{j',\beta'}}$$
$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} \frac{A(j',\beta',j'+j_1,\beta'+\beta_1)(\Phi_N,\Theta_{j'+j_1,\beta'+\beta_1})}{\Upsilon_N-\lambda_{j',\beta'}} + O(\rho^{-p\alpha}) \qquad (55)$$

and

$$(\Psi_N, \widetilde{\Theta}_{j',\beta'}) = \frac{(\Psi_N, (q(x)-Q(s))\widetilde{\Theta}_{j',\beta'})}{\Lambda_N - \widetilde{\lambda}_{j',\beta'}}$$

$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} \frac{\widetilde{A}(j',\beta',j'+j_1,\beta'+\beta_1)(\Psi_N,\widetilde{\Theta}_{j'+j_1,\beta'+\beta_1})}{\Lambda_N - \widetilde{\lambda}_{j',\beta'}} + O(\rho^{-p\alpha}), \quad (56)$$

respectively. Using (52), (55), we will find Υ_N, which is close to $\lambda_{j,\beta}$; and using (53), (56), we will find Λ_N, which is close to $\widetilde{\lambda}_{j,\beta}$, where $|j|+1<r_1$. For this, first in (52) and (53) instead of j',β', taking j and β, hence instead of r taking r_1, we get

$$(\Upsilon_N - \lambda_{j,\beta})(\Phi_N,\Theta_{j,\beta}) = (\Phi_N,(q(x)-Q(s))\Theta_{j,\beta})$$

$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)} A(j,\beta,j+j_1,\beta+\beta_1)(\Phi_N,\Theta_{j+j_1,\beta+\beta_1}) + O(\rho^{-p\alpha}) \quad (57)$$

and

$$(\Lambda_N - \widetilde{\lambda}_{j,\beta})(\Psi_N,\widetilde{\Theta}_{j,\beta}) = (\Psi_N,(q(x)-Q(s))\widetilde{\Theta}_{j,\beta})$$

$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)} \widetilde{A}(j,\beta,j+j_1,\beta+\beta_1)(\Psi_N,\widetilde{\Theta}_{j+j_1,\beta+\beta_1}) + O(\rho^{-p\alpha}), \quad (58)$$

respectively. To iterate (57) and (58) using (55) and (56), respectively, for $j'=j+j_1$ and $\beta'=\beta+\beta_1$, we will prove that there is a number N satisfying

$$|\Upsilon_N - \lambda_{j+j_1,\beta+\beta_1}| > \frac{1}{2}\rho^{\alpha_2}, \qquad |\Lambda_N - \widetilde{\lambda}_{j+j_1,\beta+\beta_1}| > \frac{1}{2}\rho^{\alpha_2}, \quad (59)$$

where $|j+j_1|+1<7r_1\equiv r_2$, since $|j|+1<r_1$ and $|j_1|<6r_1$. Then $(j+j_1,\beta+\beta_1)$ satisfies both conditions in (54). This means that, in formulas (55) and (56), the pair (j',β') can be replaced by the pair $(j+j_1,\beta+\beta_1)$. Then we get

$$(\Phi_N,\Theta_{j+j_1,\beta+\beta_1}) = O(\rho^{-p\alpha}) +$$

$$\sum_{(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)} \frac{A(j+j_1,\beta+\beta_1,j+j_1+j_2,\beta+\beta_1+\beta_2)(\Phi_N,\Theta_{j+j_1+j_2,\beta+\beta_1+\beta_2})}{\Upsilon_N - \lambda_{j+j_1,\beta+\beta_1}} \quad (60)$$

and

$$(\Psi_N,\widetilde{\Theta}_{j+j_1,\beta+\beta_1}) = O(\rho^{-p\alpha}) +$$

$$\sum_{(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)} \frac{\widetilde{A}(j+j_1,\beta+\beta_1,j+j_1+j_2,\beta+\beta_1+\beta_2)(\Psi_N,\widetilde{\Theta}_{j+j_1+j_2,\beta+\beta_1+\beta_2})}{\Lambda_N - \widetilde{\lambda}_{j+j_1,\beta+\beta_1}}, \quad (61)$$

