Turk J Math
28 (2004) , 217 – 229.

On Graded Primary Ideals

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Abstract

Let G be a group and R be a G-graded commutative ring, i.e., $R = \underset{g\in G}{\oplus} R_g$ and $R_gR_h \subseteq R_{gh}$ for all g, h $\in$G. In this paper, we study the graded primary ideals and graded primary G-decomposition of a graded ideal.

Key words and phrases: Graded rings, Graded primary ideals, Decomposition of graded ideals

0. Introduction

Let G be a group with identity e and R be a commutative ring. Then R is a G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \underset{g\in G}{\oplus} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by (R, G), and we consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum\limits_{g\in G} x_g$, where x_g is the component of x in R_g. Also, we write $h(R) = \underset{g\in G}{\cup} R_g$.

Let I be an ideal of R. Then I is a graded ideal of (R,G) if I = $\underset{g\in G}{\oplus}(I \cap R_g)$. Clearly $\underset{g\in G}{\oplus}(I \cap R_g) \subseteq I$ and hence I is graded ideal of (R,G) if $I \subseteq \underset{g\in G}{\oplus}(I \cap R_g)$.

In this paper we introduce the concepts of graded primary ideals and graded primary G-decomposition. We study these concepts in analogous way to that done for graded

1991 *Mathematics Subject Classification:* 13 A 02

ideals in [1] . However, if G is a finitely generated abelian group then G is isomorphic to the direct sums of some copies of Z_m and Z^n, and for this case the results are well known. Throughout this paper G is a non-finitely generated abelian group. Our work is a new direction in the study of graded ideals as well as an integration study of that done for the graded ideals. We believe that this work will lead to constructive ideals which introduce good tools for solving open problems of primary ideals and primary decomposition by turning them over into graded prime ideals and graded primary G-decomposition. This paper will be the primary ground to initiate more useful studies concerning the graded primary ideal and graded G-decomposition.

In section 1, we give some basic definitions and facts concerning graded primary ideals. In section 2, we define the graded primary G-decomposition of a graded ideal and study the uniqueness of this decomposition.

1. Graded Primary Ideals

In this section, we define the graded primary ideals and give some of their basic properties.

Definition 1.1 [**6**]. *Let I be a graded ideal of (R,G). Then:*

1. I is a graded prime ideal (in abbreviation, "G-prime ideal") if $I\neq R$; and whenever $rs \in I$, we have $r\in I$ or $s\in I$, where $r,s \in h(R)$.

2. I is a graded maximal ideal(in abbreviation, "G-maximal ideal") if $I\neq R$ and there is no graded ideal J of (R,G) such that $I \underset{\neq}{\subset} J \underset{\neq}{\subset} R$.

3. The graded radical of I (in abbreviation "Gr(I)") is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of (R,G), then $r\in Gr(I)$ iff $r^n \in I$ for some $n\in \boldsymbol{N}$.

Notation If $M\subseteq R$ then let V(M) denote the set of all G-prime ideals of (R,G) that contains M. Also, let GX denote the set of all G-prime ideals of (R,G).

Proposition 1.2 [**6**]. *Let I, J be graded ideals of (R, G). Then*

1. $I \subseteq Gr(I)$.

2. $Gr(Gr(I)) = Gr(I)$.

3. $Gr(I) = R$ iff $I = R$.

4. *$Gr(IJ) = Gr(I \cap J) = Gr(I) \cap Gr(J)$.*
5. *If P is a G-prime ideal of (R, G), then $Gr(P^n) = P$ for all $n > 0$.*

Definition 1.3 **[7]**. *Let I,J be ideals of R. We define the ideal quotient (I:J) by $(I:J) = \{a \in R : aJ \subseteq I\}$.*

Proposition 1.4 **[3]**. *Let $I_1, \ldots, I_n$ be graded ideals of (R, G). Let P be a G-prime ideal such that $\bigcap_{i=1}^{n} I_i \subseteq P$. Then $I_i \subseteq P$ for some $1 \le i \le n$. Also, if $P = \bigcap_{i=1}^{n} I_i$ then $P = I_i$ for some $1 \le i \le n$.*

Definition 1.5 *Let I be a graded ideal of (R,G). Then we say that I is a graded primary ideal of (R,G) (in abbreviation, "G-primary ideal") if $I \neq R$; and whenever $a,b \in h(R)$ with $ab \in I$ then $a \in I$ or $b \in Gr(I)$.*

