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On δ-I-Continuous Functions

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Abstract

In this paper, we introduce a new class of functions called δ-I-continuous functions. We obtain several characterizations and some of their properties. Also, we investigate its relationship with other types of functions.

Key words and phrases: δ-I-cluster point, R-I-open set, δ-I-continuous, strongly θ-I-continuous, almost-I-continuous, SI-R space, AI-R space.

1. Introduction

Throughout this paper Cl(A) and Int(A) denote the closure and the interior of A, respectively. Let (X,τ) be a topological space and let I an ideal of subsets of X. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) If A$\in$I and B$\subset$I, then B$\in$I ; (2) If A$\in$I and B$\in$I, then A$\cup$B$\in$I. An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I). For a subset A$\subset$X, $A^*(I)$ = {x$\in$X | U$\cap$A$\notin I$ for each neighborhood U of x} is called the local function of A with respect to I and τ [4]. We simply write A^* instead of $A^*(I)$ to be brief. X^* is often a proper subset of X. The hypothesis X = X^*[1] is equivalent to the hypothesis $\tau \cap I = \emptyset$ [5]. For every ideal topological space (X,τ,I), there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta(I,\tau)$ = {U\I: U$\in\tau$ and I$\in I$}, but in general $\beta(I,\tau)$ is not always a topology [2]. Additionally, Cl^*(A) = A$\cup(A)^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

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In this paper, we introduce the notions of δ-I-open sets and δ-I-continuous functions in ideal topological spaces. We obtain several characterizations and some properties of δ-I-continuous functions. Also, we investigate the relationships with other related functions.

2. δ-I-open sets

In this section, we introduce δ-I-open sets and the δ-I-closure of a set in an ideal topological space and investigate their basic properties. It turns out that they have similar properties with δ-open sets and the δ-closure due to Veličko [6].

Definition 2.1 *A subset A of an ideal topological space (X,τ,I) is said to be an R-I-open (resp. regular open) set if $Int(Cl^*(A)) = A$ (resp. $Int(Cl(A)) = A$). We call a subset A of X R-I-closed if its complement is R-I-open.*

Definition 2.2 *Let (X,τ,I) be an ideal topological space, S a subset of X and x a point of X.*

(1) x is called a δ-I-cluster point of S if $S \cap Int(Cl^*(U)) \neq \emptyset$ for each open neighborhood x;

(2) The family of all δ-I-cluster points of S is called the δ-I-closure of S and is denoted by $[S]_{\delta-I}$ and

(3) A subset S is said to be δ-I-closed if $[S]_{\delta-I} = S$. The complement of a δ-I-closed set of X is said to be δ-I-open.

Lemma 2.1 *Let A and B be subsets of an ideal topological space (X,τ,I). Then, the following properties hold:*

(1) $Int(Cl^*(A))$ is R-I-open;

(2) If A and B are R-I-open, then $A \cap B$ is R-I-open;

(3) If A is regular open, then it is R-I-open;

(4) If A is R-I-open, then it is δ-I-open and

(5) Every δ-I-open set is the union of a family of R-I-open sets.

Proof. (1) Let A be a subset of X and V = Int(Cl^*(A)). Then, we have Int(Cl^*(V)) = Int(Cl^*(Int(Cl^*(A))))$\subset$ Int(Cl^*(Cl^*(A))) = Int(Cl^*(A)) = V and also V = Int(V) $\subset$ Int(Cl^*(V)). Therefore, we obtain Int(Cl^*(V)) = V.

(2) Let A and B be R-I-open. Then, we have A∩B = Int(Cl^*(A))∩Int(Cl^*(B)) = Int(Cl^*(A)∩Cl^*(B)) ⊃ Int(Cl^*(A∩B)) ⊃ Int(A∩B) = A∩B. Therefore, we obtain A∩B = Int(Cl^*(A∩B)). This shows that A∩B is R-I-open.

(3) Let A be regular open. Since $\tau^* \supset \tau$, we have A = Int(A)⊂Int(Cl^*(A)) ⊂Int(Cl(A)) = A and hence A = Int(Cl^*(A)). Therefore, A is R-I-open.

(4) Let A be any R-I-open set. For each x∈A, (X-A)∩A = ∅ and A is R-I-open. Hence x∉$[X-A]_{\delta-I}$ for each x∈A. This shows that x∉(X-A) implies x∉$[X-A]_{\delta-I}$. Therefore, we have $[X-A]_{\delta-I}$⊂(X-A). Since in general, S⊂ $[S]_{\delta-I}$ for any subset S of X, $[X-A]_{\delta-I}$ = (X-A) and hence A is δ-I-open.

