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Spacelike Normal Curves in Minkowski Space $\mathbb{E}_1^3$

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Dedicated to Professor Dr. H. Hilmi Hacısalihoğlu

Abstract

In the Euclidean space $\mathbb{E}^3$, it is well known that normal curves, i.e., curves with position vector always lying in their normal plane, are spherical curves [3]. Necessary and sufficient conditions for a curve to be a spherical curve in Euclidean 3-space are given in [10] and [11].

In this paper, we give some characterizations of spacelike normals curves with spacelike, timelike or null principal normal in the Minkowski 3-space $\mathbb{E}_1^3$.

Key words and phrases: Normal Curves, Position Vector and Minkowski Space.

1. Introduction

In the Euclidean space $\mathbb{E}^3$, it is well-known that to each unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ with at least four countinuous derivatives, one can associate three mutually ortogonal unit vector fields T, N and B, called respectively the tangent, the principal normal and the binormal vector fields. At each point $\alpha(s)$ of curve α, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known respectively as the osculating plane, the rectifying plane and the normal plane. The curves $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ for wich the position vector α always lie in their rectifying plane, are for simplicity called *rectifying curves,* (see [3]). Similarly, the curves for which the position vector α always lie in their osculating plane, are for simplicity called *osculating curves;* and finally, the curves for which the position vector

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always lie in their normal plane, are for simplicity called *normal curves*. By definition, for a normal curve, the position vector α satisfies

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B(s),$$

for some differentiable functions λ and μ of $s \in I \subset \mathbb{R}$.

Characterization of rectifying curves is given in [3] and these curves are studied in Minkowski space $\mathbb{E}_1^3$ in [5]. In this paper, we characterize *spacelike normal curves*, lying fully in the Minkowski space $\mathbb{E}_1^3$.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$.

Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha^{'}(s)$ are respectively spacelike, timelike or null (lightlike). Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $\mathbb{E}_1^3$. For an arbitrary curve $\alpha(s)$ in the space $\mathbb{E}_1^3$, the following Frenet formulae are given in [4, 9].

If α is a spacelike curve with a spacelike or timelike principal normal N, then the Frenet formulae read

$$\begin{bmatrix} T^{'} \\ N^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\epsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where $g(T,T) = 1, g(N,N) = \epsilon = \pm 1, g(B,B) = -\epsilon, g(T,N) = 0, g(T,B) = 0, g(N,B) = 0$.

If α is a spacelike curve with a null (lightlike) principal normal N, the Frenet formulae are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2)$$

where $g(T,T) = 1, g(N,N) = 0, g(B,B) = 0, g(T,N) = 0, g(T,B) = 0, g(N,B) = 1$. In this case, k_1 can take only two values: $k_1 = 0$ when α is a straight line; $k_1 = 1$ in all other cases.

Let m be a fixed point in $\mathbb{E}^3_1$ and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$\mathbb{S}^2_1(m,r) = \{u \in \mathbb{E}^3_1 : g(u-m, u-m) = r^2\};$$

the pseudo-Riemannian hyperbolical space is defined by

$$\mathbb{H}^2_0(m,r) = \{u \in \mathbb{E}^3_1 : g(u-m, u-m) = -r^2\};$$

the pseudo-Riemannian lightlike cone (quadric cone) is defined by

$$C(m) = \{u \in \mathbb{E}^3_1 : g(u-m, u-m) = 0\}.$$

3. The spacelike normal curves in $\mathbb{E}^3_1$

In this section, we give some characterization theorems for spacelike normal curves.

