

A Property of Weak Convergence of Positive Contractions of Von Neumann Algebras

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Abstract

In the present paper we prove that the mixing property of positive L^1 -contraction of finite von Neumann algebras implies the property of complete mixing.

Key words and phrases: Positive contraction, mixing, completely mixing, von Neumann algebra.

1. Introduction

Let (X, \mathcal{F}, μ) be a measure space with probability measure μ . Let $L^1(X, \mathcal{F}, \mu)$ be the associated L^1 -space. Assume that $T : L^1(X, \mathcal{F}, \mu) \rightarrow L^1(X, \mathcal{F}, \mu)$ is a linear positive contraction (i.e. $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$). Then (see [7], Ch. 8, theorem 1.4) the following fact is known:

Theorem 1.1 *Let $T : L^1(X, \mathcal{F}, \mu) \rightarrow L^1(X, \mathcal{F}, \mu)$ be a positive contraction. Assume that there exists no non zero $y \in L^1(X, \mathcal{F}, \mu)$, $y \geq 0$ such that $Ty = y$. If for $z \in L^1(X, \mathcal{F}, \mu)$, the sequence $(T^n z)$ converges weakly to some element of $L^1(X, \mathcal{F}, \mu)$, then $\lim_{n \rightarrow \infty} \|T^n z\| = 0$.*

In this paper we extend this result to a non-commutative setting, since the large time behavior of quantum processes has been the subject of a number of investigations. Note that the formulated theorem is a variant of the Akcoglu and Sucheston theorem [1]. By means of theorem 1.1 they proved certain weighted ergodic theorems, namely an extension of the Blum-Houson theorem (see [7], Chapter 8). It is our hope the present

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work will find application towards the proof of weighted ergodic theorems in the frame work of van Neumann algebras, since it seems to be an area of active investigaton (see for example, [3], [8]).

2. Prelimiarles

Throughout this paper M will be a von Neumann algebra with unit $\mathbf{1}$ and τ a faithful normal finite trace on M . Therefore we omit this condition from the formulation of theorems. Recall that an element $x \in M$ is called *self-adjoint* if $x = x^*$. The set of all self-adjoint elements is denoted by M_{sa} . A self-adjoint element $p \in M$ is called a *projection* if $p^2 = p$. The set of all projections in M we will denote by ∇ . By M_* we denote a pre-dual space to M (see for definitions [2], [13]).

The map $\|\cdot\| : M \rightarrow [0, \infty)$ defined by the formula $\|x\| = \tau(|x|)$ is a norm (see [13]); here, $|x| = (x^*x)^{1/2}$. The completion of M with respect to the norm $\|\cdot\|$ is denoted by $L^1(M, \tau)$. We will use the fact [13] that the spaces $L^1(M, \tau)$ and M_* are isometrically isomorphic, therefore they can be identified.

Theorem 2.1 [10] *The space $L^1(M, \tau)$ coincides with the set*

$$L^1 = \{x = \int_{-\infty}^{\infty} \lambda de_{\lambda} : \int_{-\infty}^{\infty} |\lambda| d\tau(e_{\lambda}) < \infty\}.$$

Moreover,

$$\|x\| = \int_{-\infty}^{\infty} |\lambda| d\tau(e_{\lambda}).$$

It is known ([10]) that the equality

$$L^1(M, \tau) = L^1(M_{sa}, \tau) + iL^1(M_{sa}, \tau)$$

is valid. Note that $L^1(M_{sa}, \tau)$ is a pre-dual to M_{sa} .

Let $T : L^1(M, \tau) \rightarrow L^1(M, \tau)$ be a linear bounded operator. We say that a linear operator T is *positive* if $Tx \geq 0$ whenever $x \geq 0$. A linear operator T is said to be a *contraction* if $\|T(x)\| \leq \|x\|$ for all $x \in L^1(M_{sa}, \tau)$.

3. Main Result

Let M be a von Neumann algebra with faithful normal finite trace τ . Let $L^1(M, \tau)$ be the associated L^1 -space.

