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On A Certain Class of Bessel Integrals

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Abstract

There are many old results of integrals involving Bessel functions, currently available in handbooks, but we found no recourse in the well-known references to how they were established. In this paper, we attempt to have a clear way of proving some of these results . In fact, we consider a certain class of Bessel integrals where we prove that such integrals vanish under certain conditions. To this end some theorems regarding this class of integrals with their proofs are put forward. A computer algorithm is provided to implement some of our results. The result in this paper extend the work in [3], and it concludes by indicating the wide range of old and new results that can be obtained.

Key Words: Bessel Functions, Infinite Integrals, Integral Representations.

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1. Introduction and Statement of Results.

Bessel functions, with their manifold applications, have been studied in great detail, and extensive tables of these functions are available [2, 5, 6, 8, 9]. Infinite integrals of these functions frequently occur in the investigation of some physical and Engineering problems. There exists a considerable body of information on the subject of these integrals. Of special significance are chapter XIII of Watson's classical treaties [9] and the excellent book by Luke [6]which provides a thorough summary of results prior to 1962.

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The purpose of this paper is to derive a number of old and new infinite integrals involving Bessel functions. In fact, we investigate the class of integrals:

$$\int_0^\infty t^{\mu+1} J_v(at) \frac{F(t)}{(t^2+b^2)^m} dt, \tag{1}$$

where $J_v(at)$ is the Bessel function of the first kind, and $F(t)$ is a function that has a meromorphic extension in the right upper half-plane.

In [3], a number of infinite integrals involving Bessel functions have been investigated. One of the main results is that the class of Bessel integrals

$$\int_0^\infty t^{\mu+1} J_v(at) F(t) dt, \text{ and } \int_0^\infty t^{\mu+1} Y_v(at) F(t) dt \tag{2}$$

vanishes under the conditions that F belongs to $\mathcal{F}_\lambda$, the family of all functions F from $\mathbb{C}$ into $\mathbb{C}$, that have no singularities in the right upper half-plane, and have at most exponential growth in the sense that $|F(z)| \leq ce^{\lambda y}$, for all $y > 0$ and for some $\lambda > 0$, where $J_v(.)$ is the Bessel function of the first kind, and $Y_v(.)$ is the Neumann's function.

In the first part of this paper we find more relaxed conditions on the class of functions $F(z)$ to establish a parallel result to that in [3]. In the second part of this paper we state and prove a theorem where we allow the class of functions $F(z)$ to have singularities of different orders in the right upper half-plane.

Before stating our main results we give some basic definitions

Definition 1.1 *A meromorphic function g is said to have at most polynomial growth if*

$$|g(z)| \leq |P_g(|z|)|\, e^{\lambda y}, \qquad y > 0,\ z = x + iy \tag{3}$$

for some polynomial P_g and $\lambda > 0$.

Definition 1.2 *We denote by $\mathcal{G}_\lambda$ the family of all meromorphic functions g that have at most polynomial growth and satisfying the following two conditions:*

1. *g has no singularities in the right upper half-plane;*

2. *$g(iy) = g(-iy)$ for all $y \in \mathbb{R}$.*

Our main results are the following theorems.

Theorem 1.1 *Given that* $g \in \mathcal{G}_{\lambda}$, *with* $P_g(t) = t^d$ *for some* $d \geq 0$. *Then for* $-2-d < \mu < -3/2-d$,*and* $-2-\mu-d < \nu < 2+\mu+d$, *we have*

$$(i) \int_0^{\infty} t^{\mu+1} J_v(at) g(t) dt = 0, \quad (4)$$

$$(ii) \int_0^{\infty} t^{\mu+1} Y_v(at) g(t) dt = 0\ , \quad (5)$$

for all $a > \lambda$.

Theorem 1.2 *Suppose that* $F \in \mathcal{G}_{\lambda}$ *such that* $\deg(P_F) = d$,*and* $G(z) = \frac{F(z)}{(z^2+b^2)^m}$, $m \geq 1$, *where* b *is a positive real number. Then for* $a > \lambda, \nu > -1$, and $d < 2m - \nu - 3/2$, we have

$$\int_0^{\infty} t^{v+1} J_v(at) G(t) dt$$

$$= \sum_{\ell=0}^{m-1} \sum_{s=0}^{\ell} D_{\ell,s} F^{(\ell-s)}(ib) \frac{d^{m-\ell-1}}{dt^{m-\ell-1}} ((\frac{t}{2})^{v+1} K_v(t))\ |_{ab}, \quad (6)$$

where

$$D_{\ell,s} = \frac{(s+m-1)!}{[(m-1)!]^2} (-1)^{m-1} (-1)^s (\frac{a}{2b})^m (\frac{a}{2})^{-v-2} 2^{-s} e^{i\frac{\pi}{2}(\ell-s)} \times b^{-s} a^{-\ell} \binom{m-1}{\ell} \binom{\ell}{s}$$

It should be pointed out that many results in [4, 7] can be obtained as consequence of this theorem.

