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F-Invariant Submanifolds of Kaehlerian Product Manifold

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Abstract

In this paper, the geometry of F-invariant submanifolds of a Kaehlerian product manifold is studied. The fundamental properties of these submanifolds are investigated such as pseudo umbilical, curvature invariant, totally geodesic, mixed geodesic submanifold and locally decomposable Riemannian product manifold.

Key Words: Kaehlerian Product Manifold, Mixed Geodesic Submanifold, Locally Decomposable Riemannian Manifold and Constant Holomorphic Sectional Curved Manifold.

1. Introduction

The geometry of submanifolds of a Kaehlerian product manifold is an interesting subject which was studied by many geometers. Partially, the geometry of CR-submanifolds of a Kaehlerian product manifold was studied by M. H. Shahid [8] and he had many interesting results of this submanifold.

Also, the geometry of CR-submanifold of any Kaehlerian manifold was studied by Bejancu A., [2] and Chen B. Y. [3, 4].

The object of this note is to study F-invariant submanifolds of a Kaehlerian product manifold. In this paper, we have researched the fundamental properties of F-invariant submanifolds of a Kaehlerian product manifold. We think interesting results such as Theorem 4.5, Theorem 4.6 and Theorem 4.7 are obtained in this paper. We show that

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a F-invariant submanifold of a Kaehlerian product manifold and their distributions have the same properties.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold and $\bar{M}$ be a m-dimensional manifold isometrically immersed in M. Then $\bar{M}$ becomes a Riemannian submanifold of M with Riemannian metric induced by the Riemannian metric on M. We denote by $T\bar{M}^{\perp}$ the normal bundle to $\bar{M}$ and by g both metrics on M and $\bar{M}$. Also, we denote by $\bar{\nabla}$ and ∇ the Levi-Civita connections on $\bar{M}$ and M, respectively. Then the Gauss formula is given by

$$\nabla_X Y = \bar{\nabla}_X Y + h(X, Y) \tag{1}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $h{:}\Gamma(T\bar{M})\times\Gamma(T\bar{M}) \longrightarrow \Gamma(T\bar{M}^{\perp})$ is the second fundamental form of $\bar{M}$ in M.

Now, for any $X \in \Gamma(T\bar{M})$ and $\xi \in \Gamma(T\bar{M}^{\perp})$, we denote by $-A_\xi X$ and $\nabla_X^{\perp}\xi$ the tangent part and normal part of $\nabla_X \xi$, respectively. Then the Weingarten formula is given by

$$\nabla_X \xi = -A_\xi X + \nabla_X^{\perp}\xi, \tag{2}$$

where A_ξ is called the shape operator of $\bar{M}$ with respect to ξ.

From the Gauss and Weingarten formulas, we have

$$g(h(X, Y), \xi) = g(A_\xi X, Y) \tag{3}$$

for all $X, Y \in \Gamma(T\bar{M})$ and $\xi \in \Gamma(T\bar{M}^{\perp})$[3].

Definition 2.1 *For a submanifold $\bar{M} \subseteq M$ the mean-curvature vector field H is defined by the formula*

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i),$$

where $\{e_i\}$, $i = 1, 2, ..., m$, is a local orthonormal basis in $\Gamma(T\bar{M})$. For any $X, Y \in \Gamma(T\bar{M})$, If a submanifold $\bar{M} \subseteq M$ having one of the conditions

$$h = 0\ ,\ h(X, Y) = g(X, Y)H\ ,\ g(h(X, Y), H) = \lambda g(X, Y)\ ,\ H = 0\ ,\ \lambda \in C^{\infty}(M, \mathbb{R}),$$

then submanifold $\bar{M}$ is called totally geodesic, totally umbilical, pseudo-umbilical and minimal submanifold, respectively[3].

We recall that the length mean curvature vector field of $\bar{M}$ is constant, if $\bar{M}$ is a totally umbilical submanifold of a Riemannian manifold M[3].

