Turk J Math
29 (2005) , 175 – 192.

Constructing New K3 Surfaces

Selma Altınok

Abstract

This paper is concerned with a method based on birational geometry and produces dozens of new examples in codimensions 3, 4, 5 etc. The method is called *unprojection* by Reid. Using this method we construct new examples of K3 surfaces of codimensions 3 and 4 in weighted projective spaces from smaller codimension K3 surfaces whose rings are much simpler. This leads to the existence of almost all candidates for codimension 3 K3 surfaces in the list[1].

Introduction

K3 surfaces of codimensions 1 and 2 in weighted projective spaces were studied by Reid, Fletcher and, independently, by Yonemura. There are 95 families of K3 surfaces in codimension 1 and 83 families in codimension 2 (see Fletcher [6]). In 1997–98, we studied K3 surfaces in codimensions 3 and 4, and produced 70 families of K3 surfaces in codimension 3 and many examples in codimension 4 (see Altınok [1]). In 2001, Brown and Reid wrote the K3 database program in the computer algebra system Magma Version 2.8, which reproduces Fletcher's and Altınok's lists for codim ≤ 3 K3 surfaces and a list of codim 4 K3 surfaces effortlessly; see Altınok–Brown–Reid [3].

A technique, observed by Reid, is used to construct complicated new examples, especially K3 surfaces in codimensions 3 and 4, from smaller codimension ones. The technique is called *unprojection*. Many other examples arise from this technique, such as K3 surfaces, surfaces of general type and Fano 3 folds. There is another technique

2000 *Mathematics Subject Classification:* Primary 14J17,14J28,14E05 Secondary 14M05.
[1]Available on http://www.maths.warwick.ac.uk/~miles/doctors/Selma.

given by Kustin–Miller [8] through commutative algebra. They gave a construction of big Gorenstein rings from simple ones by constructing a resolution structure of rings. This was a first attempt to obtain a complete structure theorem on codimension 4 Gorenstein rings. The general problem still remains open. In 1974, Buchsbaum–Eisenbud [4] solved this problem for codimension 3 Gorenstein rings. They proved that an ideal I of codim 3 in a regular local ring A is Gorenstein if and only if it is the ideal of $2m$th order Pfaffians (or the submaximal Pfaffians) of some $(2m+1)\times(2m+1)$ skew-symmetric matrix M, and if $R = A/I$ is Gorenstein, then the free resolution over A of R is self dual, which gives us that the resolution of R is written as

$$0 \to A \xrightarrow{P^t} F^* \xrightarrow{M} F \xrightarrow{P} A$$

where F is a free A-module of rank $2m+1$, M a $(2m+1)\times(2m+1)$ skew-symmetric matrix, P a $(2m+1)\times 1$ matrix and P^t denotes the transpose of P. The generators of I, called the *relations* of R, are the $2m$th order Pfaffians or submaximal Pfaffians of M which are given as the Pfaffians[2] of the $2m \times 2m$ submatrices obtained by omitting the ith row and the ith column of M. A *syzygy* is a relation among the relations.

This work is mainly taken from my University of Warwick Ph.D thesis [1], written under the supervision of Miles Reid. I would like to thank him for his support.

1. Definitions and notation

Here varieties are defined over an algebraically closed field k of characteristic zero.

A K3 surface is a projective surface X with $K_X = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. By a polarised K3 surface (X, D), we mean that X is a K3 surface having at worst Du Val singularities and D an ample Weil divisor. The graded ring associated to (X, D) is

$$R(X,D) = \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD)), \quad \text{with} \quad X = \operatorname{Proj} R(X,D).$$

We study K3 surfaces, especially those whose rings R are simple. This means that R can conveniently be expressed in terms of generators and relations such as hypersurfaces, codimension 2 complete intersections, codimension 3 Pfaffians and known models for codimension 4 rings embedded in weighted projective space.

[2]The *Pfaffian* of an even sized skew-symmetric matrix is the square root of its determinant.

Let V be a surface. A singularity $P \in V$ is called a Du Val singularity if it is locally analytically isomorphic to one of the following normal forms:

$$A_n\colon x^2 + y^2 + z^{n+1} = 0 \quad \text{for } n \geq 1,$$

$$D_n\colon x^2 + y^2 z + z^{n-1} = 0 \quad \text{for } n \geq 4,$$

$$E_6\colon x^2 + y^3 + z^4 = 0,$$

$$E_7\colon x^2 + y^3 + yz^3 = 0,$$

$$E_8\colon x^2 + y^3 + z^5 = 0.$$

Type A_n singularities are also cyclic quotient singularities. To see this, let x_1, x_2 be affine coordinates of $\mathbb{A}^2$ with weights a_1, a_2 respectively. A singular point of a surface V is called a *cyclic quotient singularity* if it is locally analytically isomorphic to $(\mathbb{A}^2, 0)/\mu_{n+1}$, for some n, where μ_{n+1} is a cyclic group of $(n+1)$th roots of unity and the group action $\mu_{n+1} \times \mathbb{A}^n \to \mathbb{A}^n$ is given by

$$\varepsilon(x_1, x_2) = (\varepsilon^{a_1} x_1, \varepsilon^{a_2} x_2).$$

We denote such a singularity by $\frac{1}{r}(a_1, a_2)$. When $a_1 = 1, a_2 = -1$, this gives rise to an A_n type singularity.

The *weighted projective space* associated to the ring A is defined by

$$\mathbb{P}(a_0, \ldots, a_N) = \operatorname{Proj} A$$

where $A = k[x_0, \ldots, x_N]$ is a graded polynomial ring graded by $\operatorname{wt}(x_i) = a_i$ with positive integers a_i. For more details, see Dolgachev [5], Fletcher [6]. Taking Proj of the graded rings $R(X, D)$, it gives K3 surfaces embedded in weighted projective spaces. We denote, in general, a variety embedded in weighted projective space by $X(d_1, \ldots, d_l) \subset \mathbb{P}(a_0, \ldots, a_N)$ where the d_i are the defining equations of the variety and the a_i refer to the weights of the homogeneous coordinates.