(5) Let A be a δ-I-open set. Then (X-A) is δ-I-closed and hence (X-A) = $[X-A]_{\delta-I}$. For each x∈A, x∉$[X-A]_{\delta-I}$ and there exists an open neighborhood V_x such that Int(Cl^*(V_x))∩(X-A) = ∅. Therefore, we have x∈V_x⊂Int(Cl^*(V_x))⊂A and hence A = ∪{Int(Cl^*(V_x)) | x∈A}. By (1), Int(Cl^*(V_x)) is R-I-open for each x∈A. □

Lemma 2.2 *Let A and B be subsets of an ideal topological space (X,τ,I). Then, the following properties hold:*

(1) A⊂$[A]_{\delta-I}$;

(2) If A⊂B, then $[A]_{\delta-I}$⊂$[B]_{\delta-I}$;

(3) $[A]_{\delta-I}$ = ∩{F⊂X | A⊂F and F is δ-I-closed};

(4) If A is a δ-I-closed set of X for each α∈Δ, then ∩{A_α | α∈Δ} is δ-I-closed;

(5) $[A]_{\delta-I}$ is δ-I-closed.

Proof. (1) For any x∈A and any open neighborhood V of x, we have ∅≠A∩V⊂A∩ Int(Cl^*(V)) and hence x∈ $[A]_{\delta-I}$. This shows that A ⊂ $[A]_{\delta-I}$.

(2) Suppose that x∉$[B]_{\delta-I}$.There exists an open neighborhood V of x such that ∅ = Int(Cl^*(V))∩B; hence Int(Cl^*(V))∩A = ∅. Therefore, we have x∉$[A]_{\delta-I}$.

(3) Suppose that x∈$[A]_{\delta-I}$. For any open neighborhood V of x and any δ-I-closed set F containing A, we have ∅≠A∩Int(Cl^*(V))⊂F∩ Int(Cl^*(V)) and hence x∈$[F]_{\delta-I}$ = F. This shows that x∈∩{F⊂X | A⊂F and F is δ-I-closed}. Conversely, suppose that x∉ $[A]_{\delta-I}$. There exists an open neighborhood V of x such that Int(Cl^*(V))∩A = ∅. By Lemma 2.1, X-Int(Cl^*(V))is a δ-I-closed set which contains A and does not contain x. Therefore, we obtain x∉∩ {F⊂X | A⊂F and F is δ-I-closed}. This completes the proof. □

(4) For each $\alpha\in\Delta$, $[\bigcap_{\alpha\in\Delta} A_\alpha]_{\delta-I}\subset [A_\alpha]_{\delta-I} = A_\alpha$ and hence $[\bigcap_{\alpha\in\Delta} A_\alpha]_{\delta-I}\subset [\bigcap_{\alpha\in\Delta} A_\alpha]$. By (1), we obtain $[\bigcap_{\alpha\in\Delta} A_\alpha]_{\delta-I} = [\bigcap_{\alpha\in\Delta} A_\alpha]$. This shows that $\bigcap_{\alpha\in\Delta} A_\alpha$ is δ-I-closed.

(5) This follows immediately from (3) and (4).

A point x of a topological space (X,τ) is called a δ-cluster point of a subset S of X if Int(Cl(V))∩S≠∅ for every open set V containing x. The set of all δ-cluster points of S is called the δ-closure of S and is denoted by $Cl_\delta(S)$. If $Cl_\delta(S)$ = S, then S is said to be δ-closed [6]. The complement of a δ-closed set is said to be δ-open. It is well-known that the family of regular open sets of (X,τ) is a basis for a topology which is weaker than τ. This topology is called the *semi-regularization* of τ and is denoted by τ_S. Actually, τ_S is the same as the family of δ-open sets of (X,τ).

Theorem 2.1 *Let (X,τ,I) be an ideal topological space and $\tau_{\delta-I} = \{A\subset X \mid A$ is a δ-I-open set of (X,τ,I)$\}$. Then $\tau_{\delta-I}$ is a topology such that $\tau_S\subset\tau_{\delta-I}\subset\tau$.*

Proof. By Lemma 2.1, we obtain $\tau_S\subset \tau_{\delta-I}\subset\tau$. Next, we show that $\tau_{\delta-I}$ is a topology.

(1) It is obvious that ∅, X$\in\tau_{\delta-I}$.

(2) Let $V_\alpha\in\tau_{\delta-I}$ for each $\alpha\in\Delta$. Then X-V_α is δ-I-closed for each $\alpha\in\Delta$. By Lemma 2.2, $\bigcap_{\alpha\in\Delta}$(X-$V_\alpha$) is δ-I-closed and $\bigcap_{\alpha\in\Delta}$(X-$V_\alpha$) = X-$\bigcup_{\alpha\in\Delta} V_\alpha$. Hence $\bigcup_{\alpha\in\Delta} V_\alpha$ is δ-I-open.

