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On Intuitionistic Fuzzy Bi-Ideals of Semigroups

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Abstract

We consider the intuitionistic fuzzification of the concept of several ideals in a semigroup S, and investigate some properties of such ideals.

Key Words: Intuitionistic fuzzy $(1, 2)$-ideal, intuitionistic fuzzy bi-ideal, intuitionistic fuzzy ideal.

1. Introduction

After the introduction of fuzzy sets by L. A. Zadeh [8], several researchers explored on the generalization of the the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by K. T. Atanassov [1, 2], as a generalization of the notion of fuzzy set. In [3], N. Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals in semigroups. The concept of $(1, 2)$-ideals in semigroups was introduced by S. Lajos [5]. In this paper, we consider the intuitionistic fuzzification of the concept of several ideals in a semigroup S, and investigate some properties of such ideals.

2. Preliminaries

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset A of S such that $A^2 \subseteq A$, and by a *left* (*right*) *ideal* of S we mean a non-empty subset A of S such that $SA \subseteq A$ ($AS \subseteq A$). By *two-sided ideal* or simply *ideal*, we mean a non-empty subset of S which is both a left and a right ideal of S. A subsemigroup A of a semigroup

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S is called a *bi-ideal* of S if $ASA \subseteq A$. A subsemigroup A of S is called a $(1,2)$-*ideal* of S if $ASA^2 \subseteq A$. A semigroup S is said to be $(2,2)$-*regular* if $x \in x^2Sx^2$ for any $x \in S$. A semigroup S is said to be *regular* if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$. A semigroup S is said to be *completely regular* if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$ and $xy = yx$. For a semigroup S, note that S is completely regular if and only if S is a union of groups if and only if S is $(2,2)$-regular. A semigroup S is said to be *left* (resp. *right*) *duo* if every left (resp. right) ideal of S is a two-sided ideal of S

By a *fuzzy set* μ in a non-emptyset S we mean a function $\mu : S \to [0,1]$, and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in S given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in S$.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$.

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ in X can be identified to an ordered pair (μ_A, γ_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

3. Intuitionistic Fuzzy ideals

In what follows, let S denote a semigroup unless otherwise specified.

Definition 3.1 *[3] An IFS $A = (\mu_A, \gamma_A)$ in S is called an intuitionistic fuzzy subsemigroup of S if*

(i) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(ii) $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

for all $x, y \in S$.

Definition 3.2 *An IFS $A = (\mu_A, \gamma_A)$ in S is called an intuitionistic fuzzy left ideal of S if $\mu_A(xy) \geq \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)$ for all $x, y \in S$. An intuitionistic fuzzy right ideal* *of S is defined in an analogous way. An IFS $A = (\mu_A, \gamma_A)$ in S is called an intuitionistic fuzzy ideal of S if it is both an intuitionistic fuzzy right and an intuitionistic fuzzy left ideal of S.*

It is clear that any intuitionistic fuzzy left (right) ideal of S is an intuitionistic fuzzy subsemigroup of S.

Definition 3.3 *An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S is called an intuitionistic fuzzy bi-ideal of S if*

(i) $\mu_A(xwy) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(ii) $\gamma_A(xwy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$

for all $w, x, y \in S$.

Example 3.4 *Let $S := \{a, b, c, d, e\}$ be a semigroup with the following Cayley table:*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Define an IFS $A = (\mu_A, \gamma_A)$ in S by $\mu_A(a) = 0.6, \mu_A(b) = 0.5, \mu_A(c) = 0.4, \mu_A(d) = \mu_A(e) = 0.3$, $\gamma_A(a) = \gamma_A(b) = 0.3, \gamma_A(c) = 0.4$ and $\gamma_A(d) = 0.5, \gamma_A(e) = 0.6$. By routine calculation, we can check that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S.