2. Preliminaries

In this section we give some basic lemmas.

Lemma 2.1 *Given* $0<\varepsilon<1$,*then*

$$\sup_{|z|=\varepsilon}\left|H_v^{(1),(2)}(az)\right|\leq C(\varepsilon^{-v}+\varepsilon^{v}), \tag{7}$$

where C *is a constant independent of* ε, *and* $H_v^{(1),(2)}(\cdot)$ *are the Hankel functions defined in [2]*

Proof. It is known that

$$H_v^{(1)}(z)=\frac{J_{-v}(z)-e^{-\pi vi}J_v(z)}{i\sin v\pi}. \tag{8}$$

and

$$J_{\pm v}(z)=(\frac{z}{2})^{\pm v}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{2k}}{\Gamma(k+1)\Gamma(k\pm v+1)}. \tag{9}$$

since

$$\frac{1}{2^{\pm v}}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{2k}}{\Gamma(k+1)\Gamma(k\pm v+1)}.$$

is convergent, we have

$\left|H_v^{(1)}(z)\right|\leq C(\varepsilon^{-v}+\varepsilon^{v})$ (since $\varepsilon<1$) $\forall\ |z|=\varepsilon$.

Thus,

$$\sup_{|z|=\varepsilon}\left|H_v^{(1)}(az)\right|\leq C(\varepsilon^{-v}+\varepsilon^{v}).$$

□

Corollary 2.1 *Given* $0<\varepsilon<1$, $C_{_\varepsilon}=\left\{\varepsilon e^{i\theta}:0<\theta\leq\frac{\pi}{2}\right\}$ *and* $g\in\mathcal{G}_{_\lambda}$. *Then for* $\mu>-2$, and $-2-\mu<\nu<2+\mu$, we have

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} z^{\mu+1}H_v^{(1)}(az)g(z)dz=0. \tag{10}$$

Proof. Let $P(z) = \sum_{j=0}^{d} a_j z^j$ be such that $|g(z)| \leq C\,|P(|z|)|\,e^{\lambda y}$. Then

$$\left|\int_{C_\varepsilon} z^{\mu+1} H_v^{(1)}(az) g(z) dz\right| \leq C \cdot \varepsilon^{\mu+1}(\varepsilon^{-v} + \varepsilon^{v})(\sum_{j=0}^{d} |a_j|\,\varepsilon^j) \cdot \frac{\pi}{2}\varepsilon \cdot e^{\lambda\varepsilon} \to 0 \qquad (11)$$

as $\varepsilon \to 0$. □

Lemma 2.2 *Given $g \in \mathcal{G}_\lambda$, $a > \lambda$, $R > 0$, and $C_R = \left\{R\ e^{i\theta} : 0 < \theta \leq \frac{\pi}{2}\right\}$. Then*

$$\lim_{R\to\infty}\left|\int_{C_R} z^{\mu+1} H_v^{(1)}(az) g(z) dz\right| = 0. \qquad (12)$$

Proof. Recall that, [6, 9]

$$\begin{aligned}
\left|H_v^{(1)}(az)\right| &\leq \sqrt{\frac{2}{\pi}}\left|\frac{e^{iaz}}{\sqrt{az}}\right| \text{ for large } |z|, \\
\text{and } |g(z)| &\leq |P(|z|)|\,e^{\lambda y} \leq (\sum_{\ell=0}^{d} |a_\ell|\,R^\ell) e^{\lambda R\sin\theta} = M(R) e^{\lambda R\sin\theta} \text{ for } |z| = R.
\end{aligned}$$

Let

$$\begin{aligned}
I(R) &= \int_{C_R} z^{\mu+1} H_v^{(1)}(az) g(z) dz \\
&= \int_0^{\frac{\pi}{2}} R^{\mu+1} e^{i(v+1)\theta} H_v^{(1)}(aRe^{i\theta}) g(Re^{i\theta}) iRe^{i\theta} d\theta.
\end{aligned}$$

