Turk J Math
28 (2004) , 299 – 305.

A New Characteristic of Möbius Transformations by Use of Apollonius Points of Pentagons

Serap Bulut, Nihal Yılmaz Özgür

Abstract

In this paper, we give a new characterization of Möbius transformations. To this end, a new concept of "Apollonius points of pentagons" is used.

Key Words: Möbius transformations; Apollonius points of pentagons

1. Introduction

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function on the complex plane $\mathbb{C}$. Let us consider the following Property 1:

Property 1. $w = f(z)$ maps circles in the z-plane onto circles in the w-plane, including straight lines among circles.

The well known principle of circle transformation (see [1], [3]) reads as follows:

Theorem 1.1 *$w = f(z)$ satisfies Property 1 iff $w = f(z)$ is a Möbius transformation.*

In [2], Haruki and Rassias introduced the definition of the Apollonius point of a triangle, afterwards in [5], Piyapong Niamsup extended this definition to the (k, l)-Apollonius point of a triangle. Then, by means of these definitions, two new invariant characteristic properties of Möbius transformations were obtained. We recall that the following two definitions from [2] and [5], respectively.

2000 *Mathematics Subject Classification:* 30C35, 32A20

Definition 1.2 *[2] Let $\triangle ABC$ be an arbitrary triangle and L be a point on the complex plane. We denote by $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = by = cz$ holds, then L is said to be an Apollonius point of $\triangle ABC$.*

Definition 1.3 *[5] Let $\triangle ABC$ be an arbitrary triangle and L be a point on $\mathbb{C}$. We denote by $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = k(by) = l(cz)$ holds, where $k, l > 0$, then L is said to be a (k, l)-Apollonius point of $\triangle ABC$.*

The purpose of this paper is to give a new characterization of Möbius transformations. To do this, we introduce the notions of an Apollonius point and of a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of a pentagon in Section 2 where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^+$. Then we give the following new property:

Property 2. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon contained in R and let its $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point L be a point of R. If we set $Z_i' = f(Z_i)$ for $1 \le i \le 5$, $L' = f(L)$ and if the five different points Z_i' $(1 \le i \le 5)$ form a pentagon (i.e., any triple of Z_i' $(1 \le i \le 5)$ are not collinear), then the point L' is also a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of $Z' = Z_1'Z_2'Z_3'Z_4'Z_5'$.

Finally we prove the following theorem as a main theorem of this paper in Section 3.

Main Theorem. The following propositions are equivalent:

(i) $w = f(z)$ is a Möbius transformation.

(ii) Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. For every quadruple $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, if L is a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of the pentagon $Z = Z_1Z_2Z_3Z_4Z_5$ contained in R, *then $f(L)$ is a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of the pentagon $Z' = Z_1'Z_2'Z_3'Z_4'Z_5'$ where $Z_i' = f(Z_i), 1 \le i \le 5$.*

2. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius Points of a Pentagon

Definition 2.1 *Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon (not necessarily simple) and L be a point on $\mathbb{C}$. If the following equality holds for $2 \le k \le 5$, then L is said to be a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of Z:*

$$|L - Z_1| \cdot |Z_2 - Z_3| \cdot |Z_4 - Z_5| = \lambda_{k-1} |L - Z_k| \cdot |Z_{k+1} - Z_{k+2}| \cdot |Z_{k+3} - Z_{k+4}|,$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^+$. In the right side of the above equation, if the values depend on k are different from 5, then we consider these values in mod(5).

Remark 2.2 For $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, this definition gives the definition of Apollonius point of an arbitrary pentagon.

Theorem 2.3 *Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon on the complex plane $\mathbb{C}$ and let the positive real numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be fixed. Then the number of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius points of Z is at most 2.*

Proof. The proof follows from the Theorem of Apollonius, [2], and from the fact that if two circles meet, including straight lines among circles, then there are at most two points of intersection. □

Example 2.4 *Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary regular pentagon. Then, the center of circumscribed circle of Z is its only Apollonius point.*

For the proof of the Theorem 2.7 we need the following definition and theorem from [6].

Definition 2.5 *A hexagon $ABCDEF$ (not necessarily simple) on the complex plane for which $\overline{AB} \cdot \overline{CD} \cdot \overline{EF} = \lambda \overline{BC} \cdot \overline{DE} \cdot \overline{FA}$ holds (where the bar denotes the length of the segment) is an λ-Apollonius hexagon where $\lambda > 0$.*

Property 3. Suppose that f is analytic and univalent on a nonempty open region Δ on the complex plane. Let $ABCDEF$ be a λ-Apollonius hexagon in Δ. If we set $Z' = f(Z)$ $(Z = ABCDEF)$, then $A'B'C'D'E'F'$ is also a λ-Apollonius hexagon.

Theorem 2.6 *$w = f(z)$ satisfies Property 3 iff $w = f(z)$ is a Möbius transformation.*

Now we can give the following theorem.