Example 1.6 *Let $R = \boldsymbol{Z}[i]$ (The Gaussian integers) and let $G = \boldsymbol{Z}_2$. Then R is a G-graded ring with $R_0 = \boldsymbol{Z}$, $R_1 = i\boldsymbol{Z}$. Let $I = 2R$ be a graded prime ideal. Then I is a graded primary ideal. But I is not a primary ideal because 2 is not irreducible element of $R = \boldsymbol{Z}[i]$.*

Proposition 1.7 *Let I be a graded ideal of (R,G). Then I is a graded primary ideal of (R,G) iff R/I is not trivial and has the property that every homogeneous zero divisor in R/I is a nilpotent.*

Proof. Suppose that I is a graded primary ideal. Since I≠R, we deduce that R/I is not trivial. Let $r \in h(R) - I$ be such that $r + I \in R/I$ is a zero divisor. Then there exists $a \in R - I$ such that $a + I \neq I$ and $(a + I)(r + I) = I$, which implies $ar + I = I$ which implies $ar \in I$. Since I is a graded primary ideal and $a \notin I$, $r \in Gr(I)$. So, there exists $n \in \mathbf{N}$ such that $r^n \in I$. Hence $(r + I)^n \subseteq r^n + I = I$. Therefore, r+I is nilpotent in R/I.

Conversely, I≠R since R/I is not trivial. Let $r,s \in h(R)$ and $rs \in I$ with $r \notin I$. Then, $rs + I = I$ and hence $(r+I)(s+I) = I$. If $s \notin I$ then, $s + I$ a zero divisor in R/I and hence s+I is nilpotent in R/I. So, there exists $n \in \mathbf{N}$ such that $(s+I)^n = I$ which implies $(s+I)^n \subseteq s^n + I = I$. Hence $s^n \in I$ and then $s \in Gr(I)$. □

Lemma 1.8 *Let Q be a graded primary ideal of (R,G). Then $P = Gr(Q)$ is a graded prime ideal of (R,G), and we say that Q is a graded G-P-primary.*

Proof. Since $1 \notin$ Q we have $1 \notin$ Gr(Q) and hence Gr(Q) $\neq$ R. So, P is proper. Suppose we have a, b $\in$h(R) such that ab$\in$Gr(Q) and a $\notin$ Gr(Q). Then there exists n $\in$ **N** such that(ab)^n = a^{n}b^n $\in$ Q. Since a $\notin$Gr(Q), a^n $\notin$Gr(Q) which implies b^n $\in$Q. Hence b$\in$Gr(Q). Therefore, Gr(Q) is a graded prime ideal of (R,G). □

Corollary 1.9 *If Q is a graded G-P-primary, then P is the smallest graded prime ideal of (R,G) that contains Q.*

Proof. Let $P' \in$GX with Q$\subseteq P'$. Then P = Gr(Q)$\subseteq$Gr(P') and since P' is a graded prime ideal, Gr(P') = P'. Hence Gr(Q) = P $\subseteq P'$. Therefore, P is the unique minimal graded prime ideal of Q. □

Lemma 1.10 *Let I, J be graded ideals of (R, G). If Gr(I) + Gr(J) = R, then I + J = R.*

Proof. Direct. □

Let Q be a graded ideal such that P = Gr(Q) is a graded prime ideal of (R,G). Then Q is not necessary G-P-primary. In fact, let K be a field and R = K[X,Y] with deg X = deg Y = 1. R is Z-graded ring. Put Q = (X^2,XY). Then Q is graded ideal with Gr(Q) = (X). Since X$\notin$Q and Y $\notin$Gr(Q),Q is not G-P-primary.