When no confusion can arise, we use either $\mathbb{P}$ or $\mathbb{P}(x_0, \ldots, x_n)$ to denote $\mathbb{P}(a_0, \ldots, a_n)$. Let q_i be the point $(0, \ldots, 1, \ldots, 0) \in \mathbb{P}$, where 1 is in the ith position. We call such points *vertices* of $\mathbb{P}$. The l-plane spanned by $q_{i_1}, \ldots, q_{i_l}$ will be denoted by $q_{i_1} \ldots q_{i_l}$ and be called an $(l-1)$-stratum. The 1-dimensional strata will also be called *edges* and the 2-dimensional strata *faces*.

Let V be a closed variety of dimension m in $\mathbb{P}$, and let $\pi\colon \mathbb{A}^{n+1} - 0 \to \mathbb{P}$ be the canonical projection. The punctured affine cone $\mathrm{Cone}^*(V)$ over V is given by $\pi^{-1}(V)$ and the affine cone $\mathrm{Cone}(V)$ over V is the completion of $\mathrm{Cone}^*(V)$ in $\mathbb{A}^{n+1}$. Then V is *quasismooth* if the affine cone $\mathrm{Cone}(V)$ of V is smooth of dimension $m+1$ except the vertex 0. In other words, V has only cyclic quotient singularities.

2. Projection–unprojection

In [1] there are lists of candidates for polarised K3 surfaces in codim 3 and 4 produced by using the Hilbert function theorem (see [2]). We work on particular candidates in these lists to show their existence. Each example has a different type of construction but the strategy is the same. The existence of other candidates can be proved in a more-or-less similar way. One only has to take care of quasismoothness. The general strategy is to reduce the codimension of candidates by projecting, then lift back by using birational geometry (that is, unprojection). Especially, when we reduce the codimension 3 to the codimension 2 case the surface we get is generally in Fletcher's list (see [6]). What is nice about unprojection is that it allows us to use Bertini's theorem to prove quasismoothness. We will see clear use of it in each example.

Now we demonstrate the simplest example of projection–unprojection between two K3 surfaces of codimensions 2 and 1. In the first part of the example we consider projection, in the second part unprojection.

Example 2.1 (A) We start by assuming that a K3 surface $X = X(3,3) \subset \mathbb{P}(1,1,1,1,2)$ with an A_1 singularity and a $\mathbb{Q}$-ample Weil divisor $D = \mathcal{O}_X(1)$, where $D^2 = 9/2$, exists then we want to construct a nonsingular K3 surface $Y = X(4) \subset \mathbb{P}^3$ with an ample divisor D', containing the line $C\colon (x_1 = x_2 = 0)$.

Pick the point $(0,0,0,0,1)$, which is an A_1 singularity, and project away from this point to get $Y = X(4)$, as desired. We now give details of the construction. Let $x_1, \dots, x_4, y$ be homogeneous coordinates on $\mathbb{P}(1,1,1,1,2)$ with weights $1,1,1,1,2$. Since X has a singularity $\frac{1}{2}(1,-1)$ locally at $y = 1$, it has two local coordinates, say x_3, x_4, so that the defining equations of X can be written as

$$x_1 y = -a_2 \quad \text{and} \quad x_2 y = a_1,$$

or equivalently,

$$y = -\frac{a_2}{x_1} = \frac{a_1}{x_2}$$

where a_1 and a_2 are general homogeneous polynomials of degree 3 in $k[x_1, \dots, x_4]$. That means that y is a rational function on Y which has a pole along C. Projecting X away from the point $(0,0,0,0,1)$ gives

$$Y\colon (a_1x_1 + a_2x_2 = 0) \subset \mathbb{P}^3.$$

(B) We begin by constructing a nonsingular K3 surface $Y = X(4)$ of codim 1 in $\mathbb{P}^3$, containing the line $C\colon (x_1 = x_2 = 0)$ and then obtain a quasismooth K3 surface $X = X(3,3)$ of codim 2 with an A_1 singularity in $\mathbb{P}(1,1,1,1,2)$ from Y by using birational geometry.

Let $x_1, \dots, x_4$ be homogeneous coordinates on $\mathbb{P}^3$. Define the linear system $\mathcal{L}$ of all homogeneous polynomials of degree 4 on $\mathbb{P}^3$ containing the line C and take a sufficiently general element $f = a_1x_1 + a_2x_2 \in \mathcal{L}$, where a_1 and a_2 are sufficiently general homogeneous polynomials of degree 3 in $k[x_1, x_2, x_3, x_4]$. Denote by Y the variety defined by the equation $f = 0$ in $\mathbb{P}^3$. The base locus of the linear system $\mathcal{L}$ is the line C. By Bertini's theorem (see Remark III.10.9.2, [7]), Y is nonsingular away from C. Therefore it is sufficient to check that Y is nonsingular along C. The singular locus of Y along C is $(a_1(0,0,x_3,x_4) = a_2(0,0,x_3,x_4) = 0) \subset \mathbb{P}(x_3,x_4)$. But a_1 and a_2 are general polynomials, therefore the singular locus is empty. Hence Y is a nonsingular K3 surface in $\mathbb{P}^3$, containing the line C. Now set a new generator of degree 2

$$y = \frac{a_1}{x_2} = -\frac{a_2}{x_1}$$

and define a rational map $\varphi\colon Y \subset \mathbb{P}^3 \to X \subset \mathbb{P}(1,1,1,1,2)$, where X is the complete intersection $x_1y = -a_2$ and $x_2y = a_1$, by $(x_1,x_2,x_3,x_4) \mapsto (x_1,x_2,x_3,x_4,y)$. It can be easily observed that φ is a birational map, and in fact an isomorphism $Y \setminus C \xrightarrow{\sim} X \setminus q_4$, where $q_4 = (0,0,0,0,1)$. Indeed, the inverse map is projection from q_4. Thus it suffices to see that $\operatorname{Cone}(X)$ is nonsingular at q_4. The rank of the Jacobian matrix of $\operatorname{Cone}(X)$ is two at this point. Therefore $\operatorname{Cone}^*(X)$ is nonsingular. The singularities of X arise due to the singularities of $\mathbb{P}$ and occur only at the vertex q_4 which is a singularity of type A_1 in X. Here, unprojection is a map which contracts the curve C to an A_1 type singularity.