Theorem 2.7 *Property 1 implies Property 2.*

Proof. Let $w = f(z)$ satisfies Property 1. Suppose that $w = f(z)$ is analytic in a nonempty domain R on the z-plane. Then by Theorem 1.1 $w = f(z)$ is a Möbius transformation and so univalent in R. Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon contained in R and let its $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point L be a point of R. If we set $Z_i' = f(Z_i)$ for $1 \leq i \leq 5$, then by the univalency of $w = f(z)$, the five points Z_i' $(1 \leq i \leq 5)$ are different.

We now prove that if any triple of Z'_i $(1 \leq i \leq 5)$ are not collinear, then the point $L' = f(L)$ is also a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of $Z' = Z'_1 Z'_2 Z'_3 Z'_4 Z'_5$. Since L is a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of Z, by the Definition 2.1, for $k = 5$, we have

$$|L - Z_1| \cdot |Z_2 - Z_3| \cdot |Z_4 - Z_5| = \lambda_4 \, |L - Z_5| \cdot |Z_1 - Z_2| \cdot |Z_3 - Z_4| \, .$$

Therefore by the Definition 2.5, $LZ_1Z_2Z_3Z_4Z_5$ is a λ_4-Apollonius hexagon. By the Theorem 2.6 and [6], $L'Z'_1Z'_2Z'_3Z'_4Z'_5$ is a λ_4-Apollonius hexagon. Hence we obtain

$$|L' - Z'_1| \cdot |Z'_2 - Z'_3| \cdot |Z'_4 - Z'_5| = \lambda_4 \, |L' - Z'_5| \cdot |Z'_1 - Z'_2| \cdot |Z'_3 - Z'_4| \, . \tag{1}$$

Similarly, we have

$$\lambda_4 \, |L' - Z'_5| \cdot |Z'_1 - Z'_2| \cdot |Z'_3 - Z'_4| = \lambda_3 \, |L' - Z'_4| \cdot |Z'_5 - Z'_1| \cdot |Z'_2 - Z'_3| \, , \tag{2}$$

$$\lambda_3 \, |L' - Z'_4| \cdot |Z'_5 - Z'_1| \cdot |Z'_2 - Z'_3| = \lambda_2 \, |L' - Z'_3| \cdot |Z'_4 - Z'_5| \cdot |Z'_1 - Z'_2| \, , \tag{3}$$

$$\lambda_2 \, |L' - Z'_3| \cdot |Z'_4 - Z'_5| \cdot |Z'_1 - Z'_2| = \lambda_1 \, |L' - Z'_2| \cdot |Z'_3 - Z'_4| \cdot |Z'_5 - Z'_1| \, . \tag{4}$$

By (1) − (4), we obtain that the following products is equal for every $2 \leq k \leq 5$:

$$|L' - Z'_1| \cdot |Z'_2 - Z'_3| \cdot |Z'_4 - Z'_5| = \lambda_{k-1} \, |L' - Z'_k| \cdot \left| Z'_{k+1} - Z'_{k+2} \right| \cdot \left| Z'_{k+3} - Z'_{k+4} \right| .$$

By the Definition 2.1, we obtain that $L' = f(L)$ is also a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-Apollonius point of Z'. Consequently, $w = f(z)$ satisfies Property 2. □

3. Proof of the Main Theorem

Proof of the Main Theorem. Let $w = f(z)$ be a Möbius transformation. Then by Theorem 1.1, $w = f(z)$ satisfies *Property* 1. Thus by Theorem 2.7, $w = f(z)$ satisfies *Property* 2. This proves (ii).

Now assume that the function $w = f(z)$ satisfies (ii). Since $w = f(z)$ is analytic and univalent in the domain R, by a well known lemma

$$f'(z) \neq 0 \tag{5}$$

holds in R.

If x is an arbitrary fixed point of R, then by (5), we obtain

$$f'(x) \neq 0. \tag{6}$$

Let L be the point represented by x. Since $L \in R$, there exists a positive real number r such that the r closed circular neighborhood of L is contained in R. We denote this closed circular neighborhood by V.

Throughout the proof let $Z = Z_1Z_2Z_3Z_4Z_5$ denote an arbitrary regular pentagon which is contained in V and whose center is at L. Here the sense of Z_1, Z_2, Z_3, Z_4, Z_5 is counterclockwise. Since $Z = Z_1Z_2Z_3Z_4Z_5$ is a regular pentagon contained in V, we can represent Z_1, Z_2, Z_3, Z_4, Z_5 by complex numbers as

$$x + w_{k+1}y,$$

where $0 <\mid y \mid\leq r$ and $w_{k+1} = e^{\frac{i2\pi k}{5}}$, $0 \leq k \leq 4$.