respectively. Putting the formula (60) into (57), we obtain

$$(\Upsilon_N - \lambda_{j,\beta})c(N,j,\beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\\ (\beta_2,j_2)\in Q(\rho^\alpha,6r_2)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)c(N,j^2,\beta^2)}{\Upsilon_N - \lambda_{j^1,\beta^1}} \tag{62}$$

and putting the formula (61) into (58), we get

$$(\Lambda_N - \widetilde{\lambda}_{j,\beta})b(N,j,\beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\\ (\beta_2,j_2)\in Q(\rho^\alpha,6r_2)}} \frac{\widetilde{A}(j,\beta,j^1,\beta^1)\widetilde{A}(j^1,\beta^1,j^2,\beta^2)b(N,j^2,\beta^2)}{\Lambda_N - \widetilde{\lambda}_{j^1,\beta^1}}, \tag{63}$$

where $c(N,j,\beta) = (\Phi_N, \Theta_{j,\beta})$, $b(N,j,\beta) = (\Psi_N, \widetilde{\Theta}_{j,\beta})$ $j^k = j + j_1 + j_2 + ... + j_k$ and $\beta^k = \beta + \beta_1 + \beta_2 + ... + \beta_k$. Thus we will find a number N such that $c(N,j,\beta)$ and $b(N,j,\beta)$ are not too small and the conditions in (59) are satisfied.

Similar investigation for quasiperiodic boundary condition was made in [12]. Arguing as in that paper, one can easily obtain the following results:

Result (a) Suppose $h_1(x), h_2(x), ..., h_m(x) \in L_2(F)$, where $m = [\frac{d}{2\alpha_2}] + 1$. Then for every eigenvalue $\lambda_{j,\beta}$ of the operator $L_N(q^\delta)$, there exists an eigenvalue Υ_N of $L_N(q)$ and for every eigenvalue $\widetilde{\lambda}_{j,\beta}$ of the operator $L_D(q^\delta)$, there exists an eigenvalue Λ_N of $L_D(q)$ satisfying

(i) $|\Upsilon_N - \lambda_{j,\beta}| < 2M$, $|\Lambda_N - \widetilde{\lambda}_{j,\beta}| < 2M$, where $M = \sup|q(x)|$,

(ii) $|c(N,j,\beta)| > \rho^{-q\alpha}$, $|b(N,j,\beta)| > \rho^{-q\alpha}$, where $q\alpha = [\frac{d}{2\alpha} + 2]\alpha$,

(iii) $|c(N,j,\beta)|^2 > \frac{1}{2m}\sum_{i=1}^m |(\Phi_N, \frac{h_i}{\|h_i\|})|^2 > \frac{1}{2m}|(\Phi_N, \frac{h_i}{\|h_i\|})|^2$,
$|b(N,j,\beta)|^2 > \frac{1}{2m}\sum_{i=1}^m |(\Psi_N, \frac{h_i}{\|h_i\|})|^2 > \frac{1}{2m}|(\Psi_N, \frac{h_i}{\|h_i\|})|^2, \quad \forall i = 1,2,...,m.$

(b) Let $\gamma = \beta + j\delta \in V_\delta(\alpha) \setminus E_2$ and $(\beta_1, j_1) \in Q(\rho^\alpha, 6r_1), (\beta_k, j_k) \in Q(\rho^\alpha, 6r_k)$, where $r_k = 7r_{k-1}$ for $k = 2,3,...,p$. Then for $k = 1,2,3,...,p_1$, we have

$$|\lambda_{j,\beta} - \lambda_{j^k,\beta^k}| > \frac{3}{5}\rho^{\alpha_2}, \quad \forall \beta^k \neq \beta \tag{64}$$

and

$$|\widetilde{\lambda}_{j,\beta} - \widetilde{\lambda}_{j^k,\beta^k}| > \frac{3}{5}\rho^{\alpha_2}, \qquad \forall \beta^k \neq \beta. \tag{65}$$