Type I Unprojection

Let Y be a projectively Gorenstein variety of codimension c embedded in weighted projective space $\mathbb{P}(a_0, \dots, a_n)$, containing a projectively Gorenstein variety C of codimension $c+1$. Let $I(C)$ be the ideal of C generated by homogeneous polynomials h_i for $i = 1, \dots, l$, and $I(Y)$ the ideal of Y generated by f_i for $i = 1, \dots, k$. Then there exists a rational function v on Y which has a pole along C, that is,

$$v = \frac{m_1}{h_1} = \cdots = \frac{m_l}{h_l},$$

for some homogeneous polynomials m_i such that X defined by

$$f_1 = \cdots = f_k = vh_1 - m_1 = \cdots = vh_l - m_l = 0$$

is a projectively Gorenstein variety of codimension $c+1$ in $\mathbb{P}(a_0, \dots, a_n, a_{n+1})$, where $\mathrm{wt}(v) = a_{n+1}$. This construction is called unprojection. (See Kustin–Miller [8], Theorem 1.5 and Papadakis–Reid [9]). In the proof of Kustin–Miller v and the m_i are not given explicitly. In our construction we use a simple trick in linear algebra to get them explicitly, namely Cramer's rule.

Note that a projectively Gorenstein variety means that the projective coordinate ring of the variety is Gorenstein.

2.1. New examples in codimension 3

We can rewrite the resolution over A of a graded ring R of codimension 3 from the introduction as

$$0 \to A(-s) \xrightarrow{P^t} \bigoplus_{i=1}^{5} A(-e_i) \xrightarrow{M} \bigoplus_{i=1}^{5} A(-d_i) \xrightarrow{P} A \to R \to 0, \qquad (1)$$

where M is a 5×5 skew-symmetric matrix and P is a 5×1 vector such that the degrees of the entries are given by the degrees d_i of the relations, and the degrees of the syzygies are given by the e_i. It can be easily observed that $s - d_i = e_i$.

2.1.1. $X(2,3,3,3,3) \subset \mathbb{P}(1,1,1,1,1,2)$

To prove the existence of a quasismooth K3 surface $X = X(2,3,3,3,3)$ in $\mathbb{P}(1,1,1,1,1,2)$ with an A_1 singularity, we construct a nonsingular K3 surface $Y = X(2,3)$ of codim 2 in $\mathbb{P}^4$ containing the line $C\colon (x_1 = x_2 = x_3 = 0)$.

The first question to ask is how we know that Y and C are the right choices. In the first part of Example 2.1, we show that there is a way of getting Y and C. We start by assuming that X is given and then project X away from the singular point $q_5 = (0,0,0,0,0,1)$ to get Y and C. We briefly describe the construction. Let $x_1, \dots, x_5, y$ be homogeneous coordinates on $\mathbb{P}(1,1,1,1,1,2)$ with weights $1,1,1,1,1,2$. Since X has a singularity $\frac{1}{2}(1,-1)$, it has local coordinates, say x_4, x_5, at q_5. In other words, there are three linearly independent equations of X at q_5: $x_1 y = g_1$, $x_2 y = g_2$, $x_3 y = g_3$ for some homogeneous polynomials $g_i \in k[x_1, \dots, x_5]$. Therefore we can write down a 5×5 skew-symmetric matrix $M = (m_{ij})$ whose submaximal Pfaffians are the defining equations of X without loss of generality as follows:

$$\begin{pmatrix} 0 & x_3 & x_2 & m_{14} & m_{15} \\ & 0 & x_1 & m_{24} & m_{25} \\ & -sym & 0 & m_{34} & m_{35} \\ & & & 0 & y \\ & & & & 0 \end{pmatrix}.$$

Hence we can observe that m_{i4}, m_{i5} for $i = 1,2,3$ are in $k[x_1, \dots, x_5]$ and the degrees of the m_{i4} or the m_{i5} are either one or two. This implies that the other two equations have degrees 2,3 in $k[x_1, \dots, x_5]$ and contain the line $C\colon (x_1 = x_2 = x_3 = 0)$, and we denote Y by the zero locus of these two equations. We project X away from q_5 to get Y. From now on in all of our examples we write down Y immediately without any explanation. One can work out the details as explained above.

Now we construct X from Y. Let $x_1, \dots, x_5$ be homogeneous coordinates on $\mathbb{P}^4$. Define linear systems $\mathcal{L}_1$ and $\mathcal{L}_2$ of all homogeneous polynomials of degrees 2 and 3 respectively on $\mathbb{P}^4$, containing the line C. Let $f_i = \sum_{j=1}^{3} a_{ij} x_j \in \mathcal{L}_i$ be sufficiently general polynomials for $i = 1, 2$ where the a_{ij} are sufficiently general in $k[x_1, \dots, x_5]$. Define

$$Y\colon (f_1 = f_2 = 0) \subset \mathbb{P}^4.$$

Both of the linear systems have base locus C so that by Bertini's theorem, the general members of these linear systems are smooth outside C. To prove that Y is smooth on C, it is sufficient to show that for sufficiently general a_{ij}, the Jacobian matrix of f_1, f_2, with respect to x_1, x_2, x_3, has rank 2 at every point of the line C, that is,

$$\mathrm{rk}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\Big|_C = 2.$$

For example, take the matrix on C

$$\begin{pmatrix} 0 & x_5 & x_4 \\ x_5^2 & x_4^2 & 0 \end{pmatrix},$$

which has rank 2 at every point of $C = \mathbb{P}(x_4, x_5)$. Hence Y is nonsingular along C.