However, if Gr(Q) is a graded maximal, the following result holds:

Proposition 1.11 *Let Q be a graded ideal of (R,G) such that Gr(Q) = M is a graded maximal ideal of (R,G). Then Q is a graded G-M-primary.*

Proof. Let Q be a graded ideal of (R,G) such that Gr(Q) = M is a graded maximal ideal of (R,G). Since Q$\subseteq$Gr(Q) = M$\subset$R, Q is proper. Let a, b $\in$ h(R) such that ab$\in$Q with b$\notin$Gr(Q). Then Gr(Q) $\underset{\neq}{\subset}$ Gr(Q) + Rb and hence M + Rb = R, as M is maximal. So, Gr(Q)+Gr(Rb) = R. Hence by Lemma 1.10, Q+Rb = R and then there exist q$\in$Q, r$\in$R such that q+rb = 1. Hence a = a1 = a(q+rb) = aq+r(ab)$\in$Q since q, ab $\in$ Q. Therefore, Q is graded G-M-primary. □

Corollary 1.12 *Let M be a graded maximal ideal of (R, G). Then for n∈**N**, M^n is a graded G-M-primary.*

Proof. Let M be a graded maximal ideal of (R,G). By Proposition 1.11, $M^n$is graded G-M-primary because Gr(M^n) = M. □

Proposition 1.13 *Let I, J be graded ideals of (R,G). Then (I:J) is a graded ideal of (R,G).*

Proof. Suppose I, J are graded ideals of (R,G). Let r = $\sum_{g\in G} r_g \in$ (I:J). Our goal is to show that $r_g \in$ (I:J) for all g∈G, i.e., r_gJ⊆I for all g∈G. Without loss of generality assume $r = \sum_{i=1}^{n} r_{g_i}$ where $r_{g_i} \neq 0$ for all i = 1, . . . , n and r_g = 0 for all g∉{g_1, . . . , g_n}. Since r∈(I:J), rJ⊆I, which implies $\sum_{g\in G} r_g$J⊆I. Thus $\sum_{i=1}^{n} r_{g_i}$J⊆I. We put a = r_{g_i}. Then, it suffices to show that ac∈I for any c∈J. Since J is a graded ideal, we may show ac∈I for any c∈ J∩h(R). Now, let r = $\sum_{i=1}^{n} r_{g_i}$ and c = c_h for some h∈G. Since rc∈I, $\sum_{i=1}^{n} r_{g_i}$c ∈I, if $g_i \neq g_j$, we have g_ih≠g_jh and hence $r_{g_i}c_h$ ∈I∩$R_{gih} \subseteq I$ as I is a graded ideal. Thus a∈(I:c). and then r_{g_i}J⊆I for all i = 1, . . . , n Hence r_{g_i} ∈(I:J) for all i = 1, . . . , n. Therefore, for all g∈G, $r_g \in$ (I:J) and hence (I:J) is a graded ideal of (R,G). □

Proposition 1.14 *Let P be a graded prime ideal of (R,G) and for n≥ 1, let Q_1, . . . , Q_n be graded G-P-primary ideals of (R, G). Then $\bigcap_{i=1}^{n} Q_i$ is also graded G-P-primary.*

Proof. Let Q_1, . . . , Q_n be graded primary ideals of (R,G) with Gr(Q_i) = P for i = 1, . . . , n. By Proposition 1.2, Gr(Q_1∩ . . . ∩Q_n) = Gr(Q_1)∩ . . . ∩Gr(Q_n) = P⊂R. Suppose a, b ∈ h(R) such that ab ∈ $\bigcap_{i=1}^{n} Q_i$ with b ∉ $\bigcap_{i=1}^{n} Q_i$. Then there exists an integer j with $1 \leq j \leq n$ such that b ∉ Q_j. Since ab∈Q_j and Q_j is graded G-P-primary, we have a ∈ Gr(Q_j) = P = Gr($\bigcap_{i=1}^{n} Q_i$). Hence $\bigcap_{i=1}^{n} Q_i$ is graded G-P-primary. □

Proposition 1.15 *Let Q be a graded G-P-primary ideal of (R,G) and a∈h(R).*

(i) If a∈Q, then (Q:a) = R.

(ii) If a∉Q, then (Q:a) is graded G-P-primary, in particular Gr((Q:a)) = P.

(iii) If a∉P, then (Q:a) = Q.

Proof. (i) Since 1.a = a ∈Q, 1∈(Q:a). Thus (Q:a) = R.

(ii) Let Q be a graded G-P-primary and let a∈h(R) with a∉Q.

Claim. Q ⊆ (Q:a) ⊆P = Gr(Q).