Let $T : L^1(M, \tau) \rightarrow L^1(M, \tau)$ be a contraction. Define

$$\bar{\rho}(T) = \sup \left\{ \lim_{n \rightarrow \infty} \frac{\|T^n(u - v)\|}{\|u - v\|} : u, v \in L^1(M_{sa}, \tau), u, v \geq 0, \|u\| = \|v\| \right\},$$

and $\rho(T) = \lim_{n \rightarrow \infty} \|T^n\| - \bar{\rho}(T)$.

The magnitude $\rho(T)$ is called the asymptotic Dobrushin coefficient of ergodicity of T (see [4]). If $\bar{\rho}(T) = 0$ then T is called *completely mixing*. Recall that a positive contraction T is *mixing* if for all $x \in \{a \in L^1(M_{sa}, \tau) : \tau(a) = 0\}$ and $y \in M$, the following condition holds:

$$\lim_{n \rightarrow \infty} \tau(T^n(x)y) = 0.$$

Before proving our main theorem let us give the following auxiliary.

Lemma 3.1 *Let $x \in L^1(M, \tau)$. If the inequality*

$$\tau(xy) \geq 0 \tag{1}$$

is valid for every $y \geq 0, y \in M$, then $x \geq 0$.

Proof. Write $x = x^+ - x^-$. Let

$$x = \int_{-\infty}^{\infty} \lambda de_{\lambda}$$

be the spectral resolution of x . Set

$$p = \int_{-\infty}^0 de_{\lambda}.$$

Then according to (1) one gets $\tau(xp) \geq 0$. On the other hand we have $xp = -x^-$, hence $\tau(x^-) \leq 0$. Since $x^- \geq 0$ and τ is faithful, we infer that $x^- = 0$. Therefore $x = x^+ \geq 0$. \square

Theorem 3.2 *Let $T : L^1(M, \tau) \rightarrow L^1(M, \tau)$ be a positive contraction such that $|T(x)| \leq T(|x|)$ for every $x \in L^1(M_{sa}, \tau)$. Assume that there exists no non-zero $y \in L^1(M, \tau)$, $y \geq 0$ such that $Ty = y$. If for $z \in L^1(M, \tau)$ the sequence $(T^n z)$ converges weakly to some element of $L^1(M, \tau)$, then $\lim_{n \rightarrow \infty} \|T^n z\| = 0$. In particular, if T is mixing, then T is completely mixing.*

Proof. The contractivity of T implies that the limit

$$\lim_{\nu \rightarrow \infty} \|T^\nu z\| = \alpha$$

exists. Assume that $\alpha \neq 0$. Define $\lambda : M_{sa} \rightarrow \mathbb{R}$ by

$$\lambda(x) = \text{Lim}((\tau(x|T^n z|))_{n \in \mathbb{N}})$$

for every $x \in M_{sa}$, where Lim means a Banach limit (see, [7]). We have

$$\lambda(\mathbf{1}) = \text{Lim}((\tau(|T^n z|))_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} \|T^n z\| = \alpha \neq 0,$$

therefore $\lambda \neq 0$. Besides, λ is a positive functional, since for positive element $x \in M_{sa}$, $x \geq 0$ we have

$$\tau(x|T^n z|) = \tau(x^{1/2}|T^n z|x^{1/2}) \geq 0,$$

for every $n \in \mathbb{N}$.

For arbitrary $x \in M$, we have $x = x_1 + ix_2$ and define λ by

$$\lambda(x) = \lambda(x_1) + i\lambda(x_2).$$

Let T^{**} be the second dual of T , i.e. $T^{**} : M^{**} \rightarrow M^{**}$. Let $\lambda = \lambda_n + \lambda_s$ be a Takesaki decomposition (see [13]) of λ on normal and singular components. Now we will show that λ_n is nonzero. Consider a measure $\mu := \lambda \upharpoonright_{\nabla}$. It is clear that μ is an additive measure on ∇ . Now let us prove that it is σ -additive. To this end, it is enough to show that $\mu(p_k) \rightarrow 0$ whenever $p_{k+1} \leq p_k$ and $p_k \searrow 0$, $p_k \in \nabla$.