Since $w = f(z)$ is univalent in R, $Z_1' = f(Z_1)$, $Z_2' = f(Z_2)$, $Z_3' = f(Z_3)$, $Z_4' = f(Z_4)$, $Z_5' = f(Z_5)$ are different points. By a property of analytic functions (see [4]) and by (6) (any triple of Z_1, Z_2, Z_3, Z_4, Z_5 are not collinear on the z-plane) there exists some sufficiently small positive real number s satisfying

$$s \leq r$$

such that any triple of Z_1', Z_2', Z_3', Z_4', Z_5' are not collinear on the w-plane for all y satisfying $0 < |y| \leq s$.

Since L is the Apollonius point of the regular pentagon Z $(0 <\mid y \mid\leq s)$ (see Example 2.4) and any triple of Z_1', Z_2', Z_3', Z_4', Z_5' are not collinear, by hypothesis $L' = f(L)$ is also an Apollonius point of $Z' = Z_1'Z_2'Z_3'Z_4'Z_5'$. Hence, by definition we obtain

$$|L' - Z_1'| \,.\, |Z_2' - Z_3'| \,.\, |Z_4' - Z_5'| \tag{7}$$

$$= |L' - Z_2'| \,.\, |Z_3' - Z_4'| \,.\, |Z_5' - Z_1'| \tag{8}$$

$$= |L' - Z_3'| \,.\, |Z_4' - Z_5'| \,.\, |Z_1' - Z_2'| \tag{9}$$

$$= |L' - Z_4'| \,.\, |Z_5' - Z_1'| \,.\, |Z_2' - Z_3'| \tag{10}$$

$$= |L' - Z_5'| \, . \, |Z_1' - Z_2'| \, . \, |Z_3' - Z_4'| \, . \tag{11}$$

Let us consider (7) and (9):

$$|L' - Z_1'| \, . \, |Z_2' - Z_3'| \, . \, |Z_4' - Z_5'| = |L' - Z_3'| \, . \, |Z_4' - Z_5'| \, . \, |Z_1' - Z_2'| \, .$$

Hence we get

$$|L' - Z_1'| \, . \, |Z_2' - Z_3'| = |L' - Z_3'| \, . \, |Z_1' - Z_2'| \, .$$

Since Z_1', Z_2', Z_3', Z_4', Z_5', L' are represented by

$$f(x + w_{k+1}y), f(x),$$

where $0 \leq k \leq 4$, respectively, from the last equation we obtain

$$\begin{aligned} & |f(x) - f(x+y)| \, . \, |f(x + w_2 y) - f(x + w_3 y)| \\ = \;& |f(x) - f(x + w_3 y)| \, . \, |f(x+y) - f(x + w_2 y)| \, , \end{aligned}$$

and so

$$\left| \frac{[f(x) - f(x+y)] \, [f(x + w_2 y) - f(x + w_3 y)]}{[f(x) - f(x + w_3 y)] \, [f(x+y) - f(x + w_2 y)]} \right| = 1.$$

By a similar way in [2], after calculations we finally get

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 = 0$$

holds for all z satisfying $f'(z) \neq 0$.

Hence, the Schwarzian derivative of f vanishes for all z satisfying $f'(z) \neq 0$. Consequently, by a well-known fact f is a Möbius transformation. □

Corollary 3.1 *This theorem gives a new proof of the only if part of Theorem* 1.1.

Proof. By hypothesis $w = f(z)$ satisfies Property 1. Hence, by the Theorem 2.7, $w = f(z)$ satisfies Property 2. Consequently, by the Main Theorem, $w = f(z)$ is a Möbius transformation. □

References

[1] H. Haruki, *A proof of the principle of circle-transformations by use of a theorem on univalent functions*, Lenseign. Math., **18** (1972), 145-146.

[2] H. Haruki and T. M. Rassias, *A new characteristic of Möbius transformations by use of Apollonius points of triangles*, J. Math. Anal. Appl., **197** (1996), 14-22.

[3] Z. Nehari, *"Conformal mapping"*, McGraw-Hill Book, New York, 1952.

[4] R. Nevanlinna and V. Paatero, *"Introduction to Complex Analysis"*, Addison-Wesley, New York, 1964.

[5] P. Niamsup, *A Note on the Characteristics of Möbius Transformations*, J. Math. Anal. Appl., **248** (2000), 203-215.

[6] N. Samaris, *A new characterization of Möbius transformation by use of 2n points*, J. Nat. Geom. **22** (2002), no. 1-2, 35–38.

Serap BULUT
Balıkesir University, Faculty of Art and
Sciences, Department of
Mathematics, 10100 Balıkesir-TURKEY
e-mail: serapbulut@balikesir.edu.tr
Nihal YILMAZ ÖZGÜR
Balıkesir University, Faculty of Art and
Sciences, Department of
Mathematics, 10100 Balıkesir, TURKEY
e-mail: nihal@balikesir.edu.tr

Received 18.06.2003