We want to find a rational function on Y which has a pole along C. Notice that Y can be given by a system of linear equations

$$M\mathbf{x}^t = 0, \tag{2}$$

where $M = (a_{ij})$ is a 2×3 matrix and $\mathbf{x} = (x_1, x_2, x_3)$. Now by Cramer's rule, we can solve (2) to get

$$y\mathbf{x} = \bigwedge^2 M,$$

where $\bigwedge^2 M$ is the vector of the 2×2 minors of M. Define X by five equations $y\mathbf{x} = \bigwedge^2 M$ and $M\mathbf{x}^t = 0$ which are also the submaximal Pfaffians of the following 5×5 skew-symmetric matrix

$$\begin{pmatrix} 0 & x_1 & -x_2 & a_{13} & a_{23} \\ & 0 & x_3 & a_{12} & a_{22} \\ & & 0 & a_{11} & a_{21} \\ & -\mathrm{sym} & & 0 & -y \\ & & & & 0 \end{pmatrix}.$$

Let φ be the rational map $\varphi\colon Y \subset \mathbb{P}^4 \to X \subset \mathbb{P}(1,1,1,1,1,2)$ given by

$$(x_1, x_2, x_3, x_4, x_5) \longmapsto (x_1, x_2, x_3, x_4, x_5, y).$$

Restricting φ to $Y \setminus C$ gives a birational morphism $Y \setminus C \to X \setminus q_5$. Its inverse is projection from q_5. To show that X is quasismooth, it is sufficient to show that the cone is smooth at q_5. It can easily be observed that $y\mathbf{x} = \bigwedge^2 M$ gives three linearly independent equations locally at q_5. This implies that X is quasismooth with an A_1 singularity.

2.1.2. $X(6,6,6,7,7) \subset \mathbb{P}(1,2,3,3,3,4)$

The existence of a quasismooth K3 surface $X = X(6,6,6,7,7) \subset \mathbb{P}(1,2,3,3,3,4)$ with $3A_2$, A_3 singularities will be proved by constructing a quasismooth K3 surface $Y = X(6,6) \subset \mathbb{P}(1,2,3,3,3)$ with $4A_2$ singularities containing the line $C\colon (y = z_1 = z_2 = 0)$.

Let x, y, z_1, z_2, z_3 be homogeneous coordinates on $\mathbb{P}(1,2,3,3,3)$ of weights $1,2,3,3,3$ respectively. Let $\mathcal{L}$ be the linear system of all homogeneous polynomials of degree 6 containing C with respect to weights $1,2,3,3,3$. Let $f_1,\ f_2 \in \mathcal{L}$ be sufficiently general elements, that is,

$$f_1 = a_{11}y + a_{12}z_1 + a_{13}z_2, \quad f_2 = a_{21}y + a_{22}z_1 + a_{23}z_2,$$

where a_{ij} are sufficiently general. A general K3 surface Y of codim 2 containing C in $\mathbb{P}(1,2,3,3,3)$ is given by

$$M\mathbf{x}^t = 0, \tag{3}$$

where M is the 2×3 matrix (a_{ij}) and $\mathbf{x} = (y, z_1, z_2)$.

By Bertini's theorem, the singularities of general elements in the linear systems lie on the base locus $\mathrm{Cone}(C)$. To show that Y is quasismooth it is sufficient to prove that the cone is smooth along $\mathrm{Cone}^*(C)$. For sufficiently general a_{ij}, we can ensure that the Jacobian matrix of f_i, with respect to y, z_1, z_2, has rank 2 along $\mathrm{Cone}^*(C)$. Now we claim that Y has $4A_2$ singularities. Indeed, the singularities of Y come from the singularities of $\mathbb{P}$ and occur only on the vertices, edges and faces of $\mathbb{P}$. It is not difficult to see that the point q_4 is the only vertex which gives rise to a singularity A_2. Now we consider the face $q_2q_3q_4$ of $\mathbb{P}$. The homogeneous polynomials f_1, f_2 on $q_2q_3q_4$ can be written in the homogeneous coordinates z_1, z_2, z_3 of $\mathbb{P}(3,3,3)$. Since $\mathbb{P}(3,3,3)$ is isomorphic to $\mathbb{P}^2$ and f_1, f_2 are of degree 2 in $\mathbb{P}^2$, by Bézout's theorem, $(f_1 = f_2 = 0)$ in $\mathbb{P}^2$ consists of exactly four points counted with multiplicity, including the point q_4. By the inverse function theorem, x, y are local coordinates and so each point is of type $\frac{1}{3}(1,2)$. Hence the claim follows.

In order to construct X from Y, we can solve (3) by Cramer's rule to get a new generator t of degree 4 such that $ty = m_{23}$, $tz_1 = m_{13}$ and $tz_2 = m_{12}$, where the m_{ij} are the minors of the 2×3 matrix M. This gives three new equations. Define X by these equations plus f_1, f_2 which are just the submaximal Pfaffians of the following 5×5

skew-symmetric matrix

$$\begin{pmatrix} 0 & t & a_{11} & a_{12} & a_{13} \\ & 0 & a_{21} & a_{22} & a_{23} \\ & & 0 & z_2 & -z_1 \\ & -\mathrm{sym} & & 0 & y \\ & & & & 0 \end{pmatrix}.$$

The last thing is to show that X has only the four singularities $3A_2, A_3$. Consider the rational map $\varphi\colon Y \subset \mathbb{P}(1,2,3,3,3) \to X \subset \mathbb{P}(1,2,3,3,3,4)$ given by

$$(x, y, z_1, z_2, z_3) \longmapsto (x, y, z_1, z_2, z_3, t).$$

Restricting φ to $Y \setminus C$ gives an isomorphism $Y \setminus C \xrightarrow{\sim} X \setminus q_5$. The inverse map is projection from the point q_5. Three singularities $3A_2$ of Y excluding the point $(0,0,0,0,1)$ correspond to three points p_i of X under the isomorphism for $i = 1,2,3$. It is sufficient to check that X is quasismooth at these points p_i and q_5. We have three Pfaffians, namely $ty = m_{23}, tz_1 = m_{13}$ and $tz_2 = m_{12}$, which are linearly independent locally at q_5. This gives rise to a singularity A_3. At the p_i the Jacobian matrix of the cone has rank 3 because $f_1 = 0, f_2 = 0$ and $tz_1 = m_{13}$ form a linearly independent set locally at these points. This gives three more singularities, namely $3A_2$. Hence X is quasismooth with the desired singularities.