(3) Let A,B$\in\tau_{\delta-I}$. By Lemma 2.1, A = $\bigcup_{\alpha\in\Delta_1} A_\alpha$ and B = $\bigcup_{\beta\in\Delta_2} B_\beta$, where A_α and B_β are R-I-open sets for each $\alpha\in\Delta_1$ and $\beta\in\Delta_2$. Thus A∩B = $\cup\{A_\alpha\cap B_\beta \mid \alpha\in\Delta_1, \beta\in\Delta_2\}$. Since $A_\alpha\cap B_\beta$ is R-I-open, A∩B is a δ-I-open set by Lemma 2.1. □

Example 2.1 *Let X = $\{a,b,c,d\}$, $\tau = \{\emptyset,X,\{a\},\{c\},\{a,c\}\}$ and I = $\{\emptyset,\{a\}\}$. Then A = $\{a,c\}$ is a δ-I-open set which is not R-I-open. Since $\{a\}$ and $\{c\}$ are regular open sets, A is a δ-open set and hence δ-I-open. But A is not R-I-open. Because $A^* = \{b,c,d\}$ and $Cl^*(A) = A\cup A^* = X$. Therefore, we have $Int(Cl^*(A)) = X\neq A$.*

For some special ideals, we have the following properties.

Proposition 2.1 *Let (X,τ,I) be an ideal topological space.*

(1) If I = $\{\emptyset\}$ or the ideal N of nowhere dense sets of (X,τ), then $\tau_{\delta-I} = \tau_S$.

(2) If I = P(X), then $\tau_{\delta-I} = \tau$.

Proof. (1) Let I = {∅}, then S^* = Cl(S) for every subset S of X. Let A be R-I-open. Then A = Int(Cl^*(A)) = (A∪A^*) = Int(Cl(A)) and hence A is regular open. Therefore, every δ-I-open set is δ-open and we obtain $\tau_{\delta-I} \subset \tau_S$. By Theorem 2.1, we obtain $\tau_{\delta-I}$ = τ_S. Next, Let I = N. It is well-know that S^* = Cl(Int(Cl(S))) for every subset S of X. Let A be any R-I-open set. Then since A is open, A = Int(Cl^*(A)) = Int(A∪A^*) = Int(A∪Cl(Int(Cl(A)))) = Int(Cl(Int(Cl(A)))) = Int(Cl(A)). Hence A is regular open. Similarly to the case of I = {∅}, we obtain $\tau_{\delta-I} = \tau_S$.

(2) Let I = P(X). Then S^* = ∅ for every subset S of X. Now, let A be any open set of X. Then A = Int(A) = Int(A∪A^*) = Int(Cl^*(A)) and hence A is R-I-open. By Theorem 2.1, we obtain $\tau_{\delta-I} = \tau$. □

3. δ-I-continuous functions

Definition 3.1 *A function f:(X,τ,I)→(Y,Φ,J) is said to be δ-I-continuous if for each x∈X and each open neighborhood V of f(x), there exists an open neighborhood U of x such that f(Int(Cl^*(U)))⊂Int(Cl^*(V)).*

Theorem 3.1 *For a function f:(X,τ,I)→(Y,Φ,J), the following properties are equivalent:*

(1) f is δ-I-continuous;

(2) For each x∈X and each R-I-open set V containing f(x), there exists an R-I-open set containing x such that f(U)⊂V;

(3) f($[A]_{\delta-I}$)⊂$[f(A)]_{\delta-I}$ for every A⊂X;

(4) $[f^{-1}(B)]_{\delta-I} \subset f^{-1}([B]_{\delta-I})$ for every B⊂Y;

(5) For every δ-I-closed set F of Y, f^{-1}(F) is δ-I-closed in X;

(6) For every δ-I-open set V of Y, f^{-1}(V) is δ-I-open in X;

(7) For every R-I-open set V of Y, f^{-1}(V) is δ-I-open in X;

(8) For every R-I-closed set F of Y, f^{-1}(F) is δ-I-closed in X.

Proof. (1)⇒(2): This follows immediately from Definition 3.1.

(2)⇒(3): Let x∈X and A⊂X such that f(x)∈f($[A]_{\delta-I}$). Suppose that f(x)∉ $[f(A)]_{\delta-I}$. Then, there exists an R-I-open neighborhood V of f(x) such that f(A)∩V = ∅. By (2), there exists an R-I-open neighborhood U of x such that f(U)⊂V. Since f(A)∩f(U)⊂f(A)∩V = ∅, f(A)∩f(U) = ∅. Hence, we get that U∩A⊂f^{-1}(f(U))∩f^{-1}(f(A)) = f^{-1}(f(U)∩f(A))

$= \emptyset$. Hence we have U∩A $= \emptyset$ and x$\notin[A]_{\delta-I}$. This shows that f(x)$\notin f([A]_{\delta-I})$. This is a contradiction. Therefore, we obtain that f(x)$\in[f(A)]_{\delta-I}$.