The covariant derivative of h is defined by

$$(\nabla_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\bar{\nabla}_X Y, Z) - h(\bar{\nabla}_X Z, Y)$$

for all $X, Y, Z \in \Gamma(T\bar{M})$.

The Gauss and Codazzi equations are, respectively, given by

$$\begin{aligned} R(X,Y)Z &= \bar{R}(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y + (\nabla_X h)(Y,Z) \\ &- (\nabla_Y h)(X,Z) \end{aligned} \tag{4}$$

and

$$\{R(X,Y)Z\}^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{5}$$

for all $X, Y, Z \in \Gamma(T\bar{M})$, where R and $\bar{R}$ are the Riemannian curvature tensors of M and its submanifold $\bar{M}$, respectively, and $\{R(X,Y)Z\}^{\perp}$ denote the normal component of $R(X,Y)Z$.

$\bar{M}$ is called a curvature-invariant submanifold of M, if $R(X,Y)Z \in \Gamma(T\bar{M})$, that is, $\{R(X,Y)Z\}^{\perp} = 0$, for any $X, Y, Z \in \Gamma(T\bar{M})$[5].

An almost complex structure on a differentiable manifold M is a tensor field J of type (1,1) which is, at every point x of M, an endomorphism of $T_x(M)$ such that $J^2 = -I$, where I denotes the identity transformation of $T_x(M)$. A manifold M with an almost complex structure J is called an almost complex manifold.

We define the torsion of J to be the tensor field N of type (1,2), called the Nijenhuis torsion, which is given by

$$N(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y],$$

for any vector fields X and Y. If N vanishes identically, then an almost complex structure is called a complex structure and M is called a complex manifold.

A Hermitian metric on an almost complex manifold M is a Riemannian metric g such that

$$g(JX, JY) = g(X,Y), \quad for \quad all \quad X, Y \in \Gamma(TM).$$

An almost complex manifold(resp., a complex manifold) with Hermitian metric is called an almost Hermitian manifold(resp., a Hermitian manifold). A Hermitian manifold M is called a Kaehlerian manifold, if the almost complex structure J of M is parallel; that is, $(\nabla_X J)Y = 0$, for all $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M.

The Riemannian curvature tensor R of a Kaehlerian manifold M satisfies

$$R(X,Y)J = JR(X,Y), \quad R(JX, JY) = R(X,Y).$$

For each plane γ spanned by orthonormal vectors X and Y in the tangent space $T_x(M)$, $x \in M$, we define the sectional curvature $K(\gamma)$ by

$$K(\gamma) = K(X \wedge Y) = g(R(X,Y)X, Y).$$

If $K(\gamma)$ is a constant for all planes γ in $T_x(M)$ and for all points x of M, then M is called space of constant curvature or real space form.

Now, we consider a plane γ invariant by the almost complex structure J. In this case, we choose a basis $\{X, JX\}$ in γ, where X is a unit vector in γ. Then the sectional curvature $K(\gamma)$ is denoted by $H(X)$ and it is called the holomorphic sectional curvature of M determined by the unit vector X. Then we have

$$H(X) = g(R(X, JX)JX, X).$$

If $H(X)$ is a constant for all unit vectors in $\Gamma(TM)$ and for all points in M, then M is called a space of constant holomorphic sectional curvature and denote it by $M(c)$. In this case, The Riemannian curvature tensor of $M(c)$ is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX - g(Z,JX)JY \\ &+ 2g(X,JY)JZ\}, \end{aligned} \tag{6}$$

for any $X, Y, Z \in \Gamma(TM)$, where c is the constant holomorphic sectional curvature of M.

The holomorphic bisectional curvature for the pair of unit vectors $\{X, Y\}$ is given by

$$H_B(X \wedge Y) = g(R(X, JX)JY, Y).$$

In the rest of this paper, we will denote a Kaehlerian manifold by (M, J, g).