Therefore,

$$|I(R)| \leq C \int_0^{\frac{\pi}{2}} R^{\mu+1} \frac{e^{-Ra\sin\theta} e^{\lambda R\sin\theta}}{\sqrt{|a|}R^{\frac{1}{2}}} M(R) R d\theta.$$

Now by the fact that $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 < \theta \leq \frac{\pi}{2}$, we have

$$\begin{aligned} & \lim_{R\to\infty}\left|\int\limits_{C_R} z^{\mu+1}H_v^{(1)}(az)g(z)dz\right| \\ \leq\ & C\lim_{\varepsilon\to 0^+}\lim_{R\to\infty}\int\limits_{\varepsilon}^{\frac{\pi}{2}}\frac{R^{\mu+2}}{R^{\frac{1}{2}}}e^{-2R\frac{a\theta}{\pi}}e^{2\lambda\frac{R\theta}{\pi}}M(R)d\theta \\ =\ & C\lim_{\varepsilon\to 0^+}\lim_{R\to\infty}\left[R^{\mu+\frac{3}{2}}\frac{\pi M(R)}{2R(a-\lambda)}(e^{\frac{-2R\varepsilon}{\pi}(a-\lambda)} - e^{-R(a-\lambda)})\right] \\ =\ & 0. \end{aligned}$$

Lemma 2.3 *Given* $g \in \mathcal{G}_{\lambda}$. *Then for* $\mu > -2$, and $-2-\mu < \nu < 2+\mu$ we have

$$\text{(i)} \qquad \int\limits_0^\infty t^{\mu+1}H_v^{(1)}(at)g(t)dt = -(i)^\mu\int\limits_0^\infty t^{\mu+1}H_v^{(1)}(iat)g(it)dt. \qquad (13)$$

$$\text{(ii)} \qquad \int\limits_0^\infty t^{\mu+1}H_v^{(2)}(at)g(t)dt = -(-i)^\mu\int\limits_0^\infty t^{\mu+1}H_v^{(2)}(-iat)g(-it)dt. \qquad (14)$$

Proof. Given $0 < \varepsilon < 1$, $R > 1$ using the contour $C = C_\varepsilon + C_1 + C_2 + C_R$, by Cauchy-Goursat theorem, we get

$$\begin{aligned} & \int\limits_{C_1} z^{\mu+1}H_v^{(1)}(az)g(z)dz + \int\limits_{C_R} z^{\mu+1}H_v^{(1)}(az)g(z)dz \\ & + \int\limits_{C_2} z^{\mu+1}H_v^{(1)}(az)g(z)dz - \int\limits_{C_\varepsilon} z^{\mu+1}H_v^{(1)}(az)g(z)dz = 0. \qquad (15) \end{aligned}$$

where $C_\varepsilon = \left\{z \in \mathbb{C} : z = \varepsilon e^{i\theta} : 0 < \theta \leq \frac{\pi}{2}\right\}$, $C_1 = \{z \in \mathbb{C} : z = t, \varepsilon < t \leq R\}$, $C_2 = \{z \in \mathbb{C} : z = it, \varepsilon < t \leq R\}$ and $C_R = \left\{z \in \mathbb{C} : z = R\ e^{i\theta} : 0 < \theta \leq \frac{\pi}{2}\right\}$.

By Corollary(2.1) and Lemma(2.2), we have

$$\lim_{\varepsilon\to 0}\int_{C_3} z^{\mu+1}H_v^{(1)}(az)g(z)dz = \lim_{R\to\infty}\int_{C_R} z^{\mu+1}H_v^{(1)}(az)g(z)dz = 0. \tag{16}$$

Therefore, if $R\to\infty$ and $\varepsilon\to 0$, we have

$$\int_0^\infty t^{\mu+1}H_v^{(1)}(at)g(t)dt + \int_0^\infty (i)^{\mu+1}y^{\mu+1}H_v^{(1)}(iay)g(iy)idy = 0. \tag{17}$$

Thus

$$\begin{aligned}\int_0^\infty t^{\mu+1}H_v^{(1)}(at)g(t)dt &= \int_0^\infty (i)^{\mu+1}t^{\mu+1}H_v^{(1)}(iat)g(it)idt \\ &= -(i)^{\mu}\int_0^\infty t^{\mu+1}H_v^{(1)}(iat)g(it)dt \end{aligned} \tag{18}$$

Finally, (ii) can be obtained in a similar way.