Clearly, Q ⊆ (Q:a). Let b∈(Q:a). Then ba∈Q. Since Q is a graded G-P-primary and a ∉ Q, b ∈ Gr(Q) = P. Therefore, Q ⊆ (Q:a) ⊆P. By the claim, we have P = Gr(Q) ⊆ Gr((Q:a)) ⊆ Gr(P) = P and hence Gr((Q:a)) = P. Now, we want to show (Q:a) is G-P-primary. Suppose c, d ∈ h(R) such that cd ∈ (Q:a) and d∉P. Then cda ∈ Q. Since Q is graded G-P-primary and d∉P, ca∈Q and hence c∈(Q:a). Thus (Q:a) is graded G-P-primary.

(iii) Let b∈(Q:a). Then ba∈Q. Since a∉P and Q is graded G-P-primary, b∈Q. Thus (Q:a)⊆Q. Therefore, (Q:a) = Q. □

2. Graded Primary G-decomposition of a Graded Ideal

In this section, we define the graded primary G-decomposition of a graded ideal as well as propositions indicating the uniqueness of this decomposition side by side with propositions that study the graded primary G-decomposition in gr-Noetherian rings.

Definition 2.1 [**2**]. *We say (R,G) is gr-Noetherian if it satisfies the ascending chain condition on graded ideals of R.*

Proposition 2.2 *Let I,J be two distinct graded ideals of (R,G). Put* $\Omega = \{P' \in GX : J \supseteq P' \supseteq I\}$ *and suppose* $\Omega \neq \phi$. *Then* Ω *has a minimal member with respect to inclusion.*

Proof. (Using Zorn's Lemma). Order Ω by reverse inclusion, i.e., for $P_1, P_2 \in \Omega$, $P_1 \leq P_2$ if $P_2 \subseteq P_1$. Clearly, $(\Omega,\leq)$ is a partially ordered set. Let $\lambda = \{P_\alpha : \alpha \in \Delta\}$ be any chain of Ω. Then $P_\alpha \subseteq P_\gamma$ or $P_\gamma \subseteq P_\alpha$ for all $\alpha, \gamma \in \Delta$. Let $P = \bigcap_{\alpha\in\Delta} P_\alpha$. Then clearly, $P \in \Omega$ and P is an upper bound of λ. Therefore, by Zorn's lemma, there exists at least one maximal element M in Ω. Cleary M is a graded prime ideal of (R,G) such that J$\supseteq$M$\supseteq$I . If J $\supseteq$M $\supseteq$q $\supseteq$ I, then q $\in \Omega$. But M is maximal element in Ω and M $\leq$ q which implies M = q. Thus M is a minimal graded prime ideal of (R,G) such that J$\supseteq$M$\supseteq$I with respect to inclusion. □

Corollary 2.3 *Let I be a graded ideal of (R,G) and P$\in$GX such that P$\supseteq$I. Then*

(1) V(I) has at least one minimal member with respect to inclusion. Such a minimal member is called a minimal graded prime ideal of I.

(2) $\Lambda = \{ P' \in GX : P \supseteq P' \supseteq I \}$ *has a minimal member with respect to inclusion.*

Proof. (1) In the proof of Proposition 2.2 take J = R, then the result is clear.

The set of minimal graded prime ideals of I is denoted by G-Min(I).

(2) In the proof of Proposition 2.2 take J = P, then the result is clear. □

Note that a minimal member of Λ is a minimal graded prime ideal of I, and so deduce that there exists a minimal graded prime ideal P'' of I with $P'' \subseteq$P.

Definition 2.4 *Let I be a proper graded ideal of (R,G). A graded primary G-decomposition of I is an intersection of finitely many graded primary ideals of (R, G). Such a graded primary G-decomposition* $I = Q_1 \cap \ldots \cap Q_n$ *with* $Gr(Q_i) = P_i$ *for i = 1, . . . , n of I is said to be a minimal graded primary G-decomposition of I precisely when*

(i) $p_1, \ldots, p_n$ *are different graded prime ideals of R, and*

(ii) $Q_j \not\supseteq \bigcap_{\substack{i=1 \\ j \neq i}}^{n} Q_i$ *for all j = 1, . . . , n.*

We say I is G-decomposable graded ideal of (R,G) precisely when it has a graded primary G-decomposition.