3. Kaehlerian Product Manifolds

Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be Kaehlerian manifolds with complex dimensional n_1 and n_2(real dimension $2n_1$ and $2n_2$), respectively. We consider the Kaehlerian product manifold $M = M_1 \times M_2$, put

$$JX = J_1PX + J_2QX$$

and

$$g(X,Y) = g_1(PX,PY) + g_2(QX,QY),$$

for any vector fields X and Y on M, where P and Q denote the projection mappings of $\Gamma(T(M_1 \times M_2))$ to $\Gamma(TM_1)$ and $\Gamma(TM_2)$, that is,

$$P : \Gamma(TM) \longrightarrow \Gamma(TM_1), \quad Q : \Gamma(TM) \longrightarrow \Gamma(TM_2).$$

If we set $F = P - Q$, then we can easily see that $F^2 = I$, $g(FX,FY) = g(X,Y)$, $J_1P = PJ$, $J_2Q = QJ$, $FJ = JF$, $J^2 = -I$, $g(JX,JY) = g(X,Y)$ and $(\nabla_X J)Y = 0$, for all vector fields X, Y on M. Thus F defines an almost Riemannian product structure and J defines a Kaehlerian structure on M. We will denote the Kaehlerian product manifold $(M_1 \times M_2, J_1 \times J_2, g_1 \times g_2)$ by (M, J, g) througthout this paper. Furthermore, we have $\nabla P = \nabla Q = \nabla F = 0$ (for the detail, we refer to [7]).

If M_1 and M_2 are complex space forms with constant holomorphic sectional curvatures c_1 and c_2 and denote them by $M_1(c_1)$ and $M_2(c_2)$, respectively, then the Riemannian curvature tensor R of Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ is given by

$$\begin{aligned} R(X,Y)Z &= \frac{1}{16}(c_1 + c_2)\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &+ 2g(X,JY)JZ + 2g(FY,Z)FX - g(FX,Z)FY + g(FJY,Z)FJX \\ &- g(FJX,Z)FJY + 2g(FX,JY)FJZ\} \\ &+ \frac{1}{16}(c_1 - c_2)\{g(FY,Z)X - g(FX,Z)Y + g(Y,Z)FX - g(X,Z)FY \\ &+ g(FJY,Z)JX - g(FJX,Z)JY + g(JY,Z)FJX - g(JX,Z)FJY \\ &+ 2g(FX,JY)JZ + 2g(X,JY)JFZ\} \end{aligned} \tag{7}$$

for all $X, Y, Z \in \Gamma(TM)$[5].

Theorem 3.1 *Let $\bar{M}$ be a Kaehlerian submanifold of a Kaehlerian product manifold $M = M_1(c) \times M_2(c)$ $(c \neq 0)$. If $\bar{M}$ is curvature invariant submanifold, then $\bar{M}$ is an F-invariant or F-anti invariant submanifold [5].*

Now, we suppose that $K(X \wedge Y)$ be the sectional curvature of M determined by orthonormal vectors X and Y. Then using (7), we have

$$\begin{aligned} K(X \wedge Y) &= \frac{1}{16}(c_1 + c_2)\{1 + 3g(X, JY)^2 + 2g(FY, Y)g(FX, X) - g(FX, Y)^2 \\ &+ 3g(X, FJY)^2\} + \frac{1}{16}(c_1 - c_2)\{g(FY, Y) + g(FX, X) \\ &+ 6g(FJX, Y)g(JX, Y)\}. \end{aligned} \tag{8}$$

Let $H(X)$ be the holomorphic sectional curvature of Kaehlerian product manifold M by determined the unit vectors X and JX. Then using (8) we get

$$\begin{aligned} H(X) = K(X, JX, JX, X) &= \frac{1}{16}(c_1 + c_2)\{4 + 5g(FX, X)^2\} \\ &+ \frac{1}{2}(c_1 - c_2)\{g(FX, X)\}. \end{aligned} \tag{9}$$

