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A Note On Groups With All Subgroups Subnormal

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Abstract

We prove that if G is a periodic group with all subgroups subnormal, and if for every $x, y \in G$, $< x, y >^G$ is an FC-group, then G is nilpotent.

1. Introduction

A group G is called an $\mathbf{N_0}$-group if all subgroups of G are subnormal. Several authors have considered $\mathbf{N_0}$-groups and obtained remarkable results. For example, $\mathbf{N_0}$-groups are soluble [13], Fitting [6] groups. Some examples of non-nilpotent $\mathbf{N_0}$-groups can be found in [3], [7], [8], [10], [11]. If G is an $\mathbf{N_0}$-group, then G is nilpotent if it satisfies one of the following conditions:

(i) G is torsion-free ([4], [17]);

(ii) G is periodic hypercentral group ([14]);

(iii) G is hypercentral of length at most ω ([15]);

(iv) G has a normal nilpotent subgroup A such that G/A has finite exponent ([16], c. f. [12]);

(v) G is residually nilpotent locally finite group ([18]); and

(vi) G is a bounded Engel group ([19]).

In this note we prove the following theorem.

Theorem. *Let G be a periodic $\mathbf{N_0}$-group. If for every $x, y \in G$, $< x, y >^G$ is an FC-group, then G is nilpotent.*

Subgroups X and Y of some group is called commensurable if $\mid X : X \cap Y \mid < \infty$ and $\mid Y : X \cap Y \mid < \infty$.

Lemma. *Let G be an $\mathbf{N_0}$-p-group and let G have a normal nilpotent subgroup N such that $G/N \cong C_{p^\infty}$. If for every $x \in N, y \in G$, $< x, y >^G$ is an FC-group, then G is nilpotent.*

Proof. First we show that the center $Z(G)$ of G is non-trivial. Assume that $Z(G)$ is trivial. Let $1 \neq a \in Z(N)$. Then clearly $N \leq C_G(a^g)$ for every $g \in G$. Put $\Omega = \{a^g : g \in G\}$ and let G act on Ω via conjugation. We also have that $< \Omega >$ is an infinite proper subgroup of G, since G is a Fitting group with trivial center. Let $1 \neq b \in G$ and $B =< b^G >$. By hypothesis $< a, b >^G$ is an FC-group and whence

$$\mid B : C_B(a) \mid < \infty, \text{ then } \mid \{[b, a^x] : x \in G\} \mid < \infty.$$

We also have that

$$\mid C_G([g, a]) : C_G([g, a]) \cap C_G(a) \mid < \infty$$

for every $g \in B \setminus C_B(a)$, since $C_G([g, a]) \neq G$ and $N \leq C_G([g, a])$. Furthermore, if a^x and a^y are two conjugates of a in G, then

$$\mid C_G(a^y) : C_G(a^y) \cap C_G(a^x) \mid < \infty,$$

since $N \leq C_G(a^y) \cap C_G(a^x)$, i. e., the centralizers of the conjugates of a are commensurable. By Lemma 4 of [2], *supp*(b) is finite and this means that G acts on Ω as a finitary permutation group. Thus $G/C_G(< a^G >)$ is isomorphic to a subgroup G_1 of $FSym(\Omega)$. Since $G/N \cong C_{p^\infty}$, $G/C_G(< a^G >) \cong C_{p^\infty}$, that is , $G_1 \cong C_{p^\infty}$. But by [1] (c. f. [20]) $FSym(\Omega)$ contains no nontrivial radicable subgroup, a contradiction. Consequently the center of G is nontrivial.

Now consider the α-centre $Z_\alpha(G)$ of G for an ordinal α. By (ii) $Z_\alpha(G)$ is nilpotent. Hence we may assume that $K = Z_\alpha(G)N \neq G$. We also have that $G/K \cong C_{p^\infty}$ and that $G/Z_\alpha(G)$ provides the statement of the lemma. By the first paragraph we conclude that $Z(G/Z_\alpha(G)) \neq 1$. This implies that G is hypercentral and it is nilpotent by (ii).

Proof of the theorem. By Theorem 2.5.1 (ii) of [9] G is locally nilpotent and hence G is the direct product of primary components. So by Lemma 5 of [5] we may assume that G is a p-group for a prime p. Suppose that G is not nilpotent. We also have that G has a proper normal nilpotent subgroup N such that $G/N \cong C_{p^\infty} \times \cdots \times C_{p^\infty}$ (n factors) for a positive integer n by Theorem 1 of [5]. This means that G/N contains subgroups K_i/N such that $K_i/N \cong C_{p^\infty}$ for $i = 1, \ldots, n$ and $G/N = K_1/N \times \cdots \times K_n/N$. By the lemma each K_i is nilpotent and whence G is nilpotent.

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