Remark 2.5 *Every G-decomposable graded ideal of (R,G) has a minimal graded primary G-decomposition .*

Proposition 2.6 *Let I be a G-decomposable graded ideal of (R,G) such that* $I = Q_1 \cap \ldots \cap Q_n$ *where* $Gr(Q_i) = P_i$ *for i = 1, . . . , n, is a minimal graded primary G-decomposition of I. Let P be any graded prime ideal of (R,G). Then the following statements are equivalent.*

(1) $P = P_i$ for some i with $1 \leq i \leq n$.

(2) There exists a$\in$h(R) such that (I:a) is graded G-P-primary.

(3) There exists a$\in$h(R) such that Gr((I:a)) = P.

Proof. (1) implies (2): Suppose P = P_i for some i with $1 \leq i \leq n$. Since the graded primary G-decomposition I = $\bigcap_{j=1}^{n} Q_j$ is minimal, there exists $a_i \in \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Q_j \cap h(R) - Q_i$

and hence (I:a_i) = $(\bigcap_{j=1}^{n} Q_j$:$a_i)$ = $\bigcap_{j=1}^{n} (Q_j : a_i)$. By Proposition 1.15, ($Q_j$:$a_i$) = R for $j \neq i$ $(1 \leq j \leq n)$. While (Q_i:a_i) is graded G-P_i-primary. Since P = P_i, it follows that (I:a_i) is graded G-P-primary.

(2) implies (3): Clear.

(3) implies (1): Suppose P = Gr((I:a)) for some a$\in$h(R). Then (I:a) = $(\bigcap_{i=1}^{n} Q_i$:a) = $\bigcap_{i=1}^{n} (Q_i : a)$. By Proposition 1.15, ($Q_i$:a) = R if a$\in Q_i$, while ($Q_i$:a) is graded G-$P_i$-primary if a$\notin Q_i$. Hence, P = Gr((I:a)) = $\bigcap_{i=1}^{n} Gr(Q_i : a)$ = $\bigcap_{\substack{i=1 \\ a \notin Q}}^{n} P_i$ and by Proposition 1.4, P = $P_i$for some i. $\square$

The next Corollary shows that the number of terms appearing in a minimal graded primary G-decomposition of I is independent of the choice of minimal graded primary G-decomposition.

Corollary 2.7 *Let I be a G-decomposable graded ideal of (R,G). Suppose $I = Q_1 \cap \ldots \cap Q_n$, where $Gr(Q_i) = P_i$ for i = 1, ... ,n and $I = Q'_1 \cap \ldots \cap Q'_{n'}$, where $Gr(Q'_i) = P'_i$ for i = 1, ... ,n', are two minimal graded primary G-decompositions of I. Then $n = n'$ and $\{P_1, \ldots, P_n\} = \{P'_1, \ldots, P'_{n'}\}$.*

Proof. Since the graded G-decomposition is minimal, $\bigcap_{\substack{j=1 \\ j\neq i}}^{n} Q_j \nsubseteq Q_i$ for all i with $1 \leq i \leq n$. So there exists a $\in$ h(R)$\cap \bigcap_{\substack{j=1 \\ j\neq i}}^{n} Q_j$ and a $\notin$ Q_i. Since a $\in$ Q_j for all j $\neq$ i, (Q_j:a) = R for all j $\neq$ i. Hence (I:a) = ($\bigcap_{j=1}^{n} Q_j$:a) = $\bigcap_{j=1}^{n} (Q_j : a)$ = $\bigcap_{\substack{j=1 \\ j\neq i}}^{n} (Q_j : a)(Q_i : a)$ = (Q_i:a). Thus Gr((I:a)) = Gr((Q_i:a)) = P_i. Also, I = $Q'_1\cap \ldots \cap Q'_{n'}$ is a minimal graded primary G-decomposition of I and P = P_i satisfies Gr((I:a)) = P = P_i for some a$\in$h(R). By Proposition 2.6, there exists j$\in\{1, \ldots, n'\}$ such that P'_j = P = P_i. Therefore, $\{P_1, \ldots, P_n\} \subseteq \{P'_1, \ldots, P'_{n'}\}$. □

The other inclusion is similar and hence **n** $= n'$

Definition 2.8 *Let I be a G-decomposable graded ideal of (R,G). Let I = $Q_1\cap \ldots \cap Q_n$, where Gr(Q_i) = P_i for i = 1, ..., n, be a minimal graded primary G-decomposition of I. Then the set $\{P_1, \ldots, P_n\}$ (which is independent of the choice of minimal graded primary G-decomposition of I by corollary 2.7) is called the set of associated graded prime ideals of I and denoted by G-$\underset{R}{ass}(I)$.*