Now we prove the estimates (i), (ii) and (iii) of the Result(a) for the Neumann problem: Let A, B, C be the set of indexes N satisfying (i), (ii), (iii), respectively. Using the binding formula (17) for $L_N(q)$ and $L_N(q^{e_i})$ and the Bessel's inequality, we get

$$\begin{aligned}\sum_{N\notin A} |c(N,j,\beta)|^2 &= \sum_{N\notin A} |\frac{(\Phi_N, (q(x)-Q(s))\Theta_{j,\beta})}{\Upsilon_N - \lambda_{j,\beta}}|^2 \\ &\leq \frac{1}{4M^2}\|(q(x)-Q(s))\Theta_{j,\beta}\|^2 \leq \frac{1}{4}.\end{aligned}$$

Hence by Parseval's relation, we obtain

$$\sum_{N\in A} |c(N,j,\beta)|^2 > \frac{3}{4}.$$

Using the fact that the number of indexes N in A is less than $\rho^{d\alpha}$ and by the relation $N \notin B \quad \Rightarrow \quad |c(N,j,\beta)| < \rho^{-q\alpha}$, we have

$$\sum_{N\in A\setminus B} |c(N,j,\beta)|^2 < \rho^{d\alpha}\rho^{-q\alpha} < \rho^{-\alpha}.$$

Since $A = (A \setminus B) \bigcup (A \bigcap B)$, by above inequalities, we get

$$\frac{3}{4} < \sum_{N\in A} |c(N,j,\beta)|^2 = \sum_{N\in A\setminus B} |c(N,j,\beta)|^2 + \sum_{N\in A\bigcap B} |c(N,j,\beta)|^2,$$

which implies

$$\sum_{N\in A\bigcap B} |c(N,j,\beta)|^2 > \frac{3}{4} - \rho^{-\alpha} > \frac{1}{2}. \tag{66}$$

Now, suppose that $A \bigcap B \bigcap C = \emptyset$, i.e., for all $N \in A \bigcap B$, the condition (iii) does not hold. Then by (66) and Bessel's inequality, we have

$$\begin{aligned}\frac{1}{2} \quad &< \quad \sum_{N\in A\bigcap B} |c(N,j,\beta)|^2 \leq \sum_{N\in A\bigcap B} \frac{1}{2m}\sum_{i=1}^{m} |(\Phi_N, \frac{h_i}{\|h_i\|})|^2 \\ &= \quad \frac{1}{2m}\sum_{i=1}^{m}\sum_{N\in A\bigcap B} |(\Phi_N, \frac{h_i}{\|h_i\|})|^2 < \frac{1}{2m}\sum_{i=1}^{m} \|\frac{h_i}{\|h_i\|}\|^2 = \frac{1}{2},\end{aligned}$$

which is a contradiction.

Similarly, the estimates (i), (ii) and (iii) for the Dirichlet problem can be easily obtained.

Now we consider the following functions:

$$h_i(x) = \sum_{\substack{(j_1,\beta_1)\\(j_2,\beta_2)}} \frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j^2,\beta^2)\Theta_{j^2,\beta^2}(x)}{(\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1})^i} \tag{67}$$

and

$$\widetilde{h}_i(x) = \sum_{\substack{(j_1,\beta_1)\\(j_2,\beta_2)}} \frac{\widetilde{A}(j,\beta,j+j_1,\beta+\beta_1)\widetilde{A}(j+j_1,\beta+\beta_1,j^2,\beta^2)\widetilde{\Theta}_{j^2,\beta^2}(x)}{(\widetilde{\lambda}_{j,\beta}-\widetilde{\lambda}_{j+j_1,\beta+\beta_1})^i}, \tag{68}$$

where $(j_1,\beta_1)\in Q(\rho^{\alpha},6r_1)$ and $(j_2,\beta_2)\in Q(\rho^{\alpha},6r_2)$. Since $\{\Theta_{j^2,\beta^2}(x)\}$ is a total system and $\beta_1\neq 0$, by (50) and (64), we have $\sum_{(j',\beta')}|(h_i(x),\Theta_{j',\beta'})|^2\leq c_9\rho^{-2i\alpha_2}$, i.e.,

$$h_i(x)\in L_2(F) \quad \text{and} \quad \|h_i(x)\| = O(\rho^{-i\alpha_2}). \tag{69}$$