3. Proofs of Main Results

We start this section by presenting a proof of Theorem 1.1.

Proof of (Theorem 1.1).

Recall that

$$J_v(az) = \frac{1}{2}\left[H_v^{(1)}(az) + H_v^{(2)}(az)\right] \tag{19}$$

and

$$(-i)^v H_v^{(2)}(-iat) = -(i)^v H_v^{(1)}(iat). \tag{20}$$

By (19), (20) and Lemma (2.3), we have

$$\begin{aligned}
&\int_0^\infty t^{\mu+1} J_v(at) g(t) dt \\
&= \frac{1}{2}\left[-(i)^\mu \int_0^\infty t^{\mu+1} H_v^{(1)}(iat) g(it) dt - (-i)^\mu \int_0^\infty t^{\mu+1} H_v^{(2)}(-iat) g(-it) dt\right] \\
&= \frac{1}{2}\left[-(i)^\mu \int_0^\infty t^{\mu+1} H_v^{(1)}(iat) g(it) dt + (i)^\mu \int_0^\infty t^{\mu+1} H_v^{(1)}(iat) g(-it) dt\right] \\
&= \frac{1}{2}(i)^\mu \int_0^\infty t^{\mu+1} H_v^{(1)}(iat)\left[g(-it) - g(it)\right] dt \\
&= 0. \qquad (21)
\end{aligned}$$

Here, the last equality follows by condition 2 of Definition 1.2.

Following the same procedure we can show that $\int_0^\infty t^{\mu+1} Y_v(at) g(t) dt = 0$ holds.

Note that $\mathcal{G}_\lambda$ *contains all even polynomials and hence we obtain ,*

$$\int_0^\infty t^{\mu+n+1} J_v(at) dt = 0 \qquad (22)$$

for even n, $-2-n < \mu < -3/2-n$,and $-2-\mu-n < \nu < 2+\mu+n$,
which cannot be obtained from the results in [3].

Proof of (Theorem 1.2) Let $R > 0$ be large so that $b < R$;
let $D_R = \{z \in \mathbb{C} : |z| < R \text{ and } 0 < \arg z \leq \frac{\pi}{2}\}$,and $D_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon \text{ and } 0 < \arg z \leq \frac{\pi}{2}\}$, where ε is so small, such that $b \notin D_\varepsilon$. Let D be a small disk around ib with boundary C_b. Consider the integral

$$I = \frac{1}{2\pi i} \int_C z^{v+1} H_v^{(1)}(az) \frac{F(z)}{(z^2+b^2)^m} dz \qquad (23)$$

where $C =$ the boundary $D_R - \{D \cup D_\varepsilon\}$. Then by similar argument as in the proof of Theorem 1.1, we have

$$\frac{1}{2\pi i}\int_0^\infty t^{v+1}[H_v^{(1)}(at) - e^{v\pi i}H_v^{(1)}(ate^{\pi i})]\frac{F(t)}{(t^2+b^2)^m}dt$$

$$= \frac{1}{2\pi i}\int_{C_b} z^{v+1}H_v^{(1)}(az)\frac{F(z)}{(z^2+b^2)^m}dz \tag{24}$$

Now by [9], we have

$$(-i)^\nu H_v^{(2)}(-iat) = -(i)^\nu H_v^{(1)}(iat) \tag{25}$$

Thus

$$\frac{1}{2\pi i}\int_0^\infty t^{v+1}[H_v^{(1)}(at) + H_v^{(2)}(at)]\frac{F(t)}{(t^2+b^2)^m}dt$$

$$= \frac{1}{2\pi i}\int_{C_b} z^{v+1}H_v^{(1)}(az)\frac{F(z)}{(z^2+b^2)^m}dz. \tag{26}$$

Using the identity

$$H_v^{(1)}(az) = \frac{J_{-v}(az) - J_v(az)e^{-v\pi i}}{i\sin v\pi} \tag{27}$$

and the Bessel's representation of $J_{\pm v}(az)$, we have

$$H_v^{(1)}(az) = \frac{1}{i\sin v\pi}\left[\begin{array}{c} \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{az}{2})^{2n-v}}{\Gamma(n+1)\Gamma(-v+n+1)} \\ -e^{-v\pi i}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{az}{2})^{2n+v}}{\Gamma(n+1)\Gamma(v+n+1)}\end{array}\right].$$