(3)⇒(4): Let B⊂Y such that A = f^{-1}(B). By (3), $f([f^{-1}(B)]_{\delta-I})\subset$ $[f(f^{-1}B))]_{\delta-I}\subset[B]_{\delta-I}$. From here, we have $[f^{-1}(B)]_{\delta-I}\subset$ $f^{-1}([f(f^{-1}(B))]_{\delta-I})\subset f^{-1}([B]_{\delta-I})$. Thus we obtain that $[f^{-1}(B)]_{\delta-I}\subset$ $f^{-1}([B]_{\delta-I})$.

(4)⇒(5): Let F⊂Y be δ-I-closed. By (4), $[f^{-1}(F)]_{\delta-I}\subset f^{-1}([F]_{\delta-I}) = f^{-1}$(F) and always f^{-1}(F)$\subset[f^{-1}(F)]_{\delta-I}$. Hence we obtain that $[f^{-1}(F)]_{\delta-I} = f^{-1}(F)$. This shows that f^{-1}(F) is δ-I-closed.

(5)⇒(6): Let V⊂Y be δ-I-open. Then Y-V is δ-I-closed. By (5), f^{-1}(Y-V) = X-f^{-1}(V) is δ-I-closed. Therefore, f^{-1}(V) is δ-I-open.

(6)⇒(7): Let V⊂Y be R-I-open. Since every R-I-open set is δ-I-open, V is δ-I-open, By (6), f^{-1}(V) is δ-I-open.

(7)⇒(8): Let F⊂Y be an R-I-closed set. Then Y-F is R-I-open. By (7), f^{-1}(Y-F) = X-f^{-1}(F) is δ-I-open. Therefore, f^{-1}(F) is δ-I-closed.

(8)⇒(1): Let x∈X and V be an open set containing f(x). Now, set V_o = Int(Cl^*(V)), then by Lemma 2.1 Y-V_o is an R-I-closed set. By (8), f^{-1}(Y-V_o) = X-$f^{-1}(V_o)$ is a δ-I-closed set. Thus we have $f^{-1}(V_o)$ is δ-I-open. Since x$\in f^{-1}(V_o)$, by Lemma 2.1, there exists an open neighborhood U of x such that x∈U⊂Int(Cl^*(U))$\subset f^{-1}(V_o)$. Hence we obtain that f(Int(Cl^*(U)))⊂Int(Cl^*(V)). This shows that f is a δ-I-continuous function.

□

Corollary 3.1 *A function f:(X,τ,I)→(Y,Φ,J) is δ-I-continuous if and only if f:(X,τ,I)→(Y,Φ,J) is continuous.*

Proof. This is an immediate consequence of Theorem 2.1.

The following lemma is known in [3, as Lemma 4.3]. □

Lemma 3.1 *Let (X,τ,I) be an ideal topological space and A,B subsets of X such that B⊂A. Then* $B^*(\tau/A,I/A) = B^*(\tau,I)\cap A$.

Proposition 3.1 *Let (X,τ,I) be an ideal topological space, A, X_o subsets of X such that $A \subset X_o$ and X_o is open in X.*

(1) If A is R-I-open in (X,τ,I), then A is R-I-open in $(X_o,\tau/X_o,I/X_o)$,

(2) If A is δ-I-open in (X,τ,I), then A is δ-I-open in $(X_o,\tau/X_o,I/X_o)$.

Proof. (1) Let A be R-I-open in (X,τ,I). Then A = Int(Cl^*(A)) and $Cl^*_{X_o}(A)$ = $A \cup A^*(\tau/X_o, I/X_o) = A \cup [A^*(\tau, I) \cap X_o] = (A \cap X_o) \cup (A^* \cap X_o) = (A \cup A^*) \cap X_o = Cl^*(A) \cap X_o$. Hence we have $Int_{X_o}(Cl^*_{X_o}(A))$ = Int($Cl^*_{X_o}(A)$) = Int((Cl^*(A) $\cap X_o$) = Int((Cl^*(A))$\cap X_o$ = A. Therefore, A is R-I-open in $(X_o, \tau/X_o, I/X_o)$.

(2) Let A be a δ-I-open set of (X,τ,I). By Lemma 2.1, A = $\bigcup_{\alpha \in \Delta} A_\alpha$, where A_α is R-I-open set of (X,τ,I) for each $\alpha \in \Delta$. By (1), A is R-I- open in $(X_o,\tau/X_o,I/X_o)$ for each $\alpha \in \Delta$ and hence A is δ-I-open in $(X_o,\tau/X_o,I/X_o)$. □

Theorem 3.2 *If $f:(X,\tau,I) \to (Y,\Phi,J)$ is a δ-I-continuous function and X_o is a δ-I-open set of (X,τ,I), then the restriction $f/X_o:(X_o,\tau/X_o,I/X_o) \to (Y,\Phi,J)$ is δ-I-continuous.*