Theorem 3.5 *If $\{A_i\}_{i\in\Lambda}$ is a family of intuitionistic fuzzy bi-ideals of S, then $\cap A_i$ is an intuitionistic fuzzy bi-ideal of S, where $\cap A_i = (\wedge\mu_{A_i}, \vee\gamma_{A_i})$ and*

$$\wedge\mu_{A_i}(x) = \inf\{\mu_{A_i}(x) \mid i \in \Lambda, x \in S\},$$

$$\vee\gamma_{A_i}(x) = \sup\{\gamma_{A_i}(x) \mid i \in \Lambda, x \in S\}.$$

Proof. Let $x, y \in S$. Then we have

$$\begin{aligned}\wedge\mu_{A_i}(xy) &\geq \wedge\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\} \\ &= \min\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\}, \\ &= \min\{\min\{\mu_{A_i}(x)\}, \min\{\mu_{A_i}(y)\}\} \\ &= \min\{\wedge\mu_{A_i}(x), \wedge\mu_{A_i}(y)\},\end{aligned}$$

$$\begin{aligned}\vee\gamma_{A_i}(xy) &\leq \vee\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\}\} \\ &= \max\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\}\}, \\ &= \max\{\max\{\gamma_{A_i}(x)\}, \max\{\gamma_{A_i}(y)\}\} \\ &= \max\{\vee\gamma_{A_i}(x), \vee\gamma_{A_i}(y)\}.\end{aligned}$$

Hence $\cap A_i$ is an intuitionistic fuzzy subsemigroup of S. Next for $x, y, a \in S$, we obtain

$$\begin{aligned}\wedge\mu_{A_i}(xay) &\geq \wedge\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\} \\ &= \min\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\}, \\ &= \min\{\min\{\mu_{A_i}(x)\}, \min\{\mu_{A_i}(y)\}\} \\ &= \min\{\wedge\mu_{A_i}(x), \wedge\mu_{A_i}(y)\},\end{aligned}$$

$$\begin{aligned}\vee\gamma_{A_i}(xay) &\leq \vee\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\}\} \\ &= \max\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\}\}, \\ &= \max\{\max\{\gamma_{A_i}(x)\}, \max\{\gamma_{A_i}(y)\}\} \\ &= \max\{\vee\gamma_{A_i}(x), \vee\gamma_{A_i}(y)\}.\end{aligned}$$

Hence $\cap A_i$ is an intuitionistic fuzzy bi-ideal of S. This completes the proof. □

Theorem 3.6 *If an* IFS $A = (\mu_A, \gamma_A)$ *in* S *is an intuitionistic fuzzy bi-ideal of* S*, then so is* $\Box A := (\mu_A, \overline{\mu_A})$.

Proof. It is sufficient to show that $\overline{\mu_A}$ satisfies the condition (ii) in Definition 3.1, and (ii) in Definition 3.3. For any $a, x, y \in S$, we have

$$\overline{\mu_A}(xy) = 1 - \mu_A(xy) \leq 1 - \min\{\mu_A(x), \mu_A(y)\}$$
$$= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} = \max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}$$

and $\overline{\mu_A}(xay) = 1 - \mu_A(xay) \leq 1 - \min\{\mu_A(x), \mu_A(y)\} = \max\{1 - \mu_A(x), 1 - \mu_A(y)\} = \max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}$. Therefore $\Box A$ is an intuitionistic fuzzy bi-ideal of S. □

Definition 3.7 *An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S is called an intuitionistic fuzzy $(1,2)$-ideal of S if*

(i) $\mu_A(xw(yz)) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\}$,

(ii) $\gamma_A(xw(yz)) \leq \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}$

for all $w, x, y, z \in S$.

Theorem 3.8 *Every intuitionistic fuzzy bi-ideal is an intuitionistic fuzzy $(1,2)$-ideal.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy bi-ideal of S and let $w, x, y, z \in S$. Then

$$\begin{aligned}\mu_A(xw(yz)) &= \mu_A((xwy)z)\\ &\geq \min\{\mu_A(xwy), \mu_A(z)\}\\ &\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \mu_A(z)\}\\ &= \min\{\mu_A(x), \mu_A(y), \mu_A(z)\},\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xw(yz)) &= \gamma_A((xwy)z)\\ &\leq \max\{\gamma_A(xwy), \gamma_A(z)\}\\ &\leq \max\{\max\{\gamma_A(x), \gamma_A(y)\}, \gamma_A(z)\}\\ &= \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}.\end{aligned}$$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $(1,2)$-ideal of S. □

To consider the converse of Theorem 3.9, we need to strengthen the condition of a semigroup S.