Finally, the holomorphic bisectional curvature of Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ is given by formula

$$\begin{aligned} H_B(X \wedge Y) &= \frac{1}{16}(c_1 + c_2)\{2g(X, Y)^2 + 2g(JX, Y)^2 + 2 + 3g(FX, Y)^2 \\ &+ 2g(FJX, Y)^2 + 2g(FX, X)g(FY, Y)\} \\ &+ \frac{1}{16}(c_1 - c_2)\{2g(X, Y)g(FX, Y) + 4g(JX, Y)g(FJX, Y) \\ &+ 2g(FX, Y)g(X, Y) + 2g(FX, X) + 2g(FY, Y)\}. \end{aligned} \tag{10}$$

4. F-Invariant Submanifolds of Kaehlerian Product Manifold

Let us assume that $\bar{M}$ be a m-dimensional Riemannian manifold isometrically immersed in Kaehlerian product manifold M. For any vector field X tangent to $\bar{M}$ we put

$$FX = fX + \omega X, \tag{11}$$

where fX is the tangential part of FX and ωX is the normal part of FX. For any vector field ξ normal to $\bar{M}$, we put

$$F\xi = t\xi + s\xi, \tag{12}$$

where $t\xi$ is the tangential part of $F\xi$ and $s\xi$ is the normal part of $F\xi$. Then we have

$$\begin{aligned} f^2X &= X - t\omega X \ , \ \omega fX + s\omega X = 0 \\ s^2\xi &= \xi - \omega t\xi \ , \ ft\xi + ts\xi = 0. \end{aligned}$$

We can easily see that

$$g(fX, Y) = g(X, fY) \ , \ g(fX, fY) = g(X, Y) - g(\omega X, \omega Y)$$

for any $X, Y \in \Gamma(T\bar{M})$.

If $F(T_x\bar{M}) \subset T_x\bar{M}$, for each $x \in \bar{M}$, then $\bar{M}$ is said to be F-invariant submanifold of Kaehlerian product manifold M. Then ω vanishes idendically, $f^2 = I$ and $g(fX, fY) = g(X, Y)$. Therefore, (f, g) defines an almost Riemannian product structure on $\bar{M}$. In the rest of this paper, we assume that the submanifold $\bar{M}$ is F-invariant submanifold of Kaehlerian product manifold M. Furthermore, if $J(T_x\bar{M}) \subset T_x(\bar{M})^{\perp}$, for each $x \in \bar{M}$, then we recall that $\bar{M}$ is called anti-invariant(totally real) submanifold of a Kaehlerian product manifold M.

If $\bar{M}$ is a F-invariant submanifold of Kaehlerian product manifold M, then we can easily see that $F(T_x(\bar{M})) \subset T_x(\bar{M})$ and $F(T_x(\bar{M})^{\perp}) \subset T_x(\bar{M})^{\perp}$, for each $x \in \bar{M}$. Thus we have

$$\begin{aligned} \nabla_X FY &= F\nabla_X Y \\ \nabla_X fY &= F(\bar{\nabla}_X Y + h(X, Y)) \\ \bar{\nabla}_X fY + h(X, fY) &= f(\bar{\nabla}_X Y) + sh(X, Y), \end{aligned} \tag{13}$$

for any $X, Y \in \Gamma(T\bar{M})$. Comparing the tangential and normal parts of equation (13), we obtain

$$(\bar{\nabla}_X f)Y = 0 \ , \ sh(X, Y) = h(X, fY), \tag{14}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ and h denote also the second fundamental form of $\bar{M}$ in M.

Since f defines an almost Riemannian product structure on $\bar{M}$, $\bar{M}$ has the vertical and the horizontal distributions which are given by

$$T_1 = \{X \in \Gamma(T\bar{M}) | fX = X\}$$

and

$$T_2 = \{X \in \Gamma(T\bar{M}) | fX = -X\},$$

respectively.