Proof. Let V be any δ-I-open set of (Y,Φ,J). Since f is δ-I-continuous, f^{-1}(V) is δ-I-open in (X,τ,I). Since X_o is δ-I-open, by Theorem 2.1 $X_o \cap f^{-1}(V)$ is δ-I-open in (X,τ,I) and hence $X_o \cap f^{-1}$(V) is δ-I-open in $(X_o,\tau/X_o,I/X_o)$ by Proposition 3.1. This shows that $(f/X_o)^{-1}$(V) is δ-I-open in $(X_o,\tau/X_o,I/X_o)$ and hence f/X_o is δ-I-continuous. □

Theorem 3.3 *If $f:(X,\tau,I) \to (Y,\Phi,J)$ and $g:(Y,\Phi,J) \to (Z,\varphi,K)$ are δ-I-continuous, then so is $gof:(X,\tau,I) \to (Z,\varphi,K)$.*

Proof. It follows immediately from Cor. 3.1. □

Theorem 3.4 *If $f,g:(X,\tau,I) \to (Y,\Phi,J)$ are δ-I-continuous functions and Y is a Hausdorff space, then $A = \{x \in X : f(x) = g(x)\}$ is a δ-I-closed set of (X,τ,I).*

Proof. We prove that X-A is δ-I-open set. Let x∈X-A. Then, f(x)≠g(x). Since Y is Hausdorff, there exist open sets V_1 and V_2 containing f(x) and g(x), respectively, such that $V_1 \cap V_2 = \emptyset$. From here we have Int(Cl(V_1))∩Int(Cl(V_2)) = ∅. Thus, we obtain that Int($Cl^*(V_1)$)∩Int($Cl^*(V_2)$) = ∅. Since f and g are δ-I-continuous, there exists an open neighborhood U of x such that f(Int(Cl^*(U)))⊂Int(Cl^*(V_1)) and g(Int(Cl^*(U)))⊂Int(Cl^* (V_2)). Hence we obtain that Int(Cl^*(U))⊂f^{-1}(Int(Cl^*(V_1))) and Int(Cl^*(U))⊂g^{-1}(Int (Cl^*(V_2))). From here we have Int(Cl^*(U))⊂f^{-1}(Int(Cl^*(V_1)))∩ g^{-1}(Int(Cl^*(V_2))). Moreover f^{-1}(Int(Cl^*(V_1)))∩g^{-1}(Int(Cl^*(V_2)))∩A = ∅. Suppose that f^{-1}(Int(Cl^*(V_1))) ∩g^{-1}(Int(Cl^*(V_2)))∩A≠∅. Hence there exists a point z such that z∈f^{-1}(Int(Cl^*(V_1)))∩g^{-1} (Int(Cl^*(V_2)))∩A. Thus, f(z)∈Int(Cl^*(V_1)), g(z)∈Int(Cl^*(V_2)) and z∈A. Since z∈A, f(z) = g(z). Therefore, we have f(z)∈Int(Cl^*(V_1))∩Int(Cl^*(V_2)) and Int(Cl^*(V_1))∩ Int(Cl^* (V_2))≠ ∅. This is a contradiction to Int(Cl^*(V_1)∩Int(Cl^*(V_2) = ∅. Hence we obtain that f^{-1}(Int(Cl^*(V_1)))∩g^{-1}(Int(Cl^*(V_2)))∩A = ∅. Thus f^{-1}(Int(Cl^*(V_1)))∩g^{-1}(Int(Cl^* (V_2)))⊂X-A. Since Int(Cl^*(U))⊂f^{-1}(Int(Cl^*(V_1)))∩g^{-1}(Int(Cl^*(V_2)), we have that there exists an open neighborhood of x such that x∈U Int(Cl^*(U))⊂X-A. Therefore, X-A is a δ-I-open set. This shows that A is δ-I-closed. □

4. Comparisons

Definition 4.1 *A function f:(X,τ,I)→(Y,Φ,J) is said to be strongly θ-I-continuous (resp. θ-I-continuous, almost I-continuous) if for each x∈X and each open neighborhood V of f(x), there exists an open neighborhood U of x such that f(Cl^*(U))⊂ V (resp. f(Cl^*(U))⊂Cl^* (V), f(U)⊂Int(Cl^*(V)).*

Definition 4.2 *A function f:(X,τ,I)→(Y,Φ,J) is said to be almost-I-open if for each R-I-open set U of X, f(U) is open in Y.*

Theorem 4.1 *(1) If f:(X,τ,I)→(Y,Φ,J) is strongly θ-I-continuous and g:(Y,Φ,J)→(Z,φ,K) is almost I-continuous, then gof:(X,τ,I)→(Z,φ,K) is δ-I-continuous.*

(2) The following implications hold:

$$\text{strongly}\quad \theta-\mathrm{I}-\text{continuous} \Rightarrow \delta-\mathrm{I}-\text{continuous} \Rightarrow \text{almost}-\mathrm{I}-\text{continuous}. \quad (4.1)$$

Proof. (1) Let x∈X and W be any open set of Z containing (gof)(x). Since g is almost I-continuous, there exists an open neighborhood V⊂Y of f(x) such that g(V)⊂Int(Cl^*(W)).