Lemma 2.9 *Let I = $Q_1\cap \ldots \cap Q_n$ such that Gr(Q_i) = P_i for i = 1, ..., n be a minimal graded primary G-decomposition of I. If P$\in$V(I), then there exists $P' \in$ G-$\underset{R}{ass}(I)$ such that $P' \subseteq$P.*

Proof. Let P$\in$V(I). Then I$\subseteq$P and hence Gr(I) $\subseteq$ Gr(P) = P. Also, Gr(I) = Gr($Q_1 \cap \ldots \cap Q_n$) = Gr(Q_1)$\cap \ldots \cap$Gr(Q_n) = $\bigcap_{i=1}^{n} p_i \subseteq$P. By Proposition 1.4, $P_j \subset$ P for some j with $1 \leq j \leq n$. □

Proposition 2.10 *Let I be a G-decomposable graded ideal of (R,G), and let $P \in GX$. Then P is a minimal graded prime ideal of I iff P is a minimal member of G-$\underset{R}{ass}(I)$, with respect to inclusion. In particular, all the minimal graded prime ideals of I belongs to G-$\underset{R}{ass}(I)$.*

Proof. Assume that P is a minimal graded prime ideal of I. By Lemma 2.9, P$\supseteq P'$ for some $P' \in$G-$\underset{R}{ass}(I)$. But G-$\underset{R}{ass}(I) \subseteq$V(I), and so P = P' must be a minimal member of G-$\underset{R}{ass}(I)$ with respect to inclusion .

Conversely, suppose P is a minimal member of G-$\underset{R}{ass}(I)$. Then P$\supseteq$I, and so by Corollary 2.3, there exists a minimal graded prime ideal P' of I such that P$\supseteq P' \supseteq$I. Hence by Lemma 2.9, there exists $P'' \in$ G-$\underset{R}{ass}(I)$ such that $P' \supseteq P''$. But then P$\supseteq P' \supseteq P''$, and since P is a minimal member of G-$\underset{R}{ass}(I)$, P = $P' = P''$. Therefore, P = P' is a minimal graded prime ideal of I. □

Definition 2.11 *Let I be a G-decomposable graded ideal of (R,G). By Proposition 2.10, the minimal members of G-$\underset{R}{ass}(I)$ are precisely the minimal graded prime ideals of I. These graded prime ideals are called the minimal or G-isolated graded primes of I. The remaining associated graded primes of I, i.e, the associated graded primes of I which are not minimal, are called the G-embedded graded primes of I.*

The next Proposition shows that in a minimal graded primary G-decomposition of I, the graded primary term corresponding to a G-isolated graded prime ideal of I is *uniquely determined by I and is independent of the choice of minimal graded primary G-decomposition.*

Proposition 2.12 *Let I be a G-decomposable graded ideal of (R,G), G-$\underset{R}{ass}(I)$ = $\{P_1, \ldots, P_n\}$. Let $I = Q_1 \cap \ldots \cap Q_n$ with Gr(Q_i) = P_i for i = 1, ..., n and $I = Q'_1 \cap \ldots \cap Q'_n$ with Gr(Q'_i) = P_i for i = 1, ..., n be two minimal graded primary G-decompositions of I. If P_i is a minimal graded prime ideal belonging to I, then for each i with $1 \leq i \leq n$, Q_i = Q'_i.*

Proof. If n = 1, there is nothing to prove. We therefore, suppose that $n > 1$. Let P_i be a minimal graded prime ideal belonging to I. Then there exists a∈h(R) such that a∈ $\bigcap_{\substack{j=1 \\ j \neq i}}^{n} P_j \cap h(R) \setminus P_i$(for otherwise if $\bigcap_{\substack{j=1 \\ j \neq i}}^{n} P_j \subseteq P_i$, then $P_j \subseteq P_i$ for some j∈**N** with $1 \leq j \leq n$ and j≠i contrary to the fact that P_i is a minimal graded prime ideal belonging to I). For each j = 1, ..., n with j ≠ i, there exists $t_j \in N$ such that $a^{t_j} \in Q_j$ because $a \in P_j = Gr(Q_j)$ for j = 1, ..., n with j ≠ i. Let t≥ max{ $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ }. Then $a^t \in Q_j$ for j = 1, ... ,n and j ≠ i. Also, $a^t \notin P_i$, implies by Proposition 1.15, $(Q_j{:}a^t)$ = R for j = 1, ..., n and j ≠ i and $(Q_i{:}a^t) = Q_i$.Hence $(I{:}a^t) = (\bigcap_{j=1}^{n} Q_j{:}a^t)$