Theorem 4.1 *Let (M, J, g) be a Kaehlerian product manifold and $\bar{M}$ be a F-invariant submanifold of M. Then $\bar{M}$ is a mixed-geodesic F-invariant submanifold of M.*

Proof. Taking into account that h is symmetric and the equation (14), then we obtain $h(X, Y) = h(fX, fY)$. Thus, for any $X = X_1 \in T_1$ and $Y = Y_2 \in T_2$, we infer

$$h(X_1, Y_2) = -h(X_1, Y_2),$$

that is,

$$h(X_1, Y_2) = 0.$$

This completes the proof of the theorem. □

Moreover, we derive

$$h(X, Y) = h_1(X_1, Y_1) + h_2(X_2, Y_2). \tag{15}$$

Since M_1 and M_2 are totally geodesic submanifolds of M, it is easily seen that $h(X_1, Y_1) = h_1(X_1, Y_1)$ and $h(X_2, Y_2) = h_2(X_2, Y_2)$ are the second fundamental forms of $\bar{M}_1$ and $\bar{M}_2$ in M_1 and M_2, respectively.

From the equation (15), we have the following Corollary.

Corollary 4.2 *Let (M, J, g) be a Kaehlerian product manifold and $\bar{M}$ be a F-invariant submanifold of M. We denote the integral manifolds of the vertical and horizontal distributions of $\bar{M}$ by $\bar{M}_1$ and $\bar{M}_2$. Then $\bar{M}$ is totally geodesic submanifold of M if and only if $\bar{M}_1$ and $\bar{M}_2$ are totally geodesic submanifolds of M_1 and M_2, respectively.*

Now, we choose a local field of adapted basis

$$\{e_1, ..., e_p, e_{p+1}, ..., e_{n_1}, e^1, ..., e^q, e^{q+1}, ..., e^{n_2}\} \tag{16}$$

of $\Gamma(TM)$ with respect to g so that when restricted to the locally orthonormal basis over $\Gamma(T\bar{M})$, $\{e_1, ..., e_p\}$ are the orthonormal tangent vectors to $\Gamma(T\bar{M}_1)$ with respect to g_1, $\{e^1, ..., e^q\}$ are the orthonormal tangent vectors to $\Gamma(T\bar{M}_2)$ with respect to g_2 and $\{e_{p+1}, ..., e_{n_1}, e^{q+1}, ..., e^{n_2}\}$ are the normal vectors to $\Gamma(T\bar{M})$.

Let H be the mean curvature vector field of $\bar{M}$ in M. Then from (15) and (16), we obtain

$$mH = \sum_{i=p+1}^{n_1} trh_1 e_i + \sum_{j=q+1}^{n_2} trh_2 e^j = pH_1 + qH_2, \quad m = p + q, \tag{17}$$

where H_1 and H_2 denote the mean curvature vector fields of $\bar{M}_1$ and $\bar{M}_2$ in M_1 and M_2, respectively. The following Lemma is quite easy.

Lemma 4.3 *H_1 and H_2 are constant vectors if and only if H is constant vector. $\|H_1\|$ and $\|H_2\|$ are constant functions if and only if $\|H\|$ is constant function.*

From Lemma.4.3 and (17), we can give the following Corollary.

Corollary 4.4 *Let (M, J, g) be a Kaehlerian product manifold and $\bar{M}$ be a F-invariant submanifold of M. We denote the integral manifolds of the vertical and horizontal distributions of $\bar{M}$ by $\bar{M}_1$ and $\bar{M}_2$. Then $\bar{M}$ is a minimal submanifold of M if and only if $\bar{M}_1$ and $\bar{M}_2$ are minimal submanifolds of M_1 and M_2, respectively.*

Theorem 4.5 *Let (M, J, g) be a Kaehlerian product manifold and $\bar{M}$ be a F-invariant submanifold of M. Then $\bar{M}$ is a locally decomposable Riemannian product manifold.*