Since f is strongly θ-I-continuous, there exists an open neighborhood U$\subset$X of x such that f(Cl^*(U))$\subset$V. Hence we have g(f(Cl^*(U)))$\subset$g(V) and g(f(Int(Cl^*(U))))$\subset$g(f(Cl^*(U)))$\subset$g(V) $\subset$Int(Cl^*(W)). Thus, we obtain g(f(Int(Cl^*(U))))$\subset$Int(Cl^*(W)). This shows that gof is δ-I-continuous.

(2) Let f be strongly θ-I-continuous. Let x$\in$X and V be any open neighborhood of f(x). Then, there exists an open neighborhood U$\subset$X of x such that f(Cl^*(U))$\subset$V. Since always f(Int(Cl^*(U)))$\subset$f(Cl^*(U)), f(Int(Cl^*(U)))$\subset$V. Since V is open, we have f(Int(Cl^*(U)))$\subset$Int(Cl^*(V)). Thus, f is δ-I-continuous. Let f be δ-I-continuous. Now we prove that f is almost I-continuous. Then, for each x$\in$X and each open neighborhood V$\subset$Y of f(x), there exists an open neighborhood U$\subset$X of x such that f(Int(Cl^*(U)))$\subset$Int(Cl^*(V)). Since U$\subset$Int(Cl^*(U)), f(U)$\subset$Int(Cl^*(V)). Thus, f is almost I-continuous. □

Remark 4.1 *The following examples enable us to realize that none of these implications in Theorem 4.1 (2) is reversible.*

Example 4.1 *Let* $X = \{a,b,c\}$, $\tau = \{\emptyset,X,\{a\},\{a,c\}\}$, $I = \{\emptyset,\{c\}\}$, $\Phi = \{\emptyset,X,\{a,b\}\}$ *and* $J = \{\emptyset,\{b\},\{c\},\{b,c\}\}$. *The identity function* $f:(X,\tau,I)\rightarrow(X,\Phi,J)$ *is* δ*-I-continuous but it is not strongly* θ*-I-continuous.*

(i) Let $a\in X$ *and* $V = \{a,b\}\in\Phi$ *such that* $f(a)\in V$. $V^* = (\{a,b\})^* = \{a,b,c\} = X$, $Cl^*(V) = V\cup V^* = X$ *and* $Int(Cl^*(V)) = Int(X) = X$. *Then, there exists an open* $U = \{a,c\}\subset X$ *such that* $a\in U$. *We have* $U^* = (\{a\})^* = \{a,b,c\}$, $Cl^*(U) = U\cup U^* = \{a,b,c\}$ *and* $Int(Cl^*(U)) = \{a,c\}$. *Since* $f(Int(Cl^*(U))) = f(\{a,c\}) = \{a,c\}$ *and* $\{a,c\}\subset Int(Cl^*(V)) = X$.

(ii) Let $b\in X$ *and* $V = \{a,b\}\in\Phi$ *such that* $f(b)\in V$. $V^* = (\{a,b\})^* = \{a,b,c\} = X$, $Cl^*(V) = V\cup V^* = X$ *and* $Int(Cl^*(V)) = Int(X) = X$. *Then, there exists an open* $U = X$ *such that* $b\in U$. *We have* $Cl^*(U) = Cl^*(X) = X$ *and* $Int(Cl^*(U)) = Int(X)$. *Since* $f(Int(Cl^*(U))) = f(X) = X$ *and* $X\subset Int(Cl^*(V)) = X$.

(iii) Let $x = a$, b *or* c *and* $V = X\in\Phi$ *such that* $f(x)\in V$. $Cl^*(V) = V\cup V^* = X$ *and* $Int(Cl^*(V)) = Int(X) = X$. *Then, there exists an open* $U = X$ *such that* $x\in U$. *We have* $Cl^*(U) = Cl^*(X) = X$ *and* $Int(Cl^*(U)) = Int(X)$. *Since* $f(Int(Cl^*(U))) = f(X) = X$ *and* $X\subset Int(Cl^*(V)) = X$. *By (i), (ii) and (iii), f is* δ*-I-continuous. On the other hand by (i), since* $f(Cl^*(U)) = f(Cl^*(\{a\}) = f(\{a,b,c\})) = \{a,b,c\}$ *is not subset of* $V = \{a,b\}$, *f is not strongly* θ*-I-continuous.*