$$= \bigcap_{j=1}^{n} (Q_j : a^t) = (Q_i{:}a^t) = Q_i.$$ Also, Gr(Q'_i) = P_i for i = 1, ..., n. So, in a similar way there exists m∈**N** such that $(I{:}a^m) = Q'_i$. Let s≥ max{t,m}. Then $(I : a^s) = Q_i = Q'_i$. □

Definition 2.13 *Let I be a graded ideal of (R,G). We say that I is G-irreducible iff I is proper and whenever $I = I_1 \cap I_2$ with I_1, I_2 graded ideals of (R,G), then $I = I_1$ or $I = I_2$.*

Proposition 2.14 *Let R be a G-graded ring being gr-Noetherian ring. Then every proper graded ideal of (R,G) can be expressed as an intersection of finitely many G-irreducible graded ideals of (R,G).*

Proof. Let Ω be the set of all proper graded ideals of (R,G) that can not be expressed as an intersection of finitely many G-irreducible graded ideals of (R,G). Our aim is to show that $\Omega = \phi$. Suppose that this is not the case. Then, since (R,G) is gr-Noetherian, Ω has a maximal member I with respect to inclusion. Then I itself is not G-irreducible, for we can write I = I$\cap$I and I would not be in Ω. Since I is proper, it therefore, follows that I = $I_1 \cap I_2$ for some graded ideals I_1, I_2 of (R,G) for which $I \underset{\neq}{\subset} I_1$ and $I \underset{\neq}{\subset} I_2$. Note that this implies both of I_1 and I_2 are proper graded ideals. By choice of I, we must have $I_i \notin \Omega$ for i = 1, 2. Since both I_1 and I_2 are proper, it follows that both can be expressed as intersection of finitely many G-irreducible graded ideals of (R,G). Hence I = $I_1 \cap I_2$ has the same property, and this is a contradiction. Hence $\Omega = \phi$. □

Proposition 2.15 *Let (R,G) be a gr-Noetherian ring and let I be a G-irreducible graded ideal of (R,G). Then I is graded primary.*

Proof. By definition of G-irreducible graded ideal, I is proper. Suppose a, b $\in$ h(R) such that ab $\in$ I and b $\notin$ I. Consider (I:a) $\subseteq$(I:a^2)$\subseteq \ldots \subseteq$(I:$a^n$)$\subseteq \ldots$, to be an ascending chain of graded ideals of (R,G). Since (R,G) is

gr-Noetherian, there exists n $\in$ **N** such that (I:a^n) = (I:a^{n+i}) for all i $\in$ **N**. We show I = (I+Ra^n) $\cap$ (I+Rb). It is clear that I$\subseteq$(I+Ra^n)$\cap$(I+Rb). Let r$\in$(I+Ra^n)$\cap$(I+Rb), then we can write r = g + ca^n = h + db, for some g, h $\in$ I and c, d $\in$R. Thus ra = ga + ca^{n+1} = ha + dab, so that since ab, g, h$\in$I, we have ca^{n+1} = ha + dab– ga$\in$I. Hence c$\in$(I:a^{n+1}) = (I:a^n)(by choice of n). So, r = g + ca^n $\in$I implies I = (I+ Ra^n)$\cap$(I+ Rb). Now since I is G-irreducible and I$\neq$I+Rb, I = I+Ra^n which implies a^n $\in$I and hence a$\in$Gr(I). Therefore, I is graded primary. □

Corollary 2.16 *Let (R,G) be a gr-Noetherian ring. If I is proper graded ideal of (R,G), then I has a graded primary G-decomposition.*

Proof. Let I be a proper graded ideal of (R,G). Then by Proposition 2.14, I can be expressed as an intersection of finitely many G-irreducible graded ideals of (R,G) and a G-irreducible graded ideal of (R,G) is graded primary by Proposition 2.15. Therefore, I has a graded primary G-decomposition. □

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Received 07.01.2003

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