Proof. We denote the integral manifolds of the vertical distribution T_1 and the horizontal distribution T_2 by $\bar{M}_1$ and $\bar{M}_2$, respectively. Then from (14) we have

$$f\bar{\nabla}_Z X_1 = \bar{\nabla}_Z f X_1 = \bar{\nabla}_Z X_1,$$

for all $Z \in \Gamma(T\bar{M})$ and $X_1 \in T_1$. Thus the distribution T_1 is parallel. In the same way, the distribution T_2 is also parallel.

$$\begin{aligned} f[X_1, Y_1] &= f\bar{\nabla}_{X_1} Y_1 - f\bar{\nabla}_{Y_1} X_1 = \bar{\nabla}_{X_1} f Y_1 - \bar{\nabla}_{Y_1} f X_1 \\ &= \bar{\nabla}_{X_1} Y_1 - \bar{\nabla}_{Y_1} X_1 = [X_1, Y_1] \end{aligned}$$

for any $X_1, Y_1 \in T_1$, that is, the distribution T is involutive. In the same way, the distribution T_2 is also involutive. Since $\bar{\nabla}$ is a Levi-Civita connection, T_1 and T_2 are orthogonal distributions, we have

$$g(\bar{\nabla}_{X_1} Y_1, Z_2) = -g(\bar{\nabla}_{X_1} Z_2, Y_1) = 0,$$

which implies that $\bar{M}_1$ is the totally geodesic submanifold of $\bar{M}$. In the same way, we get $\bar{M}_2$ is also totally geodesic submanifold of $\bar{M}$. Thus $\bar{M}$ is a locally decomposable Riemannian product manifold. □

Now, from $P + Q = I$ and $F = P - Q$ we have

$$PX_1 = \frac{1}{2}(I + F)X_1 = \frac{1}{2}(X_1 + FX_1) = \frac{1}{2}(X_1 + fX_1) = X_1$$

and

$$QX_1 = \frac{1}{2}(I - F)X_1 = \frac{1}{2}(X_1 - FX_1) = \frac{1}{2}(X_1 - fX_1) = 0$$

for any $X_1 \in T_1$. Similarly, we get $PX_2 = 0$ and $QX_2 = X_2$ for any $X_2 \in T_2$. Hence $\bar{M}_1$ and $\bar{M}_2$ are the submanifolds of M_1 and M_2, respectively.

Furthermore, if $\bar{M}_1$ and $\bar{M}_2$ are Kaehlerian submanifolds of M_1 and M_2, respectively. Then $\bar{M}$ is a Kaehlerian product manifold of $\bar{M}_1 \times \bar{M}_2$.

Theorem 4.6 *Let (M, J, g) be a Kaehlerian product manifold and $\bar{M}$ be a F-invariant submanifold of M. We denote the integral manifolds of the vertical and horizontal distributions of $\bar{M}$ by $\bar{M}_1$ and $\bar{M}_2$, respectively. Then $\bar{M}$ is a pseudo-umbilical submanifold of $M = M_1 \times M_2$ if and only if $\bar{M}_1$ and $\bar{M}_2$ are pseudo-umbilical submanifolds of M_1 and M_2, respectively.*

Proof. We suppose that $\bar{M}$ be pseudo-umbilical submanifold of M. Then there exists a smooth function λ on $\bar{M}$ such that

$$g(h(X, Y), H) = \lambda g(X, Y) \tag{18}$$

for any $X, Y \in \Gamma(T\bar{M})$.