Theorem 3.1 *Let $\alpha = \alpha(s)$ be a unit speed spacelike normal curve in $\mathbb{E}^3_1$ with spacelike or timelike principal normal N and with curvatures $k_1(s) > 0$, $k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then the following statements hold:*

(i) The curvatures $k_1(s)$ and $k_2(s)$ satisfy the following equality

$$\frac{1}{k_1(s)} = c_1 \cosh\Big(\int k_2(s)ds\Big) + c_2 \sinh\Big(\int k_2(s)ds\Big), \quad c_1, c_2 \in \mathbb{R};$$

(ii) The principal normal and binormal component of the position vector of the curve are given respectively by

$$g(\alpha(s), N) = a_1 \cosh\Big(\int k_2(s)ds\Big) + a_2 \sinh\Big(\int k_2(s)ds\Big)$$

$$g(\alpha(s), B) = a_1 \sinh\Big(\int k_2(s)ds\Big) + a_2 \cosh\Big(\int k_2(s)ds\Big), \quad a_1, a_2 \in \mathbb{R};$$

(iii) If the position vector of the curve is null vector, then α lies on pseudo-Riemannian lightlike cone $C(m)$ and the curvatures $k_1(s)$ and $k_2(s)$ satisfy

$$\frac{1}{k_1(s)} = c_1[\cosh(\int k_2(s)ds) \pm \sinh(\int k_2(s)ds)].$$

Conversely if $\alpha(s)$ is a unit speed spacelike curve in $\mathbb{E}_1^3$ with spacelike or timelike principal normal N, the curvatures $k_1(s) > 0$, $k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$ and one of the statements (i), (ii) and (iii) hold, then α is a normal curve or congruent to a normal curve.

Proof. Let us first suppose that $\alpha(s)$ is a unit speed spacelike normal curve in $\mathbb{E}_1^3$ with spacelike or timelike principal normal N, where s is pseudo arclength parameter. Then by definition we have

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B(s).$$

Differentiating this with respect to s and using the corresponding Frenet equations (1), we find

$$\epsilon\lambda k_1 = -1, \quad \lambda^{'} + \mu k_2 = 0, \quad \mu^{'} + \lambda k_2 = 0. \tag{3}$$

From the first and second equation in (3), we get

$$\lambda = -\frac{\epsilon}{k_1}, \quad \mu = \frac{\epsilon}{k_2}\left(\frac{1}{k_1}\right)^{'}. \tag{4}$$

Thus

$$\alpha(s) = -\frac{\epsilon}{k_1}N + \frac{\epsilon}{k_2}\left(\frac{1}{k_1}\right)^{'}B. \tag{5}$$

Further, from the third equation in (3) and using (4), we find the following differential equation

$$\left[\frac{1}{k_2}\left(\frac{1}{k_1}\right)^{'}\right]^{'} - \frac{k_2}{k_1} = 0. \tag{6}$$

Putting $y(s) = \frac{1}{k_1}$ and $p(s) = \frac{1}{k_2}$, equation (6) can be written as

$$(p(s)y^{'}(s))^{'} - \frac{y(s)}{p(s)} = 0.$$

If we change variables in the above equation as $t = \int \frac{1}{p(s)} ds$, then we get

$$\frac{d^2y}{dt^2} - y = 0.$$

The solution of the previous differential equation is

$$y = c_1 \cosh(t) + c_2 \sinh(t),$$

where $c_1, c_2 \in \mathbb{R}$. Therefore,

$$\frac{1}{k_1(s)} = c_1 \cosh(\int k_2(s)ds) + c_2 \sinh(\int k_2(s)ds). \tag{7}$$

Thus we have proved statement (i). Next, substituting (7) into (4) and (5), we get

$$\lambda = -\epsilon[c_1 \cosh(\int k_2(s)ds) + c_2 \sinh(\int k_2(s)ds)],$$
$$\mu = \epsilon[c_1 \sinh(\int k_2(s)ds) + c_2 \cosh(\int k_2(s)ds)],$$

and

$$\begin{aligned} \alpha = & -\epsilon(c_1 \cosh(\int k_2(s)ds) + c_2 \sinh(\int k_2(s)ds))N \\ & +\epsilon(c_1 \sinh(\int k_2(s)ds) + c_2 \cosh(\int k_2(s)ds))B. \end{aligned} \tag{8}$$