So far we have showed how to get a K3 surface of codim 3 from a K3 surface of codim 2. Now we give an interesting construction of a K3 surface of codim 3 from a K3 surface of codim 1 called Type II unprojection. There are two candidates in the codim 3 list whose existence can be proved by this type of construction.

Type II unprojection

The construction we give for the Type II unprojection is based on notes of Reid (see Reid [12] Section 9). We start by constructing a hypersurface $Y \subset \mathbb{A}^4$ containing a nonnormal curve to obtain a codim 3 Gorenstein $X \subset \mathbb{A}^6$.

Let $\Gamma \subset \mathbb{A}^4$ be the nonnormal variety parametrised by $x = r^2, y = r^3, z = s$ and $t = rs$. Its four defining equations can be given by the nongeneric determinantal form

$$\operatorname{rk}\begin{pmatrix} x & z & y & t \\ y & t & x^2 & xz \end{pmatrix} \le 1. \tag{4}$$

A general hypersurface Y containing Γ is defined by the zero locus of

$$f = A(xt - yz) + B(y^2 - x^3) + C(x^2z - yt) + D(t^2 - xz^2),$$

where A, B, C, D are general polynomials in $k[x, y, z, t]$. By analogy with the Type I unprojection, we are looking for rational sections of $\mathcal{O}_Y(n)$ with a pole along Γ. Consider the following matrix N associated to f:

$$N = \begin{pmatrix} x & y & xC - zD \\ z & t & xB \\ y & x^2 & xA + tD \\ t & xz & zA - yB + tC \end{pmatrix}.$$

We are not concerned here with where this matrix originally came from. Notice that the 3×3 minors of N equal f times x, z, y, t respectively, and that

$$(t, -y, z, -x)N = (0, 0, 0, 0) \quad \text{and} \quad (xz, -x^2, t, -y)N = (0, 0, f).$$

Since the 3×3 minors of N are divisible by f and $\operatorname{rk} N = 2$ on Y the equations

$$(u, v, 1)N^T = 0 \quad \text{on } Y \tag{5}$$

have a unique solution $u, v \in K(Y)$. Clearly, u, v are rational functions which become regular on multiplying by any 2×2 minors of the matrix in (4).

This construction leads to a codim 3 Gorenstein subvariety X whose defining equations are the submaximal Pfaffians of a 5×5 skew-symmetric matrix M, where

$$M = \begin{pmatrix} 0 & x & z & y & t \\ & 0 & v & D & -u \\ & & 0 & u + C & B \\ & -\text{sym} & & 0 & A + xv \\ & & & & 0 \end{pmatrix}$$

in $k[x, y, z, t, u, v]$. In other words, we add u, v to the ring $R(Y)$ of Y in order to get the Gorenstein ring $R(Y)[u, v]$ of codimension 3, the ring of X.

To be more precise, the four submaximal Pfaffians Pf_i for $i = 2, \dots, 4$ are the four equations in (5), and Pf_1 can be written as a combination of the others since $M(\mathrm{Pf}_1, \dots, \mathrm{Pf}_5) = 0$. Now consider the map $Y \subset \mathbb{A}^4 \to X \subset \mathbb{A}^6$ via

$$(x, y, z, t) \longmapsto (x, y, z, t, u, v).$$

This is a birational map and an isomorphism $Y \setminus \Gamma \to X \setminus q_5$. Hence $\operatorname{codim} X = 3$. Since X is given by the maximal Pfaffians of M, it is Gorenstein. Because of the isomorphism it is sufficient to check that X is smooth at q_5. Since we have the 3 Pfaffians, namely $vy = \dots, vt = \dots$, and $vA = \dots$, X is smooth at q_5 if and only if A contains a nonzero linear term in x, z.

Now we apply this construction to K3 surfaces.

2.1.3. $X(7,8,8,9,10) \subset \mathbb{P}(2,3,3,4,4,5)$

We want to construct $X = X(7,8,8,9,10) \subset \mathbb{P}(2,3,3,4,4,5)$ having singularities $3A_1, 3A_2, A_3$ via constructing a codim 1 K3 surface $Y = X(12) \subset \mathbb{P}(2,3,3,4)$ with singularities $3A_1, 4A_2$ containing a projectively nonnormal curve Γ.

Let x, y_1, y_2, z_1 be homogeneous coordinates on $\mathbb{P}(2,3,3,4)$ with weights $2,3,3,4$ respectively. The curve Γ is given in parametric form by $x = r^2, y_1 = r^3, y_2 = s, z_1 = rs$. The defining equations of Γ can be put into nongeneric determinantal form

$$\operatorname{rk}\begin{pmatrix} x & y_2 & y_1 & z_1 \\ y_1 & z_1 & x^2 & xy_2 \end{pmatrix} \le 1.$$

Now define the linear system $\mathcal{L}$ of all homogeneous polynomials of degree 12 with respect to weights $2,3,3,4$ containing Γ:

$$\mathcal{L} = \big\{A(xz_1 - y_2y_1) + B(x^3 - y_1^2) + C(x^2y_2 - y_1z_1) + D(xy_2^2 - z_1^2)\big| A, B, C, D \text{ are homogeneous polynomials in degrees } 6,6,5,4 \text{ resp. in } k[x, y_1, y_2, z_1]\big\}.$$

Let $f \in \mathcal{L}$ be a general element, that is,

$$f = A(xz_1 - y_2y_1) + B(x^3 - y_1^2) + C(x^2y_2 - y_1z_1) + D(xy_2^2 - z_1^2),$$

where A, B, C, D are general homogeneous polynomials. Define a general hypersurface $Y \subset \mathbb{P}(2,3,3,4)$ by the zero locus of f. By Bertini's theorem, $\operatorname{Cone}(Y)$ is nonsingular away from $\operatorname{Cone}(\Gamma)$. It can be seen that $\operatorname{Cone}(Y)$ is smooth along $\operatorname{Cone}^*(\Gamma)$. Under the $\mathbb{C}^*$-action Y has singularities $3A_1, 4A_2$.