We consider the basis in (16) and taking $X = Y = e_1, e_2, ..., e_p$ in (18), then we obtain

$$\begin{aligned} g(pH_1, H) &= \lambda \sum_{i=1}^{p} g(e_i, e_i) \\ pg(H_1, H) &= \lambda p, \end{aligned}$$

that is,

$$\lambda = g(H_1, H) = \frac{p}{m} g_1(H_1, H_1).$$

In the same way, if we choose $X = Y = e^1, e^2, ..., e^q$ in (18), then we get

$$\begin{aligned} g(qH_2, H) &= \lambda \sum_{j=1}^{q} g(e^j, e^j) \\ qg(H_2, H) &= \lambda q, \end{aligned}$$

or,

$$\lambda = g(H_2, H) = \frac{q}{m} g_2(H_2, H_2).$$

Thus, using the $\lambda = \frac{p}{m} g_1(H_1, H_1)$, $h(X, Y) = h_1(X_1, Y_1) + h_2(X_2, Y_2)$ and taking $X = X_1, Y = Y_1 \in T_1$ in (18), we have

$$\begin{aligned} g(h_1(X_1, Y_1), H) &= \frac{p}{m} g_1(H_1, H_1) g(X_1, Y_1) \\ g_1(h_1(X_1, Y_1), H_1) &= g_1(H_1, H_1) g(X_1, Y_1). \end{aligned}$$

It follows that $\bar{M}_1$ is pseudo-umbilical submanifold of M_1.

Similarly, making use of $\lambda = \frac{q}{m} g_2(H_2, H_2)$ and taking $X = X_2, Y = Y_2 \in T_2$ in (18), then we obtain

$$\begin{aligned} g(h_2(X_2, Y_2), H) &= \frac{q}{m} g_1(H_2, H_2) g(X_2, Y_2) \\ g_2(h_2(X_2, Y_2), H_2) &= g_2(H_2, H_2) g(X_2, Y_2). \end{aligned}$$

Hence $\bar{M}_2$ is also pseudo-umbilical submanifold of M_2.

Conversely, we suppose that $\bar{M}_1$ and $\bar{M}_2$ are pseudo-umbilical submanifold of M_1 and M_2, respectively. Then we have

$$g_1(h_1(X_1, Y_1), H_1) = g_1(H_1, H_1) g(X_1, Y_1)$$

for any $X_1, Y_1 \in T_1$ and

$$g_2(h_2(X_2, Y_2), H_2) = g_2(H_2, H_2) g(X_2, Y_2)$$

for any $X_2, Y_2 \in T_2$.

From the equations (15), (17) and making use of projection mappings

$$P : \Gamma(TM) \longrightarrow \Gamma(TM_1)$$

and

$$Q : \Gamma(TM) \longrightarrow \Gamma(TM_2),$$

we get

$$\frac{m}{p} g_1(h_1(X_1, Y_1), PH) = \frac{m^2}{p^2} g_1(PH, PH) g(X_1, Y_1)$$

and

$$\frac{m}{q} g_2(h_2(X_2, Y_2), QH) = \frac{m^2}{q^2} g_2(QH, QH) g(X_2, Y_2).$$

Hence we have

$$g_1(Ph(X, Y), PH) = \frac{m}{p} g_1(PH, PH) g(X_1, Y_1) \tag{19}$$

and

$$g_2(Qh(X, Y), QH) = \frac{m}{q} g_2(QH, QH) g(X_2, Y_2). \tag{20}$$

We sum equations (19), (20) and using

$$g(H, H) = \frac{p}{m} g_1(H_1, H_1) + \frac{q}{m} g_2(H_2, H_2),$$

we obtain

$$\begin{aligned} g(h(X, Y), H) &= \frac{p}{m} g_1(H_1, H_1) g(X_1, Y_1) + \frac{q}{m} g_2(H_2, H_2) g(X_2, Y_2) \\ &= g(H, H)\{g(X_1, Y_1) + g(X_2, Y_2)\} \\ &= g(H, H) g(X, Y), \end{aligned}$$

which implies that $\bar{M}$ is pseudo-umbilical submanifold of Kaehlerian product manifold M. □