Therefore, from (8) we easily find that

$$g(\alpha, \alpha) = \epsilon(c_1^2 - c_2^2), \tag{9}$$

$$g(\alpha, N) = a_1 \cosh(\int k_2(s)ds) + a_2 \sinh(\int k_2(s)ds), \tag{10}$$

$$g(\alpha, B) = a_1 \sinh(\int k_2(s)ds) + a_2 \cosh(\int k_2(s)ds), \tag{11}$$

where $a_1 = -c_1 \in \mathbb{R}, a_2 = -c_2 \in \mathbb{R}$. Consequently, we have proved (ii).

Next, suppose that α is a normal curve with a null (lightlike) position vector. Then we have $g(\alpha, \alpha) = 0$. Substituting this into equation (9), we obtain $c_1^2 = c_2^2$. Then (7) becomes

$$\frac{1}{k_1(s)} = c_1[\cosh(\int k_2(s)ds) \pm \sinh(\int k_2(s)ds)]. \tag{12}$$

On the other hand, let us consider the vector

$$m = \alpha(s) + \frac{\epsilon}{k_1}N - \frac{\epsilon}{k_2}\left(\frac{1}{k_1}\right)' B.$$

Differentiating this with respect to s and using corresponding Frenet equations (1), we find $m' = 0$, and therefore $m =$ constant. Then $g(\alpha - m, \alpha - m) = 0$, which means that α lies on $C(m)$. Consequently, we have proved statement (iii).

Conversely, suppose that statement (i) holds. Then we have

$$\frac{1}{k_1(s)} = c_1 \cosh(\int k_2(s)ds) + c_2 \sinh(\int k_2(s)ds).$$

Differentiating this with respect to s, we get

$$\left[\frac{1}{k_2}\left(\frac{1}{k_1}\right)'\right]' = \frac{k_2}{k_1}.$$

By applying Frenet equations (1), we obtain

$$\frac{d}{ds}\left[\alpha(s) + \frac{\epsilon_1}{k_1}N - \frac{\epsilon_1}{k_2}\left(\frac{1}{k_1}\right)' B\right] = 0.$$

Consequently, α is congruent to a normal curve. Next, assume that statement (ii) holds. Then the equations (9) and (10) are satisfied. Differentiating (9) with respect to s and using (10), we find $g(\alpha, T) = 0$, which means that α is normal curve. Finally, assume that statement (iii) holds. Then α lies on light cone $C(m)$ with vertex at m, $m =$ constant and curvatures $k_1(s)$ and $k_2(s)$ satisfy the equation (12). Hence we have

$$g(\alpha - m, \alpha - m) = 0.$$

Differentiating this four times with respect to s and using Frenet equations (1), we get

$$\alpha(s) - m = -\frac{\epsilon}{k_1}N + (\frac{\epsilon}{k_2})(\frac{1}{k_1})'B.$$

This means that, up to a translation for vector m, curve α is congruent to a normal curve. Let us put $m = 0$. Then using (12) we easily find $g(\alpha, \alpha) = 0$, which proves the theorem.

□

Theorem 3.2 *Let $\alpha = \alpha(s)$ be unit speed spacelike normal curve in $\mathbb{E}^3_1$ with curvatures $k_1(s) > 0$, $k_2(s) \neq 0$, non-null principal normal N and non–null position vector. Then:*

(i) The position vector α is spacelike if and only if the curve α lies on the pseudo–Riemannian sphere $\mathbb{S}^2_1(m, r)$ and there holds

$$\frac{1}{k_1(s)} = \pm\sqrt{c^2 + \epsilon r^2}\cosh(\int k_2(s)ds) + c\sinh(\int k_2(s)ds), \quad c \in \mathbb{R}, \quad \epsilon = \pm 1; \quad (13)$$