Similarly, using the fact that $\{\widetilde{\Theta}_{j^2,\beta^2}(x)\}$ is a total system, by (51) and (65), we get

$$\widetilde{h}_i(x)\in L_2(F) \quad \text{and} \quad \|\widetilde{h}_i(x)\| = O(\rho^{-i\alpha_2}). \tag{70}$$

Theorem 1 *a) For every eigenvalue $\lambda_{j,\beta}$ of the operator $L_N(q^{\delta})$ with $\beta+j\delta\in V_{\delta}(\rho^{\alpha_1})\setminus E_2$, there exists an eigenvalue Υ_N of the operator $L_N(q)$ satisfying*

$$\Upsilon_N = \lambda_{j,\beta}+O(\rho^{-\alpha_2}). \tag{71}$$

b) For every eigenvalue $\widetilde{\lambda}_{j,\beta}$ of the operator $L_D(q^{\delta})$ with $\beta+j\delta\in V_{\delta}(\rho^{\alpha_1})\setminus E_2$, there exists an eigenvalue Λ_N of the operator $L_D(q)$ satisfying

$$\Lambda_N = \widetilde{\lambda}_{j,\beta}+O(\rho^{-\alpha_2}). \tag{72}$$

Proof. a) By Result (a), for the chosen $h_i(x), i=1,2,...,m$ in (67), there exists a number N, satisfying (i), (ii), (iii). Since $\beta_1\neq 0$, by (64), we have

$$|\lambda_{j,\beta}-\lambda_{j^1,\beta^1}|>c_{10}\rho^{\alpha_2}.$$

The above inequality together with (i) imply

$$|\Upsilon_N-\lambda_{j^1,\beta^1}|>c_{11}\rho^{\alpha_2}.$$

Using the following well known decomposition

$$\frac{1}{|\Upsilon_N-\lambda_{j^1,\beta^1}|}=\sum_{i=1}^{m}\frac{|\Upsilon_N-\lambda_{j,\beta}|^{i-1}}{|\lambda_{j,\beta}-\lambda_{j^1,\beta^1}|^i}+O(\rho^{-(m+1)\alpha_2}),$$

we see that the formula (62) can be written as

$$\begin{aligned}(\Upsilon_N-\lambda_{j,\beta})c(N,j,\beta)&=O(\rho^{-p\alpha})\\&+\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)}}\frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j^2,\beta^2)c(N,j^2,\beta^2)}{\Upsilon_N-\lambda_{j+j_1,\beta+\beta_1}}\\&=\sum_{i=1}^{m}|\Upsilon_N-\lambda_{j,\beta}|^{i-1}(\Phi_N,\frac{h_i}{\|h_i\|})\|h_i\|+O(\rho^{-(m+1)\alpha_2}).\end{aligned}$$

Now dividing both sides of the last equation by $c(N,j,\beta)$ and using (ii), (iii), we have

$$\begin{aligned}|\Upsilon_N-\lambda_{j,\beta}| &\le \frac{|(\Phi_N,\frac{h_1}{\|h_1\|})|}{|c(N,j,\beta)|}\|h_1\|+\frac{|\Upsilon_N-\lambda_{j,\beta}||(\Phi_N,\frac{h_2}{\|h_2\|})|}{|c(N,j,\beta)|}\|h_2\|\\&\quad+...+\frac{|\Upsilon_N-\lambda_{j,\beta}|^{(m-1)}|(\Phi_N,\frac{h_m}{\|h_m\|})|}{|c(N,j,\beta)|}\|h_m\|+O(\rho^{-(m+1)\alpha_2+q\alpha})\\&\le\|h_1\|+2M\|h_2\|+...+(2M)^{m-1}\|h_m\|+O(\rho^{-(m+1)\alpha_2+q\alpha}).\end{aligned}$$

Hence by (69), we obtain

$$\Upsilon_N=\lambda_{j,\beta}+O(\rho^{-\alpha_2}),$$

since $(m+1)\alpha_2-q\alpha>\alpha_2$.