As in the construction of Type II unprojection above, we can write down the 4×3 matrix N associated to f in order to get two new generators, say t, z_2, of degrees 5,4 and the 5×5 matrix M whose submaximal Paffians define X. The original hypersurface Y can be obtained by eliminating t and z_2 from the defining equations of X.

It can be observed as in the previous examples that X is quasismooth with singularities $3A_1, 3A_2, A_3$ in $\mathbb{P}(2,3,3,4,4,5)$ by considering a rational map $\varphi\colon Y \to X$ given by $(x, y_1, y_2, z_1) \mapsto (x, y_1, y_2, z_1, z_2, t)$.

The next example can be done in a similar way. Here we just state the result and give the machinery.

2.1.4. $X(10,11,12,13,14) \subset \mathbb{P}(3,4,5,5,6,7)$

$X(10,11,12,13,14) \subset \mathbb{P}(3,4,5,5,6,7)$ with $2A_2, A_3, A_4, A_4$ singularities is a quasismooth K3 surface of codim 3. This follows from first constructing a general hypersurface $X(18) \subset \mathbb{P}(3,4,5,6)$ with $3A_2, A_3, A_1, A_4$ singularities, containing the nonnormal curve Γ given by nongeneric determinantal form

$$\operatorname{rk}\begin{pmatrix} z & y & u & t \\ u & t & z^2 & yz \end{pmatrix} \leq 1,$$

where y, z, t, u are homogeneous coordinates on $\mathbb{P}(3,4,5,6)$ with weights $3,4,5,6$, respectively.

2.2. Codimension 4 case

Let X be a projectively Gorenstein subscheme of codim 3, containing a projectively Gorenstein subscheme C of codim 4, that is the canonical sheaves of X and C are $\omega_X = \mathcal{O}_X(k_X)$ and $\omega_C = \mathcal{O}_C(k_C)$ for some $k_X, k_C \in \mathbb{Z}$ respectively. If $l = k_C - k_X < 0$, then there exists a homomorphism $s\colon \mathcal{I}_C \to \mathcal{O}(-l)$ such that for each $p \in C$, s_p is a basis of $\mathcal{H}om(\mathcal{I}_C, \mathcal{O}_X(-l))_p$ (see Lemma 1.1, Papadakis–Reid [9]).
Note that this can also be stated for any codimension, in particular, to get codim ≥ 5 examples. Moreover, it leads to the existence of codim ≥ 4 rings, but it does not explicitly give the defining equations of the surfaces we construct. In the case of codim 4, we work on a particular example under some mild conditions; which we can give explicitly by defining equations, and furthermore, by using birational geometry, we show that the surfaces we construct are quasismooth.

The next example leads to the general construction of codim 4 K3 surfaces with explicit equations from successive constructions in codimensions 2 and 3.

Example 2.2 We want to construct a codim 4 K3 surface X having an A_6 singularity in $\mathbb{P}(1,1,3,4,5,6,7)$ by constructing successively codim 2 and codim 3 K3 surfaces.

Let x_1, x_2, z, t, u be homogeneous coordinates on $\mathbb{P}(1,1,3,4,5)$ with weights $1,1,3,4,5$ respectively. Now define two linear systems $\mathcal{L}_1$, $\mathcal{L}_2$ of all homogeneous polynomials of degrees $6,8$ respectively in $k[x_1, x_2, z, t, u]$ with respect to weights $1,1,3,4,5$, containing the line

$$C_1\colon (x_1 = z = t = 0).$$

Let $f_i = x_1 a_{i1} + z a_{i2} + t a_{i3} \in \mathcal{L}_i$ be sufficiently general for $i = 1, 2$, where the a_{ij} are sufficiently general homogeneous polynomials in $k[x_1, x_2, z, t, u]$. Define a general surface

$$Z = X(6,8)\colon (f_1 = f_2 = 0) \subset \mathbb{P}(1,1,3,4,5). \tag{6}$$

By Bertini's theorem the singularities of $\mathrm{Cone}(Z)$ lie on the base locus $\mathrm{Cone}(C_1)$. We want Z to have an ordinary double point $q_1 = (0,1,0,0,0)$ and an A_4 singularity along C_1, where A_4 corresponds to the point $q_4 = (0,0,0,0,1)$. Obviously, this puts some restrictions on the a_{ij}. Clearly, locally at q_4 we have

$$f_1 = x_1 + \cdots \quad \text{and} \quad f_2 = z + \cdots \subset \mathbb{A}^4 / \langle \varepsilon \rangle,$$

where ε is a primitive 5th root of unity and $\cdots$ refers to other terms. By the inverse function theorem x_2, t are local coordinates. This implies that q_4 is a singularity of type A_4. The surface Z having a singularity q_1 is equivalent to $\mathrm{rk}\, J(Z)_{|q_1} \le 1$, where the Jacobian matrix $J(Z)$ of Z along C_1 is

$$\frac{\partial(f_1, f_2)}{\partial(x_1, z, t)}\Big|_{C_1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\Big|_{C_1}.$$

Since the a_{ij} are general $\mathrm{rk}\, J(Z)_{|q_1} = 1$. By replacing f_2 by f_2 with a suitable multiple of $x_2^2 f_1$ subtracted, we can assume that the a_{2j} do not contain any pure powers of x_2. From here it can be observed that $\mathrm{Cone}(Z)$ is nonsingular on $\mathrm{Cone}(C_1)$ except at the origin and q_1.