Theorem 3.9 *If S is a regular semigroup, then every intuitionistic fuzzy $(1,2)$-ideal of S is an intuitionistic fuzzy bi-ideal of S.*

Proof. Assume that a semigroup S is regular and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy $(1,2)$-ideal of S. Let $w, x, y \in S$. Since S is regular, we have $xw \in (xSx)S \subseteq xSx$, which implies that $xw = xsx$ for some $s \in S$. Thus

$$\begin{aligned}\mu_A(xwy) &= \mu_A((xsx)y) = \mu_A(xs(xy))\\ &\geq \min\{\mu_A(x), \mu_A(x), \mu_A(y)\}\\ &= \min\{\mu_A(x), \mu_A(y)\},\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xwy) &= \gamma_A((xsx)y) = \gamma_A((xs(xy))\\ &\leq \max\{\gamma_A(x), \gamma_A(x), \gamma_A(y)\}\\ &= \max\{\gamma_A(x), \gamma_A(y)\}.\end{aligned}$$

Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S. □

Theorem 3.10 *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy bi-ideal of S. If S is a completely regular, then $A(a) = A(a^2)$ for all $a \in S$.*

Proof. Let $a \in S$. Then there exists $x \in S$ such that $a = a^2xa^2$. Hence

$$\begin{aligned}\mu_A(a) &= \mu_A(a^2xa^2) \geq \min\{\mu_A(a^2), \mu_A(a^2)\}\\ &= \mu_A(a^2) \geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(a) &= \gamma_A(a^2xa^2) \leq \max\{\gamma_A(a^2), \gamma_A(a^2)\}\\ &= \gamma_A(a^2) \leq \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a).\end{aligned}$$

It follows that $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$ so that $A(a) = A(a^2)$. □

Theorem 3.11 *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of S. If S is an intra-regular, then $A(a) = A(a^2)$ for all $a \in S$.*

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a = xa^2y$. Hence since $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal,

$$\begin{aligned}\mu_A(a) &= \mu_A(xa^2y) \geq \mu_A(xa^2) \\ &\geq \mu_A(a^2) \geq \{\mu_A(a), \mu_A(a)\} = \mu_A(a)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(a) &= \gamma_A((xa^2y)) = \gamma_A(xa^2) \leq \gamma_A(a^2) \\ &\leq \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a).\end{aligned}$$

Hence we have $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$. Therefore $A(a) = A(a^2)$ for all $x, y \in S$. □

Theorem 3.12 *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of S. If S is an intra-regular, then $A(ab) = A(ba)$ for all $a, b \in S$.*

Proof. Let $a, b \in S$. Then by Theorem 3.14, we have

$$\begin{aligned}\mu_A(ab) &= \mu_A((ab)^2) \geq \mu_A(a(ba)b) \\ &\geq \mu_A(ba) = \mu_A((ba)^2) \geq \mu_A(b(ab)a) \geq \mu_A(ab)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(ab) &= \gamma_A((ab)^2) = \gamma_A(a(ba)b) \leq \gamma_A(ba) \\ &= \gamma_A((ba)^2) = \gamma_A(b(ab)a) \leq \gamma_A(ab).\end{aligned}$$

So we have $\mu_A(ab) = \mu_A(ba)$ and $\gamma_A(ab) = \gamma_A(ba)$. Therefore $A(ab) = A(ba)$. □