The part b) of the theorem can be proved similarly. Theorem is proved. □

It follows from (64),(65), (71) and (72) that the triples (N,j^k,β^k) for $k=1,2,...,p_1$, satisfy the iterability conditions in (54). In (55) and (56), instead of j',β' and r taking j^2,β^2 and r_3 , we have

$$c(N,j^2,\beta^2)=\sum_{(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)}\frac{A(j^2,\beta^2,j^3,\beta^3)(\Phi_N,\Theta_{j^3,\beta^3})}{\Upsilon_N-\lambda_{j^2,\beta^2}}+O(\rho^{-p\alpha})\qquad(73)$$

and

$$b(N,j^2,\beta^2)=\sum_{(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)}\frac{\widetilde{A}(j^2,\beta^2,j^3,\beta^3)(\Psi_N,\widetilde{\Theta}_{j^3,\beta^3})}{\Lambda_N-\widetilde{\lambda}_{j^2,\beta^2}}+O(\rho^{-p\alpha}),\qquad(74)$$

respectively.

To obtain the other terms of the asymptotic formulas of Υ_N and Λ_N, we iterate the formulas (52) and (53), respectively.

Now we isolate the terms with multiplicands $c(N,j,\beta)$ in the right hand side of (62); hence we get

$$\begin{aligned}(\Upsilon_N-\lambda_{j,\beta})c(N,j,\beta)&=O(\rho^{-p\alpha})\\&+\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)=(j,\beta)}}\frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j,\beta)}{\Upsilon_N-\lambda_{j^1,\beta^1}}c(N,j,\beta)\\&+\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)\neq(j,\beta)}}\frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)}{\Upsilon_N-\lambda_{j^1,\beta^1}}c(N,j^2,\beta^2).\qquad(75)\end{aligned}$$

Substituting the equation (73) into the second sum of the equation (75), we get

$$\begin{aligned}(\Upsilon_N-\lambda_{j,\beta})c(N,j,\beta)&=O(\rho^{-p\alpha})\\&+\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)=(j,\beta)}}\frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j,\beta)}{\Upsilon_N-\lambda_{j^1,\beta^1}}c(N,j,\beta)+\\&\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)\neq(j,\beta)\\(j_3,\beta_3)\in Q(\rho^\alpha,6r_3)}}\frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j^3,\beta^3)}{(\Upsilon_N-\lambda_{j^1,\beta^1})(\Upsilon_N-\lambda_{j^2,\beta^2})}c(N,j^3,\beta^3).\qquad(76)\end{aligned}$$

Again isolating the terms $c(N,j,\beta)$ in the last sum of the equation (76), we obtain

$$(\Upsilon_N-\lambda_{j,\beta})c(N,j,\beta)=[\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)=(j,\beta)}}\frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j,\beta)}{\Upsilon_N-\lambda_{j^1,\beta^1}}$$

$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)\\(j^2,\beta^2)\neq(j,\beta)\\(j^3,\beta^3)=(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j,\beta)}{\Upsilon_N-\lambda_{j^1,\beta^1}}]c(N,j,\beta)$$

$$\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j_3,\beta_3)\in Q(\rho^\alpha,6r_3)\\(j^2,\beta^2)\neq(j,\beta)\\(j^3,\beta^3)\neq(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j^3,\beta^3)}{(\Upsilon_N-\lambda_{j^1,\beta^1})(\Upsilon_N-\lambda_{j^2,\beta^2})}c(N,j^3,\beta^3)+O(\rho^{-p\alpha}). \quad (77)$$