Now by Cramer's rule we can solve (6) to get $v\mathbf{x} = \bigwedge^2 M$, where $M = (a_{ij})$ is a 2×3 matrix and $\mathbf{x} = (x_1, z, t)$. Define $Y = X(6,7,8,9,10) \subset \mathbb{P}(1,1,3,4,5,6)$ by the five equations

$$v\mathbf{x} = \bigwedge^2 M \quad \text{and} \quad M\mathbf{x}^t = 0$$

which are also the submaximal Pfaffians of the following 5×5 skew-symmetric matrix

$$N=\begin{pmatrix} 0 & -t & z & a_{21} & -a_{11}\\ & 0 & -x_1 & a_{22} & -a_{12}\\ & & 0 & a_{23} & -a_{13}\\ & -\mathrm{sym} & & 0 & -v\\ & & & & 0\end{pmatrix},$$

where the entries of N are taken from the ring $k[x_1,x_2,z,t,u,v]$. We want to show that Y is a quasismooth K3 surface with A_5, containing the line

$$C_2\colon (x_1=z=t=u=0)$$

in $\mathbb{P}(1,1,3,4,5,6)$. Since the monomial x_2^α, for any nonzero α, does not appear in a_{2j}, Y contains the line C_2. Define a rational map $\varphi\colon Z\to Y$ by

$$(x_1,x_2,z,t,u)\longmapsto(x_1,x_2,z,t,u,v).$$

The map φ restricted to $Z\setminus C_1$ gives an isomorphism $Z\setminus C_1\xrightarrow{\sim} Y\setminus C_2$. Indeed, consider the map from $Y\setminus q_5$ to Z, which is projection from the point q_5, and so the inverse map is the restriction of this map to $Y\setminus C_2$. To see that $\mathrm{Cone}(Y)$ is smooth it is enough to consider $\mathrm{Cone}(Y)$ on $\mathrm{Cone}(C_2)$ excluding the origin since $\mathrm{Cone}(Z)$ is nonsingular outside $\mathrm{Cone}(C_1)$. Now observe that $(f_1=0)$ is nonsingular at q_1 and $(f_2=0)$ is singular at q_1. Consider $(f_2=0)$ on $(f_1=0)$ and calculate the tangent cone of this at q_1. One can prove that $\mathrm{Cone}(Y)$ being smooth corresponds to the tangent cone of $(f_2=0)$ on $(f_1=0)$ at q_1 being nondegenerate. Locally at q_5 there are three linearly independent equations of Y, which give rise to a singularity A_5. Since $l=k_{C_2}-k_Y=-7<0$ there exists a generator w of degree 7 which has a pole along C_2. This means that

$$w=\frac{A}{x_1}=\frac{B}{z}=\frac{C}{t}=\frac{D}{u}$$

is a rational function of weight 7 on Y, having a pole along C_2 for some homogeneous polynomials $A,B,C,D\in k[x_1,x_2,z,t,u,v]$. This gives four relations, namely:

$$x_1w=A,\quad zw=B,\quad tw=C\quad\text{and}\quad uw=D.$$

Now denote by X the variety given by the nine equations:

$$v\mathbf{x}=\bigwedge^2 M,\quad M\mathbf{x}^t=0\quad\text{and}\quad \mathbf{y}w=(A,B,C,D),$$

where $\mathbf{x} = (x_1, z, t)$ and $\mathbf{y} = (x_1, z, t, u)$. Notice that the first five equations are the defining equations of Y. Since a_{2j} does not contain a pure power of x_2 for $j = 1, 2, 3$ we can write

$$a_{2j} = x_1\lambda_1^j + z\lambda_2^j + t\lambda_3^j + u\lambda_4^j,$$

where $\lambda_i^j \in k[x_1, x_2, z, t, u]$ for $i = 1, 2, 3, 4$. To find A, B, C, D explicitly we rewrite the defining equations Pf_k of Y in terms of λ_i^j by substituting for the a_{2j}:

$$(\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5) = \mathbf{y}H$$

where

$$H = \begin{pmatrix} v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3 & a_{13}\lambda_1^1 - a_{11}\lambda_1^3 & a_{12}\lambda_1^1 - a_{11}\lambda_1^2 & a_{11} & a_{21} \\ a_{13}\lambda_2^2 - a_{12}\lambda_2^3 & a_{13}\lambda_2^1 - a_{11}\lambda_2^3 - v & a_{12}\lambda_2^1 - a_{11}\lambda_2^2 & a_{12} & a_{22} \\ a_{13}\lambda_3^2 - a_{12}\lambda_3^3 & a_{13}\lambda_3^1 - a_{11}\lambda_3^3 & a_{12}\lambda_3^1 - a_{11}\lambda_3^2 + v & a_{13} & a_{23} \\ a_{13}\lambda_4^2 & a_{13}\lambda_4^1 & a_{12}\lambda_4^1 - a_{11}\lambda_4^2 & 0 & 0 \end{pmatrix}.$$

First we begin by finding A and B. Since $zA - x_1B \in I(Y)$, some combination $\beta_1 \mathrm{Pf}_1 + \beta_2 \mathrm{Pf}_1$ of Pf_1 and Pf_2 can be made to contain a term which is a multiple of $a_{13}t$. Then we can add a multiple of Pf_4 to $\beta_1 \mathrm{Pf}_1 + \beta_2 \mathrm{Pf}_1$ to make it zero along $x_1 = z = 0$. This is possible since Pf_4 is zero along C_1. Explicitly:

$$-(a_{13}\lambda_4^1)\mathrm{Pf}_1 + (a_{13}\lambda_4^2)\mathrm{Pf}_2 - (-\lambda_4^1(a_{13}\lambda_3^2 - a_{12}\lambda_3^3) + \lambda_4^2(a_{13}\lambda_3^1 - a_{11}\lambda_3^3))\mathrm{Pf}_4 = x_1A + zB$$

where

$$\begin{aligned} A =& a_{13}\lambda_4^1(a_{13}\lambda_2^2 - a_{12}\lambda_2^3 - a_{12}\lambda_3^2) - a_{13}\lambda_4^2(a_{13}\lambda_2^1 - a_{11}\lambda_2^3 - v - a_{12}\lambda_3^1) \\ &+ a_{12}\lambda_3^3(a_{12}\lambda_4^1 - a_{11}\lambda_4^2), \\ B =& - a_{13}\lambda_4^1(v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3 - a_{11}\lambda_3^2) + a_{13}\lambda_4^2(a_{13}\lambda_1^1 - a_{11}\lambda_1^3 - a_{11}\lambda_3^1) \\ &+ a_{11}\lambda_3^3(-a_{12}\lambda_4^1 + a_{11}\lambda_4^2). \end{aligned}$$