Theorem 3.13 *An* IFS *$A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S if and only if the fuzzy sets μ_A and $\overline{\gamma_A}$ are fuzzy bi-ideals of S.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy bi-ideal of S. Then clearly μ_A is a fuzzy bi-ideal of S. Let $x, a, y \in S$. Then

$$\begin{aligned}\overline{\gamma_A}(xy) &= 1 - \gamma_A(xy)\\ &\geq 1 - \max\{\gamma_A(x), \gamma_A(y)\}\\ &= \min\{(1 - \gamma_A(x)), (1 - \gamma_A(y))\}\\ &= \min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}, \text{ and}\end{aligned}$$

$$\begin{aligned}\overline{\gamma_A}(xay) &= 1 - \gamma_A(xay)\\ &\geq 1 - \max\{\gamma_A(x), \gamma_A(y)\}\\ &= \min\{(1 - \gamma_A(x)), (1 - \gamma_A(y))\}\\ &= \min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}.\end{aligned}$$

Hence $\overline{\gamma_A}$ is a fuzzy bi-ideal of S.

Conversely, suppose that μ_A and $\overline{\gamma_A}$ are fuzzy bi-ideals of S. Let $a, x, y \in S$. Then

$$\begin{aligned}1 - \gamma_A(xy) = \overline{\gamma_A}(xy) &\geq \min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}\\ &= \min\{(1 - \gamma_A(x)), (1 - \gamma_A(y))\}\\ &= \max\{\gamma_A(x), \gamma_A(y)\}, \text{ and}\end{aligned}$$

$$\begin{aligned}1 - \gamma_A(xay) &= \overline{\gamma_A}(xay)\\ &\geq \min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}\\ &= 1 - \max\{\gamma_A(x), \gamma_A(y)\}\end{aligned}$$

which imply that $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$ and $\gamma_A(xay) \leq \max\{\gamma_A(x), \gamma_A(y)\}$. This completes the proof. □

Corollary 3.14 *An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S if and only if $\Box A = (\mu_A, \overline{\mu_A})$ and $\Diamond A = (\overline{\gamma_A}, \gamma_A)$ are intuitionistic fuzzy bi-ideals of S.*

Proof. It is straightforward by Theorem 3.14. □

Let f be a map from a set X to a set Y. If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IFSs in X and Y respectively, then the *preimage* of B under f, denoted by $f^{-1}(B)$, is an IFS in X defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)).$$

Theorem 3.15 *Let $f : S \to T$ be a homomorphism of semigroups. If $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy bi-ideal of T, then the preimage $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ of B under f is an intuitionistic fuzzy bi-ideal of S.*

Proof. Assume that $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy bi-ideal of T and let $x, y, \in S$. Then

$$\begin{aligned} f^{-1}(\mu_B)(xy) &= \mu_B(f(xy)) \\ &= \mu_B(f(x)f(y)) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\ &= \min\{f^{-1}(\mu_B(x)), f^{-1}(\mu_B(y))\}, \text{ and} \end{aligned}$$

$$\begin{aligned} f^{-1}(\gamma_B)(xy) &= \gamma_B(f(xy)) \\ &= \gamma_B(f(x)f(y)) \\ &\leq \max\{\gamma_B(f(x)), \gamma_B(f(y))\} \\ &= \max\{f^{-1}(\gamma_B(x)), f^{-1}(\gamma_B(y))\}. \end{aligned}$$

Hence $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic fuzzy subsemigroup of S. For any $a, x, y \in S$ we have

$$\begin{aligned} f^{-1}(\mu_B)(xay) &= \mu_B(f(xay)) \\ &= \mu_B(f(x)f(a)f(y)) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\ &= \min\{f^{-1}(\mu_B(x)), f^{-1}(\mu_B(y))\} \text{ and} \end{aligned}$$

$$\begin{aligned} f^{-1}(\gamma_B)(xay) &= \gamma_B(f(xay)) \\ &= \gamma_B(f(x)f(a)f(y)) \\ &\le \max\{\gamma_B(f(x)), \gamma_B(f(y))\} \\ &= \max\{f^{-1}(\gamma_B(x)), f^{-1}(\gamma_B(y))\}. \end{aligned}$$

Therefore $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic fuzzy bi-ideal of S. □

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