Turk J Math
28 (2004) , 383 – 386.

A Generalization of a Result on Torsion-Free Groups With all Subgroups Subnormal

Tahire Özen

Abstract

The main result in this paper is the following: Let G be a torsion-free locally nilpotent group and let F be a finitely generated subgroup of G. If every subgroup of G containing F is subnormal in G, then G is nilpotent.

1. Introduction

In [1] and [5] it is proved that a torsion-free group with all subgroups subnormal is nilpotent. A generalization of this result is given by Smith in [8].

In this note we consider locally nilpotent torsion-free groups such that every subgroup containing a fixed finitely generated subgroup is subnormal and prove the following result :

Theorem. *Let G be a torsion-free locally nilpotent group and let F be a finitely generated subgroup of G. If every subgroup of G containing F is subnormal in G, then G is nilpotent.*

To prove the theorem we mainly exploit Möhres', Casolo's and Detomi's ideas in [4], [1] and [3], respectively.

2. Proof of the Theorem

Let G be a group and H be a subgroup of G. The isolator of H in G is the set defined by $I_G(H) = \{g \in G : g^n \in H$, for some non-negative natural number $n\}$. See [4] for some interesting properties of isolators.

The following Lemma is analogous to Lemma 4 of [4].

Lemma 1. *Let G be a non-nilpotent, locally nilpotent, torsion-free group and let F be a finitely generated subgroup of G. If every subgroup of G containing F is subnormal in G, then there exist a non-negative natural number r, a non-nilpotent subgroup K of G and a finitely generated subgroup H of K containing F such that the subnormality index in K of every subgroup of K containing H is at most r. If G is countable, then K can be chosen such that $I_G(K) = G$.*

Proof. Since G is a non-nilpotent group, G has a non-nilpotent countable subgroup containing F. So we may assume that G is countable. Suppose that the assertion is false. Put $H_0 = F$ and let $x_0 \in G \setminus F$. By Lemma 2 of [4], there exists a subgroup K_1 of G containing F such that $x_0 \notin K_1$ and $I_G(K_1) = G$. Since G is a non-nilpotent group, by Lemma 1.3.5 of [3], K_1 is a non-nilpotent subgroup of G. Hence by Lemma 3 of [4] there exists a finitely generated subgroup H_1 of G such that $H_0 \le H_1 \le K_1$ and $s(K_1 : H_1) > 1$, where $s(K_1 : H_1)$ is the subnormality index of H_1 in K_1. This implies that there exists an element $x_1 \in [K_1, H_1] \setminus H_1$ and clearly $x_0, x_1 \notin H_1$. Assume that there exist a non-nilpotent subgroup K_{i-1} of G and a finitely generated subgroup H_{i-1} of K_{i-1} such that $x_0, \ldots, x_{i-1} \notin H_{i-1}$ for a natural number $i \ge 1$. Then we can obtain a non-nilpotent subgroup K_i of K_{i-1} containing a finitely generated subgroup H_i such that $s(K_i : H_i) > i$ and $H_{i-1} \le H_i$ as above. Thus there exists $x_i \in [K_{i,i} H_i] \setminus H_i$ and clearly $x_0, \ldots, x_i \notin H_i$. Now put $H = \cup_{i=1}^{\infty} H_i$. Since $F \le H$, H is subnormal in G by hypothesis. If d is the subnormality index of H in G, then $x_d \in [K_{d,d} H_d] \le [G,_d H] \le H$. Hence $x_d \in H_i$ for $i > d$, a contradiction. That completes the proof. □

Lemma 2. *Let G be a locally nilpotent, torsion-free, metabelian group and let F be a finitely generated subgroup of G. If every subgroup of G containing F is subnormal in G, then G is nilpotent.*

Proof. Assume that G is not nilpotent. Hence we may assume that G is countable. By Lemma 1 there exist a natural number r, a non-nilpotent subgroup K of G and a finitely generated subgroup H of K containing F such that every subgroup of K containing H has defect at most r in K.