Now

$$\frac{1}{\pi i}\int_0^\infty t^{v+1}J_v(at)\frac{F(t)}{(t^2+b^2)^m}dt$$
$$= \frac{-1}{2\pi\sin v\pi}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+1)}\left\{\frac{A_n}{\Gamma(-v+n+1)}-\frac{e^{-v\pi i}B_n}{\Gamma(v+n+1)}\right\}\right], \quad (28)$$

where

$$A_n = (\frac{a}{2})^{2n-v}\int_{C_b}\frac{z^{2n+1}F(z)}{(z^2+b^2)^m}dz. \quad (29)$$

$$B_n = (\frac{a}{2})^{2n+v}\int_{C_b}\frac{z^{2n+2v+1}F(z)}{(z^2+b^2)^m}dz. \quad (30)$$

By using *Leibnitz's rule*,

$$(fgh)^{(k)} = \sum_{\ell=0}^{k}\sum_{s=0}^{\ell}\binom{k}{s}\binom{\ell}{s}f^{(k-\ell)}g^{(s)}h^{(\ell-s)}. \quad (31)$$

First with $f = z^{2n+1}, h = F$, and $g = (z+ib)^{-m}$ we get,

$$A_n = \frac{2\pi i}{(m-1)!}(\frac{a}{2})^{2n-v}\sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}C_{n,m,\ell,s}(ib)^{2n-2m+\ell-s+2}F^{(\ell-s)}(ib) \quad (32)$$

where

$$C_{n,m,\ell,s} = \frac{(2n+1)!}{(2n+2-m+\ell)!}\cdot\frac{(m+s-1)!}{(m-1)!}(-1)^s\binom{m-1}{\ell}\binom{\ell}{s}\cdot 2^{-m-s}. \quad (33)$$

Thus by simplifying $C_{n,m,\ell,s}$, we get,

$$A_n = \left(\frac{2\pi i}{[(m-1)!]^2}(\frac{a}{2})^{2n-v}2^{-m}(ib)^{2n-2m+2}(2n+1)!\right) \cdot \sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}\frac{(-1)^s(2ib)^{-s}(ib)^{\ell}\binom{m-1}{\ell}\binom{\ell}{s}(m+s-1)!F^{(\ell-s)}(ib)}{(2n+2-m+\ell)!}. \tag{34}$$

Second, with $f = z^{2n+2v+1}$, $g = (z+ib)^{-m}$, $h = F$ we get,

$$B_n = \left(\frac{2\pi i}{[(m-1)!]^2}(\frac{a}{2})^{2n+v}2^{-m}(ib)^{2n+2v-2m+2}\Gamma(2n+2v+2)\right) \cdot \sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}\frac{(-1)^s(2ib)^{-s}(ib)^{\ell}\binom{m-1}{\ell}\binom{\ell}{s}(m+s-1)!F^{(\ell-s)}(ib)}{\Gamma(2n+2v+3-m+\ell)}. \tag{35}$$

Thus (25) becomes,

$$\frac{1}{\pi i}\int_0^{\infty} J_v(at)\frac{t^{v+1}F(t)}{(t^2+b^2)^m}dt$$

$$\begin{aligned}
= \quad & \frac{-1}{2\pi\sin v\pi}\sum_{n=0}^{\infty}\sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}(-1)^n\{\frac{2\pi i}{[(m-1)!]^2}(\frac{a}{2})^{2n-v}2^{-m}(ib)^{2n-2m+2} \\
& \times\Gamma(2n+2)\cdot\frac{(-1)^s(2ib)^{-s}(ib)^{\ell}\binom{m-1}{\ell}\binom{\ell}{s}(m+s-1)!F^{(\ell-s)}(ib)}{\Gamma(-v+n+1)\Gamma(n+1)(2n+2-m+\ell)!}\} \\
& +\frac{e^{-v\pi i}}{2\pi\sin v\pi}\sum_{n=0}^{\infty}\sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}(-1)^n\{\frac{2\pi i}{[(m-1)!]^2}(\frac{a}{2})^{2n+v}2^{-m}(ib)^{2n+2v-2m+2} \\
& \times\Gamma(2n+2v+2)\cdot\frac{(-1)^s(2ib)^{-s}(ib)^{\ell}\binom{m-1}{\ell}\binom{\ell}{s}(m+s-1)!F^{(\ell-s)}(ib)}{\Gamma(n+1)\Gamma(v+n+1)\Gamma(2n+2v+3-m+\ell)}\},
\end{aligned}$$