In this way, iterating $2p$ times, we get

$$(\Upsilon_N-\lambda_{j,\beta})c(N,j,\beta)=[\sum_{k=1}^{2p}S'_k]c(N,j,\beta)+C'_{2p}+O(\rho^{-p\alpha}), \quad (78)$$

where

$$S'_k(\Upsilon_N,\lambda_{j,\beta})=\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\dots,\\(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,6r_{k+1})\\(j^{k+1},\beta^{k+1})=(j,\beta)\\(j^s,\beta^s)\neq(j,\beta),s=2,\dots,k}} (\prod_{i=1}^{k}\frac{A(j^{i-1},\beta^{i-1},j^i,\beta^i)}{(\Upsilon_N-\lambda_{j^i,\beta^i})})A(j^k,\beta^k,j,\beta) \quad (79)$$

and

$$C'_k=\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\dots,\\(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,6r_{k+1})\\(j^s,\beta^s)\neq(j,\beta),s=2,\dots,k+1}} (\prod_{i=1}^{k}\frac{A(j^{i-1},\beta^{i-1},j^i,\beta^i)}{(\Upsilon_N-\lambda_{j^i,\beta^i})})A(j^k,\beta^k,j^{k+1},\beta^{k+1})c(N,j^{k+1},\beta^{k+1}). \quad (80)$$

Similarly, we isolate the terms with multiplicands $b(N,j,\beta)$ in the right hand side of (63), substitute the equation (74) into the obtained equation and iterate $2p$ times, we obtain

$$(\Lambda_N-\widetilde{\lambda}_{j,\beta})b(N,j,\beta)=[\sum_{k=1}^{2p}S''_k]b(N,j,\beta)+C''_{2p}+O(\rho^{-p\alpha}), \quad (81)$$

where

$$S''_k(\Lambda_N,\widetilde{\lambda}_{j,\beta})=\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\dots,\\(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,6r_{k+1})\\(j^{k+1},\beta^{k+1})=(j,\beta)\\(j^s,\beta^s)\neq(j,\beta),s=2,\dots,k}} (\prod_{i=1}^{k}\frac{\widetilde{A}(j^{i-1},\beta^{i-1},j^i,\beta^i)}{(\Lambda_N-\widetilde{\lambda}_{j^i,\beta^i})})\widetilde{A}(j^k,\beta^k,j,\beta) \quad (82)$$

and

$$C_k'' = \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\dots,\\ (j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,6r_{k+1})\\ (j^s,\beta^s)\neq(j,\beta),s=2,\dots,k+1}} (\prod_{i=1}^{k} \frac{\widetilde{A}(j^{i-1},\beta^{i-1},j^i,\beta^i)}{(\Lambda_N-\widetilde{\lambda}_{j^i,\beta^i})})\widetilde{A}(j^k,\beta^k,j^{k+1},\beta^{k+1})b(N,j^{k+1},\beta^{k+1}). \tag{83}$$

First we estimate S_k' and C_k'. For this, we consider the terms which appear in the denominators of (79) and (80). By the conditions under the summations in (79) and (80), we have $j_1+j_2+...+j_i\neq 0$ or $\beta_1+\beta_2+...+\beta_i\neq 0$, for $i=2,3,...,k$.

If $\beta_1+\beta_2+...+\beta_i\neq 0$, then by (64) and (71), we have

$$|\Upsilon_N-\lambda_{j^i,\beta^i}|>\frac{1}{2}\rho^{\alpha_2}. \tag{84}$$

If $\beta_1+\beta_2+...+\beta_i=0$, i.e., $j_1+j_2+...+j_i\neq 0$, then by well-known theorem

$$|\lambda_{j,\beta}-\lambda_{j^i,\beta^i}|=|\mu_j-\mu_{j^i}|>c_{13},$$

hence by (71), we obtain

$$|\Upsilon_N-\lambda_{j^i,\beta^i}|>\frac{1}{2}c_{13}. \tag{85}$$

Since $\beta_k\neq 0$ for all $k\leq 2p$, the relation $\beta_1+\beta_2+...+\beta_i=0$ implies $\beta_1+\beta_2+...+\beta_{i\pm1}\neq 0$. Therefore the number of multiplicands $\Upsilon_N-\lambda_{j^i,\beta^i}$ in (84) is no less than p. Thus by (50), (84) and (85), we get