Let $A = HK'$ and $L = I_K(A')$. Since H is subnormal in A and K' is nilpotent, by Lemma 1 of [7] A is nilpotent and so $I_K(A')$ is a nilpotent group by Lemma 1.3.5 of [3]. So K/L is non-trivial and torsion-free. Since AL/L is an abelian normal subgroup of K/L and every subgroup of K/L containing HL/L has defect in K/L at most r, K/L

is nilpotent by Lemma 3 of [1]. Hence we obtain that K is nilpotent by Lemma 4 of [1]. This contradiction completes the proof. □

Lemma 3. *Let G be a locally nilpotent, torsion-free group such that G' is nilpotent and let F be a finitely generated subgroup of G. If every subgroup of G containing F is subnormal in G, then G is nilpotent.*

Proof. Since $G/I_G(G'')$ is nilpotent by Lemma 2 and G' is nilpotent, G is nilpotent by Lemma 4 of [1].

Proof of the Theorem. Assume that the assertion is false. Then we may assume that G is countable and by Lemma 1 there exist a non-negative natural number r, a non-nilpotent subgroup K of G and a finitely generated subgroup H of K containing F such that the subnormality index in K of every subgroup of K containing H is at most r. We proceed by induction on r. If $r = 1$, then every subgroup containing H is normal in K. By 5.3.7 (Dedekind-Baer) of [6] $K/I_K(H)$ is abelian. By Lemma 1.3.5 of [3] $I_K(H)$ is nilpotent and Lemma 3 gives the contradiction that K is nilpotent. If $r > 1$ then $N = H^K$ is nilpotent. Because if R is a subgroup of N such that $R \geq H$, then $R^K = N$ and so R has defect at most $r-1$ in $H^K = N$. Thus by induction hypothesis, N is a nilpotent group. By Corollary to Theorem 1 (Roseblade Theorem) of [2], $K/I_K(N)$ is a nontrivial soluble group. Let the derived length of $K/I_K(N)$ be d. Now we prove that K is nilpotent by induction on d. If $d = 1$ then by Lemma 1 (v) of [4], $K/I_K(N')$ is a metabelian group which satisfies the conditions of Lemma 2. Thus $K/I_K(N')$ is nilpotent. By Lemma 4 of [1] K is nilpotent. If $d > 1$ then since the derived length of $K'I_K(N)/I_K(N)$ is $d-1$, by induction hypothesis $K'I_K(N)$ is nilpotent. Since $K/I_K((K'I_K(N))')$ is a metabelian group which satisfies the conditions of Lemma 2, it is nilpotent. Thus by Lemma 4 of [1] K is nilpotent. But this is a contradiction. □

The author is grateful to Professor Carlo Casolo for his helpful suggestions in proof of the Theorem.

References

[1] C. Casolo. Torsion-free groups with all subgroups subnormal. *Rend. Circ. Mat. Palermo* **2** 50 (2001), 321–324.

[2] J.E.Roseblade. On groups in which every subgroup is subnormal. *J. Algebra* **2** (1965), 402-412.

[3] E. Detomi. Generalizing subnormality and nilpotency in infinite groups. PhD Thesis, Firenze University.

[4] W. Möhres. Torsionsfreie Gruppen, deren Untergruppen alle subnormal sind. *Math. Ann.* **284** (1989), 245-249.

[5] H. Smith. Torsion-free groups with all subgroups subnormal. *Arch. Math.(Basel)* **76** (2001),1-6.

[6] D.J.S. Robinson. A course in the theory of groups.Springer-Verlag New York. Heidelberg. Berlin, (1982).

[7] W. Möhres. Torsionsgruppen, deren Untergruppen alle subnormal sind. *Geom.Dedicata* **31** (1989), 237-244.

[8] H. Smith. Torsion-free groups with all non-nilpotent subgroups subnormal. *Topics in infinite groups*, *Quad. Mat.* **8**, Aracne, Rome (2001), 297-308.

Tahire ÖZEN
Faculty of Arts and Sciences,
Department of Mathematics,
06500, Ankara-TURKEY
e-mail: tahire@gazi.edu.tr

Received 27.06.2003