Let $\varepsilon > 0$. From $p_n \searrow 0$ we infer that $\tau(p_n) \rightarrow 0$ as $n \rightarrow \infty$. It follows that there exists $k_\varepsilon \in \mathbb{N}$ such that $\tau(p_k) < \varepsilon$ for all $k \geq k_\varepsilon$. Since $T^n z$ converges weakly then the set $\{T^n z\}$ is relatively compact in $L^1(M, \tau)$, hence according to Theorem 5.4 Ch.3 [13] we obtain

$$\tau(p_k|T^n z|) < \varepsilon, \quad \forall k \geq k_\varepsilon,$$

for every $n \geq n_0$. From a property of the Banach limit, one gets

$$\lambda(p_k) = \text{Lim}((\tau(p_k|T^n z|))_{n \in \mathbb{N}}) < \varepsilon \quad \text{for every } k \geq k_\varepsilon.$$

This means $\mu(p_k) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, we conclude that the restriction of λ_n on ∇ coincides with μ . Since

$$\tau(p^\perp|T^n z|) > \tau(|T^n z|) - \varepsilon \geq \inf \|T^n z\| - \varepsilon = \alpha - \varepsilon$$

and ε has been arbitrary, so $\alpha - \varepsilon > 0$, and hence $\mu(p^\perp) > 0$ for all $p \in \nabla$ such that $\tau(p) < \delta$. Therefore $\mu \neq 0$ and, consequently, $\lambda_n \neq 0$.

From this, we infer that there exists a positive element $y \in L^1(M, \tau)$ such that

$$\lambda_n(x) = \tau(yx), \quad \forall x \in M.$$

The property of T implies

$$\begin{aligned} \tau(Tyx) &= \tau(yT^*x) \\ &= \text{Lim}((\tau(|T^n z|T^*x))_{n \in \mathbb{N}}) \\ &= \text{Lim}((\tau(T|T^n z|x))_{n \in \mathbb{N}}) \\ &\geq \text{Lim}((\tau(|T^{n+1}z|x))_{n \in \mathbb{N}}) = \tau(yx) \end{aligned}$$

for all $x \geq 0$. Hence, for every $x \geq 0$ we have

$$\tau((Ty - y)x) \geq 0.$$

According to Lemma 3.1 we infer that $Ty \geq y$. Since T is a contraction one gets $Ty = y$. But this contradicts the assumption of the theorem. \square

Remark 3.3 *The proved theorem is a non-commutative analog of Theorem 1.1. Certain similar results have been obtained in [9],[5] for quantum dynamical semigroups in von Neumann algebras.*

Corollary 3.4 *Let $\alpha : M \rightarrow M$ be a normal Jordan automorphism of finite von Neumann algebra such that there exists no non-zero $y \in L^1(M, \tau)$, $y \geq 0$ such that $\alpha^*y = y$, where α^* is the conjugate operator to α . If for $z \in L^1(M, \tau)$ the sequence $((\alpha^*)^n z)$ converges weakly to some element of $L^1(M, \tau)$, then $\lim_{n \rightarrow \infty} \|(\alpha^*)^n z\| = 0$.*

The proof immediately comes from Theorem 3.2 since for Jordan automorphisms the equality $|\alpha(x)| = \alpha(|x|)$ is valid for all $x \in M_{sa}$ (see [2]).

Remark 3.5 *Note that an analogous theorem has been recently proved by A.Katz [6] for *-automorphisms of an arbitrary von Neumann algebra. It is known ([2]) that not every Jordan automorphism is an *-automorphism of M ; therefore Corollary 3.4 extends the result of [6] to Jordan automorphisms, but only for finite von Neumann algebras. Here it should be also noted that linear mappings of von Neumann algebras which satisfy the condition $|\alpha(x)| = \alpha(|x|)$ have been studied in [12], [11].*

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