$$S_1'=O(\rho^{-\alpha_2}),\qquad C_{2p}'=O(\rho^{-p\alpha_2}). \tag{86}$$

By similar calculations and considerations, it can be easily obtained that

$$S_1''=O(\rho^{-\alpha_2}),\qquad C_{2p}''=O(\rho^{-p\alpha_2}). \tag{87}$$

Theorem 2 *(a) For every eigenvalue $\lambda_{j,\beta}$ of $L_N(q^\delta)$ and for every eigenvalue $\widetilde{\lambda}_{j,\beta}$ of $L_D(q^\delta)$ such that $\beta+j\delta\in V_\delta(\rho^{\alpha_1})\setminus E_2$, there exists an eigenvalue Υ_N of the operator $L_N(q)$ and an eigenvalue Λ_N of the operator $L_D(q)$ satisfying*

$$\Upsilon_N=\lambda_{j,\beta}+E_{k-1}+O(\rho^{-k\alpha_2}) \tag{88}$$

and

$$\Lambda_N = \widetilde{\lambda}_{j,\beta} + E_{k-1} + O(\rho^{-k\alpha_2}), \tag{89}$$

respectively, where $E_0 = 0$, $E_s = \sum_{k=1}^{2p} S'_k(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta})$, $\widetilde{E}_0 = 0$, $\widetilde{E}_s = \sum_{k=1}^{2p} S''_k(\widetilde{E}_{s-1} + \widetilde{\lambda}_{j,\beta}, \widetilde{\lambda}_{j,\beta})$, $s = 1, 2, ...$

(b) If

$$|\Upsilon_N - \lambda_{j,\beta}| < c_{14}, \qquad |\Lambda_N - \widetilde{\lambda}_{j,\beta}| < c_{15} \tag{90}$$

and

$$|c(N, j, \beta)| > \rho^{-n\alpha}, \qquad |b(N, j, \beta)| > \rho^{-n\alpha} \tag{91}$$

then Υ_N *satisfies (88) and* Λ_N *satisfies (89).*

Proof. By Result (a)–(b), there exists N satisfying the conditions (90) and (91) in part (b). Hence it suffices to prove part (b). By (64), (65) and (90), the triples (N, j^k, β^k) satisfy the iterability conditions in (54). Hence we can use (78), (81), (86) and (87). Now, we prove the theorem by induction:

For $k = 1$, to prove (88), we divide both sides of the equation (78) by $c(N, j, \beta)$ and use the estimations (86). Similarly, to prove (89) for $k = 1$, we divide both sides of the equation (81) by $b(N, j, \beta)$ and use the estimations (87).

Suppose that (88) and (89) hold for $k = s$, i.e.,

$$\Upsilon_N = \lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \tag{92}$$

$$\Lambda_N = \widetilde{\lambda}_{j,\beta} + \widetilde{E}_{s-1} + O(\rho^{-s\alpha_2}). \tag{93}$$

First we prove that (88) holds for $k = s+1$. For this, we substitute the formula (92) into the expression $\sum_{k=1}^{2p} S'_k(\Upsilon_N, \lambda_{j,\beta})$ in equation (78) , then we get

$$\begin{aligned}(\Upsilon_N - \lambda_{j,\beta})c(N, j, \beta) = &(\sum_{k=1}^{2p} S'_k(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}))c(N, j, \beta) \\ &+C'_{2p} + O(\rho^{-p\alpha})\end{aligned} \tag{94}$$

Dividing both sides of (94) by $c(N, j, \beta)$ using (91) and (86), we have

$$\Upsilon_N = \lambda_{j,\beta} + \sum_{k=1}^{2p} S'_k(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}) + O(\rho^{-(p-q)\alpha}). \tag{95}$$