Now we denote the Riemannian curvature tensor of Kaehlerian product manifold M by R. Then from $\nabla F = 0$ and the properties of R, we can easily see that $R(PX, QY) = 0$ for any $X, Y \in \Gamma(TM)$. Using the first Bianchi's identity for R and $\nabla P = \nabla Q = 0$, by direct calculations, we obtain

$$R(X, Y)Z = R_1(PX, PY)PZ + R_2(QX, QY)QZ. \tag{21}$$

for all $X, Y, Z \in \Gamma(TM)$, where R_1 and R_2 are the Riemannian curvature tensors of Kaehlerian manifolds M_1 and M_2, respectively. Moreover, The equation of Gauss is given by

$$\begin{aligned} R(X,Y)Z &= \bar{R}(X,Y)Z + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \\ &+ A_{h(X,Z)}Y - A_{h(Y,Z)}X \end{aligned} \tag{22}$$

for all $X, Y, Z \in \Gamma(T\bar{M})$, where A and $\bar{R}$ denote the shape operator and Riemannian curvature tensor of $\bar{M}$, respectively. Thus we can give the following theorem.

Theorem 4.7 *Let (M, J, g) be a Kaehlerian product manifold and $\bar{M}$ be F-invariant submanifold of M. We denote the integral manifolds of the vertical and horizontal distributions of $\bar{M}$ by $\bar{M}_1$ and $\bar{M}_2$, respectively. Then $\bar{M}$ is a curvature-invariant submanifold of M if and only if $\bar{M}_1$ and $\bar{M}_2$ are curvature-invariant submanifolds of M_1 and M_2, respectively.*

Proof. By direct calculations, from the equations (4), (21) and (22), we conclude

$$\begin{aligned} (\nabla_X h)(Y,Z) &- (\nabla_Y h)(X,Z) = (\nabla_{X_1} h_1)(Y_1,Z_1) - (\nabla_{Y_1} h_1)(X_1,Z_1) \\ &+ (\nabla_{X_2} h_2)(Y_2,Z_2) - (\nabla_{Y_2} h_2)(X_2,Z_2), \end{aligned} \tag{23}$$

where $X = X_1 + X_2, Y = Y_1 + Y_2, Z = Z_1 + Z_2 \in \Gamma(T\bar{M})$, h_1 and h_2 denote the second fundamental forms of $\bar{M}_1$ and $\bar{M}_2$ in M_1 and M_2, respectively. From the equation (23), we derive

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0$$

if and only if

$$(\nabla_{X_1} h_1)(Y_1,Z_1) - (\nabla_{Y_1} h_1)(X_1,Z_1) = 0$$

and

$$(\nabla_{X_2} h_2)(Y_2,Z_2) - (\nabla_{Y_2} h_2)(X_2,Z_2) = 0,$$

which proves our assertion. □

Theorem 4.8 *Let (M, J, g) be a Kaehlerian product manifold and $\bar{M}$ be a F-invariant and anti-invariant(with respect to J) submanifold of $M = M_1(c_1) \times M_2(c_2)$. We denote* *the integral manifolds of the vertical and horizontal distributions of $\bar{M}$ by $\bar{M}_1$ and $\bar{M}_2$, respectively. Then $\bar{M}_1$ and $\bar{M}_2$ are curvature invariant submanifolds of M_1 and M_2, respectively.*

Proof. If $\bar{M}$ is a F-invariant and anti-invariant submanifold of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$, then from the equations (4) and (7), for any $X, Y, Z \in \Gamma(T\bar{M})$, we have

$$\begin{aligned}
\bar{R}(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y &= \frac{1}{16}(c_1+c_2)\{g(Y,Z)X - g(X,Z)Y \\
&+ 2g(FY,Z)FX - g(FX,Z)FY\} \\
&+ \frac{1}{16}(c_1-c_2)\{g(FY,Z)X - g(FX,Z)Y \\
&+ g(Y,Z)FX - g(X,Z)FY\}
\end{aligned}$$

and

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0,$$

which implies that $\bar{M}_1$ and $\bar{M}_2$ are curvature invariant submanifolds of Kaehlerian manifolds $M_1(c_1)$ and $M_2(c_2)$. □

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