(ii) The position vector α is timelike if and only if the curve α lies on the pseudohyperbolical space $\mathbb{H}^2_0(m, r)$ and there holds

$$\frac{1}{k_1(s)} = \pm\sqrt{c^2 - \epsilon r^2}\cosh(\int k_2(s)ds) + c\sinh(\int k_2(s)ds), \quad c \in \mathbb{R}, \quad \epsilon = \pm 1. \quad (14)$$

Proof. Let us first assume that the position vector α is spacelike. Then $g(\alpha, \alpha) = r^2$, $r \in \mathbb{R}^+$. Substituting this into (9), we get $c_1 = \pm\sqrt{c_2^2 + \epsilon r^2}$. By using the last equation and (7), we obtain that (13) holds. Next, let us consider the vector

$$m = \alpha + (\epsilon/k_1)N - (\epsilon/k_2)(1/k_1)'B.$$

Differentiating this and using the corresponding Frenet equations, we get $m' = 0$. Consequently, m = constant. It follows that $g(\alpha - m, \alpha - m) = r^2$, which means that α lies on pseudo–Riemannian sphere $S^2_1(m, r)$ with center m and of radius r. Conversely, assume that (13) holds and that α lies on $\mathbb{S}^2_1(m, r)$. Then $g(\alpha - m, \alpha - m) = r^2$, where $r \in \mathbb{R}^+$. Differentiating this four times with respect to s and using Frenet equations, we find

$$\alpha - m = -(\epsilon/k_1)N + (\epsilon/k_2)(1/k_1)'B.$$

Therefore, up to a translation for a vector m, α is congruent to a normal curve. In particular, let us put $m = 0$. Then (13) implies that $g(\alpha, \alpha) = r^2$, which proves statement (i).

The proof of statement (ii) is analogous to the proof of statement (i). □

Remark. The spacelike curves with a null principal normal N, in the space $\mathbb{E}^3_1$ can have the first curvature $k_1 = 0$ or $k_1 = 1$ [7]. If $k_1 = 0$, then $\alpha(s)$ is straight line. Therefore $\alpha(s)$ is in direction of $T(s)$ for each s. For straight line we have $N = B = 0$, so we do not have normal plane $\{N, B\}$. Therefore, if $k_1 = 0$ then $\alpha(s)$ can not be normal curve.

Theorem 3.3 *Let $\alpha(s)$ be unit speed spacelike normal curve in $\mathbb{E}^3_1$ with a null principal normal N and $k_1 = 1$. Then α is normal curve if and only if the principal normal and binormal component of the position vector are, respectively, $g(\alpha, N) = -1, \quad g(\alpha, B) = c, \quad c \in \mathbb{R}$.*

Proof. Let us first assume that $\alpha(s)$ is normal curve. Then we have

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B(s). \tag{15}$$

Differentiating this with respect to s and using Frenet equations (2), we get

$$\mu = -1, \quad \lambda' + \lambda k_2 = 0 \quad \text{and} \quad \mu' - \mu k_2 = 0 \tag{16}$$

We obtain from the third equation in (16) that $k_2 = 0$. Then the second equation in (16) implies $\lambda' = 0$. Thus $\lambda = c, \quad c \in \mathbb{R}$ and therefore

$$\alpha = cN - B. \tag{17}$$

Finally, we obtain $g(\alpha, N) = -1, \; g(\alpha, B) = c$.

Conversely, let $g(\alpha, N) = -1, \; g(\alpha, B) = c$. Then differentiating with respect to s, we find $k_2 = 0$ and $g(\alpha, T) = 0$, which means that α is normal curve. □

Theorem 3.4 *Let $\alpha(s)$ be unit speed spacelike normal curve in $\mathbb{E}^3_1$ with a null principal normal N and $k_1 = 1$. Then α lies on pseudo-Riemannian sphere $\mathbb{S}^2_1(m, r)$ if and only if α is plane normal curve with the equation $\alpha - m = -\frac{r^2}{2}N - B$.*

Proof. Suppose that α lies on pseudo-Riemannian sphere $\mathbb{S}^2_1(m,r)$. Then we have

$$g(\alpha - m, \alpha - m) = r^2, \qquad r \in \mathbb{R}^+.$$

Differentiating this and applying Frenet formulae, we find

$$k_2 g(N, \alpha - m) = 0.$$

Thus $k_2 = 0$, and α is plane curve. We will prove that it is normal curve. Decompose the vector $\alpha - m$ by

$$\alpha - m = aT + bN + cB,$$

where $a = a(s), b = b(s), c = c(s)$ are arbitrary functions of s.