Now we add and subtract the term $\sum_{k=1}^{2p} S_k'(E_{s-1}+\lambda_{j,\beta},\lambda_{j,\beta})$ in (95) then we have

$$\begin{aligned}\Upsilon_N &= \lambda_{j,\beta}+E_s+O(\rho^{-(p-q)\alpha})\\ &+[\sum_{k=1}^{2p} S_k'(\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2}),\lambda_{j,\beta})-\sum_{k=1}^{2p} S_k'(E_{s-1}+\lambda_{j,\beta},\lambda_{j,\beta})] \qquad (96)\end{aligned}$$

Now, we first prove that $E_j=O(\rho^{-\alpha_2})$ by induction. $E_0=0$. Suppose that $E_{j-1}=O(\rho^{-\alpha_2})$, then $a=\lambda_{j,\beta}+E_{j-1}$ satisfies (84) and (85). Hence we get

$$S_1'(a,\lambda_{j,\beta})=O(\rho^{-\alpha_2})\Rightarrow E_j=O(\rho^{-\alpha_2}). \qquad (97)$$

So to prove (88) for $k=s+1$, we need to show that the expression in the square brackets in (96) is equal to $O(\rho^{-(s+1)\alpha_2})$. This can be easily checked by (97) and the obvious relation

$$\frac{1}{\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2})-\lambda_{j^k,\beta^k}}-\frac{1}{\lambda_{j,\beta}+E_{s-1}-\lambda_{j^k,\beta^k}}=O(\rho^{-(s+1)\alpha_2}),$$

for $\beta^k\neq\beta$. The formula (89) for $k=s+1$ can be proved similarly. The theorem is proved. □

References

[1] Atılgan, Ş.& Karakılıç, S. & Veliev. O.A.(2002). Asymptotic Formulas for the Eigenvalues of the Schrödinger Operator. Turk J Math,26,215-227.

[2] Feldman,J.& Knorrer,H.& Trubowitz,E.(1990)The Perturbatively Stable Spectrum of the Periodic Schrodinger Operator. Invent. Math.,100, 259-300.

[3] Feldman,J.& Knorrer,H.& Trubowitz,E.(1991)The Perturbatively unstable Spectrum of the Periodic Schrodinger Operator. Comment.Math.Helvetica, 66,557-579.

[4] Friedlanger,L.(1990).On the Spectrum for the Periodic Problem for the Schrodinger Operator. Communications in Partial Differential Equations,15,1631-1647.

[5] Hald,O.H. & McLaughlin,J.R.(1996). Inverse Nodal Problems: Finding the Potential from Nodal Lines. Memoirs of AMS. 572,119,0075-9266.

[6] Karpeshina,Yu.E.(1992). Perturbation Theory for the Schrödinger Operator with a non-smooth Periodic Potential. Math.USSR-Sb,71,701-123.

[7] Karpeshina,Yu.E.(1996). Perturbation series for the Schrödinger Operator with a Periodic Potential near Planes of Diffraction. Communication in Analysis and Geometry,4,3,339-413.

[8] Veliev,O.A.(1983). On the Spectrum of the Schrödinger Operator with Periodic Potential. Dokl.Akad.Nauk SSSR 268,no.6,1289-1292.

[9] Veliev,O.A. and Molchanov, S.A.(1985). Structure of the Spectrum of the Periodic Schrödinger Operator on a Euclidean Torus. Functsional Anal. i Prilozhen, 19, no3, 86-87.

[10] Veliev,O.A.(1987). Asymptotic Formulas for the Eigenvalues of the Periodic Schrödinger Operator and the Bethe-Sommerfeld Conjecture. Functsional Anal. i Prilozhen, 21, no.2,1-15.

[11] Veliev,O.A.(1988). The Spectrum of Multidimensional Periodic Operators. Teor.Funktsional Anal. i Prilozhen,49,17-34.

[12] Veliev,O.A.(2001). The Periodic Multidimensional Schrodinger Operator, Part 1, Asymptotic Formulae for Eigenvalues, University of Texas, Mathematics Department, Mathematical Physics Preprint Archive, 01-446.Dec.4.

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