Example 4.2 *Let* $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X,\{a,c\},\{d\},\{a,b,c\},\{a,c,d\}\}$, $I = \{\emptyset,\{d\}\}$ *and* $J = \{\emptyset,\{c\},\{d\},\{c,d\}\}$. *The identity function* $f:(X,\tau,I)$ (X,τ,J) *is almost I-continuous but it is not δ-I-continuous. (i) Let* $x = a$ *or* $c\in X$ *and* $V = \{a,c\}\in\Phi = \tau$ *such that* $f(x)\in V$. $V^* = (\{a,c\})^* = \{a,b,c\}$, $Cl^*(V) = V\cup V^* = \{a,b,c\}$ *and* $Int(Cl^*(V)) = \{a,c\}$. *Then, there exists an open* $U = \{a,c\}\subset X$ *such that* $x\in U$. *We have* $U^* = (\{a,c\})^* = \{a,b,c\}$ *and* $Int(Cl^*(U)) = Int(\{a,b,c\}) = \{a,b,c\}$. *Since* $f(U) = f(\{a,c\}) = \{a,c\}\subset Int(Cl^*(V)) = \{a,c\}$.

(ii) Let $x = a$, c *or* $d\in X$ *and* $V = \{a,c,d\}\in\Phi = \tau$ *such that* $f(x)\in V$. $V^* = (\{a,c,d\})^* = \{a,b,c\}$ *and* $Cl^*(V) = V\cup V^* = \{a,b,c,d\}$ *and* $Int(Cl^*(V)) = X$. *Then, there exists an open* $U = \{a,c,d\}\subset X$ *such that* $x\in U$. *We have* $U^* = (\{a,c,d\})^* = \{a,b,c,d\} = X$ *and* $Int(Cl^*(U)) = Int(X) = X$. *Since* $f(U) = f(\{a,c,d\}) = \{a,c,d\} \subset Int(Cl^*(V)) = \{a,b,c,d\}$.

(iii) Let $x = a$, b *or* $c\in X$ *and* $V = \{a,b,c\}\in\Phi = \tau$ *such that* $f(x)\in V$. $V^* = (\{a,b,c\})^* = \{a,b,c\}$ *and* $Cl^*(V) = V\cup V^* = \{a,b,c\}$ *and* $Int(Cl^*(V)) = \{a,b,c\}$. *Then, there exists an open* $U = \{a,b,c\}\subset X$ *such that* $x\in U$. *We have* $U^* = (\{a,b,c\})^* = \{a,b,c\}$ *and* $Int(Cl^*(U)) = \{a,b,c\}$. *Since* $f(U) = f(\{a,b,c\}) = \{a,b,c\}\subset Int(Cl^*(V)) = \{a,b,c\}$.

(iv) Let $d\in X$ *and* $V = \{d\}\in\Phi = \tau$ *such that* $f(d)\in V$. $V^* = (\{d\})^* = \emptyset$ *and* $Cl^*(V) = V\cup V^* = \{d\}$ *and* $Int(Cl^*(V)) = \{d\}$. *Then, there exists an open* $U = \{d\}\subset X$ *such that* $d\in U$. *We have* $U^* = (\{d\})^* = \emptyset$ *and* $Int(Cl^*(U)) = \{d\}$. *Since* $f(U) = f(\{d\}) = \{d\}\subset Int(Cl^*(V)) = \{d\}$. *By (i), (ii), (iii) and (iv), f is almost I-continuous. On the other hand by (i), since* $f(Int(Cl^*(U))) = f(\{a,b,c\}) = \{a,b,c\}$ *is not subset of* $Int(Cl^*(V))$ *and* $Int(Cl^*(V)) = \{a,c\}$, *f is not δ-I-continuous.*

Definition 4.3 *An ideal topological space* (X,τ,I) *is said to be an SI-R space if for each* $x\in X$ *and each open neighborhood V of x, there exists an open neighborhood U of x such that* $x\in U\subset Int(Cl^*(U))\subset V$.

Theorem 4.2 *For a function* $f:(X,\tau,I)\to(Y,\Phi,J)$, *the following are true:*

(1) If Y is an SI-R space and f is δ-I-continuous, then f is continuous.

(2) If X is an SI-R space and f is almost I-continuous, then f is δ-I-continuous.

Proof. (1) Let Y be an SI-R space. Then, for each open neighborhood V of f(x), there exists an open neighborhood V_o of f(x) such that f(x)$\in V_o\subset$Int($Cl^*(V_o)$)$\subset$V. Since f is δ-I-continuous, there exists an open neighborhood U_o of x such that f(Int($Cl^*(U_o)$))$\subset$Int(Cl^* (V_o)). Since U_o is an open set, f(U_o)$\subset$f(Int($Cl^*(U_o)$))$\subset$ Int($Cl^*(V_o)$)$\subset$V. Thus, f(U_o)$\subset$V and hence f is continuous.