Similarly,

$$(-a_{11}\lambda_4^2 + a_{12}\lambda_4^1)\mathrm{Pf}_1 - (a_{13}\lambda_4^2)\mathrm{Pf}_3 + (\lambda_4^1(-a_{13}\lambda_2^2 + a_{12}\lambda_2^3) + \lambda_4^2(-a_{11}\lambda_2^3 + a_{13}\lambda_2^1))\mathrm{Pf}_4 = x_1C - tA$$

and

$$(-a_{13}\lambda_2^1 + a_{11}\lambda_2^3 + v + a_{12}\lambda_3^1)\operatorname{Pf}_1 + (a_{13}\lambda_2^2 - a_{12}\lambda_2^3 - a_{12}\lambda_3^2)\operatorname{Pf}_2 \\ + (a_{12}\lambda_3^3)\operatorname{Pf}_3 + (E)\operatorname{Pf}_4 = -x_1 D + uA,$$

where

$$\begin{aligned} C =& (-a_{11}\lambda_4^2 + a_{12}\lambda_4^1)(v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3 + a_{11}\lambda_2^3) \\ & - a_{13}\lambda_4^2(a_{12}\lambda_1^1 - a_{11}\lambda_1^2 - a_{11}\lambda_2^1) - a_{13}\lambda_4^1(\lambda_2^2 a_{11}), \\ D =& -(-a_{13}\lambda_2^1 + a_{11}\lambda_2^3 + v + a_{12}\lambda_3^1)(v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3) \\ & - (a_{13}\lambda_2^2 - a_{12}\lambda_2^3 - a_{12}\lambda_3^2)(a_{13}\lambda_1^1 - a_{11}\lambda_1^3) \\ & - (a_{12}\lambda_3^3)(a_{12}\lambda_1^1 - a_{11}\lambda_1^2) - a_{11}(-\lambda_3^1(-a_{12}\lambda_2^3 + a_{13}\lambda_2^2) \\ & + \lambda_3^2(-a_{11}\lambda_2^3 + a_{13}\lambda_2^1 - v) - \lambda_3^3(-a_{11}\lambda_2^2 + a_{12}\lambda_2^1)), \\ E =& -\lambda_3^1(-a_{12}\lambda_2^3 + a_{13}\lambda_2^2) + \lambda_3^2(-a_{13}\lambda_2^3 + a_{13}\lambda_2^1 - v) - \lambda_3^3(-a_{11}\lambda_2^2 + a_{12}\lambda_2^1). \end{aligned}$$

The rational map $\varphi\colon Y \to X$ given by $(x_1, x_2, z, t, u, v) \mapsto (x_1, x_2, z, t, u, v, w)$ is birational and an isomorphism $Y \setminus C_2 \xrightarrow{\sim} X \setminus q_6$. Therefore it suffices to check that X is quasismooth at q_6. Clearly, X has an A_6 singularity since locally at this point q_6 there are four linearly independent equations $x_1 = A, z = B, t = C$ and $u = D$. Again, the inverse map is projection from the point q_6.

References

[1] S. Altınok, *Graded rings corresponding to polarised K3 surfaces and $\mathbb{Q}$-Fano 3 folds*, Ph.D. thesis, University of Warwick (1998).

[2] S. Altınok, *Hilbert Series and Applications to Graded Rings*, IJMMS **7** (2003), 397–403.

[3] S. Altınok, G. Brown, M. Reid, *Fano 3-folds, K3 surfaces and graded rings*, submitted to SISTAG conference proceedings (Singapore, Jul 2001), Contemporary Math. **314** (2002), 25–53.

[4] D.A. Buchsbaum and D. Eisenbud, *Algebra Structures for Finite Free Resolutions, and Some Structure Theorems for Ideals of Codimension 3*, Amer. J. Math **99** (1977), 447–485.

[5] I. Dolgachev, *Weighted Projective Space*, Group Actions and Vector Fields, Proc. Vancouver 1981, LNM **956**, 34–71, Springer–Verlag.

[6] A.R. Fletcher, *Working with Weighted Complete Intersection*, Explicit biratonal geomety of 3-folds, 101–173, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.

[7] R. Hartshorne, *Algebraic Geometry*, Springer G.T.M. **52** (1977).

[8] A.R. Kustin, M. Miller, *Constructing Big Gorenstein ideals from small ones*, Journal of Algebra **85** (1983), 303–322.

[9] S. Papadakis, M. Reid, *Kustin–Miller unprojection without complexes*, to appear in J. Algebraic Geometry, math. AG/0011094, 15 pp.

[10] M. Reid, *Young Person's Guide to Canonical Singularities*, Proc. Symposia in Pure Maths **46** (1987), 345–414.

[11] M. Reid, *Graded rings over* K3*s*, Conference on Commutative Algebra to the Memory of Prof. H. Matsumura, Nagoya, August 1996, 50–60.

[12] M. Reid, *Graded rings and birational geometry*, in proc of algebraic geometry symposium (Kinosaki, Oct. 2000), K. Ohno (Ed.), 1-72, available from www.maths.warwick.ac.uk/∼miles/3folds.

[13] B. Saint-Donat, *Projective Models of* K3 *Surfaces*, American Journal of Mathematics **96** (1974), 602–639.

Selma ALTINOK
Department of Mathematics,
Faculty of Art and Sciences,
Adnan Menderes University,
09010 Aydın-TURKEY
e-mail: saltinok43@yahoo.com

Received 17.02.2004