Then $g(\alpha - m, T) = 0 = a,\ g(\alpha - m, N) = c = -1,\ g(\alpha - m, B) = b$. Differentiating $g(\alpha - m, B) = b$, we get $b = b_0 =$ constant. We obtain that

$$\alpha - m = b_0 N - B,$$

and since $g(\alpha - m, \alpha - m) = r^2$, we have $g(\alpha - m, \alpha - m) = -2b_0 = r^2$ and $b_0 = -\dfrac{r^2}{2}$.

Finally, α has the equation

$$\alpha - m = -\frac{r^2}{2} N - B,$$

and it is congruent to a normal curve.

Conversely, if α is plane normal curve with the equation $\alpha - m = -\dfrac{r^2}{2} N - B$ where $r \in \mathbb{R}^+$ and $m = (m_1, m_2, m_3) \in \mathbb{E}^3_1$, then we have $k_2 = 0$. Next, we get that $m = \alpha + \dfrac{r^2}{2} N + B$ which differentiating in s gives $m' = 0$. Thus $m =$ constant $\in \mathbb{E}^3_1$, (i.e. m is constant vector). Therefore, α lies on $\mathbb{S}^2_1(m,r)$. □

Theorem 3.5 *Let $\alpha(s)$ be unit speed spacelike normal curve in $\mathbb{E}^3_1$ with a null principal normal N and $k_1 = 1$. Then α lies on pseudo-Riemannian hyperbolical space $\mathbb{H}^2_0(m,r)$ if and only if α is plane normal curve with the equation $\alpha - m = \dfrac{r^2}{2} N - B$, where $r \in \mathbb{R}^+$*

Proof. The proof is similar with the proof of theorem 3.4. □

Theorem 3.6 *Let $\alpha(s)$ be unit speed spacelike normal curve in $\mathbb{E}_1^3$ with a null principal normal N and $k_1 = 1$. Then α lies on light cone $C(m)$ with vertex at m if and only if α is congruent to a normal curve with the equation $\alpha(s) = -B(s)$.*

Proof. Suppose that α lies on light cone $C(m)$ with vertex at point $m \in \mathbb{E}_1^3$. Then

$$g(\alpha - m, \alpha - m) = 0.$$

Differentiating the previous equation and using Frenet equations (2), we get $g(\alpha - m, T) = 0, g(\alpha - m, N) = -1$ and $k_2 = 0$. Next, decompose the vector $\alpha - m$ by

$$\alpha - m = aT + bN + cB,$$

where $a = a(s), b = b(s), c = c(s)$ are arbitrary functions of s.

Then $g(\alpha - m, T) = 0 = a$, $g(\alpha - m, N) = c = -1$, $g(\alpha - m, B) = b$. Differentiating $g(\alpha - m, B) = b$, we get $b = b_0 =$ constant. It follows that

$$\alpha - m = b_0 N - B.$$

Since $g(\alpha - m, \alpha - m) = 0 = -2b_0$, we get $b_0 = 0$. Thus $\alpha - m = -B$. Therefore, up to a translation for the vector m, α is congruent to a normal curve and $\alpha = -B$.

Conversely, assume that α is congruent to a normal curve with the equation $\alpha = -B$. Differentiating this we get $k_2 = 0$. Let us consider the vector $m = \alpha + B$. Taking the derivative of the last equation, we find $m =$ constant and finally $g(\alpha - m, \alpha - m) = 0$, which means that α lies on the light cone $C(m)$. □

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