$$= \frac{-(2\pi i)2^{-m}(ib)^{-2m+2}}{[(m-1)!]^2(2\pi\sin v\pi)}\left[\sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}k_{\ell,s}\left\{S_1(ab)-S_2(ab)\right\}F^{(\ell-s)}(ab)\right],$$

where

$$k_{\ell,s} = (-1)^s (2ib)^{-s} (ib)^{\ell} \binom{m-1}{\ell} \binom{\ell}{s} (m+s-1)!, \tag{36}$$

$$S_1(ab) = \sum_{n=0}^{\infty} \frac{(\frac{a}{2})^{2n-v}(b)^{2n}\Gamma(2n+2)}{\Gamma(-v+n+1)\Gamma(n+1)\Gamma(2n+3-m+\ell)} , \tag{37}$$

$$S_2(ab) = \sum_{n=0}^{\infty} \frac{(\frac{a}{2})^{2n+v}(b)^{2n+2v}\Gamma(2n+2v+2)}{\Gamma(v+n+1)\Gamma(n+1)\Gamma(2n+2v+3-m+\ell)} . \tag{38}$$

Now

$$(\frac{z}{2})^{v+1} I_{-v}(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{2n+1}}{\Gamma(n+1)\Gamma(-v+n+1)}. \tag{39}$$

Thus

$$\frac{d^{m-\ell-1}}{dz^{m-\ell-1}} \left[(\frac{z}{2})^{v+1} I_{-v}(z) \right] |_{ab}$$

$$\begin{aligned} &= \frac{1}{2^{m-\ell-1}} \sum_{n=0}^{\infty} \frac{(\frac{ab}{2})^{2n+2-m+\ell}\Gamma(2n+2)}{\Gamma(n+1)\Gamma(-v+n+1)\Gamma(2n+3-m+\ell)} \\ &= \frac{1}{2^{m-\ell-1}} (\frac{ab}{2})^{-m+\ell+2} (\frac{a}{2})^{v} \sum_{n=0}^{\infty} \frac{(\frac{a}{2})^{2n-v}(b)^{2n}\Gamma(2n+2)}{\Gamma(n+1)\Gamma(-v+n+1)\Gamma(2n+3-m+\ell)}. \end{aligned} \tag{40}$$

Therefore,

$$S_1(ab) = 2^{m-\ell-1} (\frac{ab}{2})^{m-\ell-2} (\frac{a}{2})^{-v} \frac{d^{m-\ell-1}}{dz^{m-\ell-1}} \left[(\frac{z}{2})^{v+1} I_{-v}(z) \right] \Big|_{ab}. \tag{41}$$

Next, since

$$(\frac{z}{2})^{v+1} I_{v}(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{2n+2v+1}}{\Gamma(n+1)\Gamma(v+n+1)}, \tag{42}$$

we have

$$\frac{d^{m-\ell-1}}{dz^{m-\ell-1}}\left[(\frac{z}{2})^{v+1}I_v(z)\right]|_{ab}$$

$$=\frac{1}{2^{m-\ell-1}}\sum_{n=0}^{\infty}\frac{(\frac{ab}{2})^{2n+2v-m+\ell+2}\Gamma(2v+2n+2)}{\Gamma(n+1)\Gamma(v+n+1)\Gamma(2v+2n+3-m+\ell)}$$

$$S_2(ab)=2^{m-\ell-1}(\frac{ab}{2})^{m-\ell-2}(\frac{a}{2})^{-v}\frac{d^{m-\ell-1}}{dz^{m-\ell-1}}\left[(\frac{z}{2})^{v+1}I_v(z)\right]|_{ab} \quad (43)$$