(2) Let x∈X and V be an open neighborhood of f(x). Since f is almost I-continuous, there exists an open neighborhood U of x such that f(U)⊂Int(Cl^*(V)). Since X is an SI-R space, there exists an open neighborhood U_1 of x such that Int($Cl^*(U_1)$)⊂U. Thus f(Int($Cl^*(U_1)$))⊂f(U)⊂Int(Cl^*(V)). Therefore f is δ-I-continuous. □

Corollary 4.1 *If (X,τ,I) and (Y,Φ,J) are SI-R spaces, then the following concepts on a function f:(X,τ,I)→(Y,Φ,J): δ-I-continuity, continuity and almost I-continuity are equivalent.*

Definition 4.4 *An ideal topological space (X,τ,I) is said to be an AI-R space if for each R-I-closed set F⊂X and each x∉F, there exist disjoint open sets U and V in X such that x∈U and F⊂V.*

Theorem 4.3 *An ideal topological space (X,τ,I) is an AI-R space if and only if each x∈X and each R-I-open neighborhood V of x, there exists an R-I-open neighborhood U of x such that x∈U⊂Cl^*(U)⊂Cl(U)⊂V.*

Proof. **Necessity.** Let x∈V and V be R-I-open. Then {x}∩(X-V) = ∅. Since X is an AI-R space, there exist open sets U_1 and U_2 containing x and (X-V), respectively, such that U_1∩U_2 = ∅. Then Cl(U_1)∩U_2 = ∅ and hence $Cl^*(U_1)$⊂Cl(U_1)⊂(X-U_2)⊂V. Thus x∈U_1⊂$Cl^*(U_1)$⊂Cl(U_1)⊂V and we obtain that U_1⊂ Int($Cl^*(U_1)$)⊂$Cl^*(U_1)$. Let Int($Cl^*(U_1)$) = U. Thus, we have Cl(U) = Cl(Int($Cl^*(U_1)$))⊂Cl($Cl^*(U_1)$)⊂Cl(Cl(U_1)) = Cl(U_1)⊂Cl(U) and U_1⊂U⊂Cl^*(U)⊂$Cl^*(U_1)$⊂Cl(U_1)⊂V. Therefore, there exists an R-I-open set U such that x∈U⊂Cl^*(U)⊂Cl(U)⊂V.

Sufficiency. Let x∈X and an R-I-closed set F such that x∉F. Then, X-F is an R-I-open neighborhood of x. By hypothesis, there exists an R-I-open neighborhood V of x such that x∈V⊂Cl^*(V)⊂Cl(V)⊂X-F. From here we have F⊂X-Cl(V)⊂(X-Cl^*(V)), where X-Cl(V) is an open set. Moreover, we have that V∩(X-Cl(V)) = ∅ and V is open. Therefore, X is an AI-R space. □

Theorem 4.4 *For a function f:(X,τ,I)→(Y,Φ,J), the following are true:*

(1) If Y is an AI-R space and f is θ-I-continuous, then f is δ-I-continuous.

(2) If X is an AI-R space, Y is an SI-R space and f is δ-I-continuous, then f is strongly θ-I-continuous.

Proof. (1) Let Y be an AI-R space. Then, for each x∈X and each R-I-open neighborhood V of f(x), there exists an R-I-open neighborhood V_o of f(x) such that f(x)$\in V_o \subset Cl^*(V_o) \subset$V. Since f is θ-I-continuous, there exists an open neighborhood U_o of x such that f($Cl^*(U_o)$)$\subset Cl^*(V_o)$. Hence, we obtain that f(Int($Cl^*(U_o)$))⊂f($Cl^*(U_o)$)$\subset Cl^*(V_o)$ ⊂V and thus f(Int($Cl^*(U_o)$))⊂V. By Theorem 3.1, f is δ-I-continuous.

(2) Let X be an AI-R space and Y an SI-R space. For each x∈X and each open neighborhood V of f(x), there exists an open set V_o such that f(x)$\in V_o \subset$Int($Cl^*(V_o)$)⊂V since Y is an SI-R space. Since f is δ-I-continuous, there exists an open set U containing x such that f(Int(Cl^*(U)))⊂Int($Cl^*(V_o)$). By Lemma 2.1, Int(Cl^*(U)) is R-I-open and since X is AI-R, by Theorem 4.3 there exists an R-I-open set U_o such that x$\in V_o \subset Cl^*(U_o) \subset$Int($Cl^*$(U)). Every R-I-open set is open and hence U_o is open. Moreover, we have f($Cl^*(U_o)$)⊂V. This shows that f is strongly θ-I-continuous. □

Theorem 4.5 *If a function f:(X,τ,I)→(Y,Φ,J) is θ-I-continuous and almost-I-open, then it is δ-I-continuous.*

Proof. Let x∈X and V be an open neighborhood of f(x). Since f is θ-I-continuous, there exists an open neighborhood of x such that f(Cl^*(U))$\subset Cl^*$(V); therefore, f(Int(Cl^*(U)))⊂ Cl^*(V). Since f is almost-I-open, we have f(Int(Cl^*(U)))⊂Int(Cl^*(V)). This shows that f is δ-I-continuous. □

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