Hence

$$\frac{1}{\pi i}\int_0^{\infty}J_v(at)\frac{t^{v+1}F(t)}{(t^2+b^2)^m}dt$$

$$=\frac{2^{-m}(ib)^{-2m+2}}{[(m-1)!]^2(i\sin v\pi)}\left[\begin{array}{c}\sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}k_{\ell,s}F^{(\ell-s)}(ib)2^{m-\ell-1}(\frac{ab}{2})^{m-\ell-2}(\frac{a}{2})^{-v}\\ \cdot\frac{d^{m-\ell-1}}{dz^{m-\ell-1}}\left[(\frac{z}{2})^{v+1}I_{-v}(z)-(\frac{z}{2})^{v+1}I_v(z)\right]|_{ab}\end{array}\right]$$

$$=\frac{(ib)^{-2m+2}(\frac{ab}{2})^{m-2}(\frac{a}{2})^{-v}}{[(m-1)!]^2(i\sin v\pi)}\left[\begin{array}{c}\sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}k_{\ell,s}F^{(\ell-s)}(ib)2^{-\ell-1}(\frac{ab}{2})^{-\ell}\\ \cdot\frac{d^{m-\ell-1}}{dz^{m-\ell-1}}\left[(\frac{z}{2})^{v+1}I_{-v}(z)-(\frac{z}{2})^{v+1}I_v(z)\right]|_{ab}\end{array}\right].$$

The last equation combined with the identity $K_v(z)=\frac{\pi}{2\sin v\pi}(I_{-v}(z)-I_v(z))$, we have

$$\int_0^{\infty}t^{v+1}J_v(at)\frac{F(t)}{(t^2+b^2)^m}dt$$

$$=\frac{2(ib)^{-2m+2}(\frac{ab}{2})^{m-2}(\frac{a}{2})^{-v}}{[(m-1)!]^2}\times\left[\begin{array}{c}\sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}k_{\ell,s}2^{-\ell-1}(\frac{ab}{2})^{-\ell}\\ \cdot F^{(\ell-s)}(ib)\frac{d^{m-\ell-1}}{dz^{m-\ell-1}}((\frac{z}{2})^{v+1}K_v(z))|_{ab}\ .\end{array}\right]$$

$$= \frac{(-1)^{m-1}(\frac{a}{2b})^m(\frac{a}{2})^{-v-2}}{[(m-1)!]^2}$$
$$\times \left[\begin{array}{c} \sum_{\ell=0}^{m-1}\sum_{s=0}^{\ell}(-1)^s 2^{-s} e^{\frac{i\pi}{2}(\ell-s)} b^{-s} a^{-\ell}\binom{m-1}{\ell}\binom{\ell}{s}(m+s-1)! \\ \cdot F^{(\ell-s)}(ib)\frac{d^{m-\ell-1}}{dt^{m-\ell-1}}((\frac{t}{2})^{v+1}K_v(t)) \mid_{ab} . \end{array} \right].$$

Thus we obtain (5). □

Example: Direct application of theorem 1.2 gives

$$\int_0^\infty z^{v+1}J_v(at)\frac{1}{(t^2+b^2)^m}dz = \frac{b^{\nu-m+1}a^{m-1}}{2^{m-1}\Gamma(m)}K_{\nu-m+1}(ab),$$

$a, b > 0$, $-1 < \nu < 2m - 3/2$.

Finally we provide an algorithm that implement the result in theorem 1.2

Algorithm: *To implement the result of theorem 1.2 for the class of integrals:*

$I = \int_0^\infty t^{v+1}J_v(at)\frac{F(z)}{(z^2+b^2)^m}dt$

using MATLAB 6.0.

INPUT*: The Function $F(z)$, m the order of the poles, v the order of Bessel function, a and b.*

OUTPUT: *The value of integral I.*

STEP 1*: for $i = 0$ to $(m-1)$*

for $j = 0$ to i,

$$D(i,j) = \frac{(j+m-1)!}{[(m-1)!]^2}(-1)^{m-1}(-1)^j(\frac{a}{2b})^m(\frac{a}{2})^{-v-2}2^{-j}e^{i\frac{\pi}{2}(i-j)}$$
$$\times b^{-j}a^{-i}\binom{m-1}{i}\binom{i}{j},$$

STEP 2: *Compute the derivative $F^{(i-j)}(z)$ at $z = ib$,*

STEP 3: *Compute $\frac{d^{m-i-1}}{dt^{m-i-1}}((\frac{t}{2})^{v+1}K_v(t))$ at $t = ab$,*

$$I = D(i,j) * F^{(i-j)}(z) * \frac{d^{m-i-1}}{dt^{m-i-1}}((\frac{t}{2})^{v+1}K_v(t))$$

STEP 4*: OUTPUT I.*

STOP.

4. Summary

The central result is a closed form evaluation of a cerain class of infinite integrals that involves Bessel functions. This in turn leads to several Bessel integrals that confirm, extend and add to known identities in handbooks and the literature.

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