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Conjugacy Classes of Finite Subgroups of Certain Mapping Class Groups

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Abstract

We give a complete description of conjugacy classes of finite subgroups of the mapping class group of the sphere with r marked points. As a corollary we obtain a description of conjugacy classes of maximal finite subgroups of the hyperelliptic mapping class group. In particular, we prove that, for a fixed genus g, there are at most five such classes.

Key words and phrases: Hyperelliptic mapping class group, finite subgroups of mapping class group.

1. Introduction

Let $\mathcal{M}_{0,r}$ be the mapping class group of the sphere with $r \geq 3$ marked points, where we allow maps to permute the set of marked points. In [3] Gillette and Van Buskirk proved, using purely algebraic methods, the following theorem.

Theorem 1 *$\mathcal{M}_{0,r}$ contains an element of finite order n if and only if n divides one of r, $r-1$, $r-2$.*

Later on the stronger version of this theorem was obtained as a by-product of certain considerations of Harvey and Maclachlan [5].

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Theorem 2 *Every element of finite order in $\mathcal{M}_{0,r}$ is contained in a maximal cyclic subgroup of order r, $r-1$ or $r-2$, and all such subgroups of the same order are conjugate.*

In this paper we extend the above results. For every finite subgroup N of $\mathcal{M}_{0,r}$ we find a maximal finite subgroup M of $\mathcal{M}_{0,r}$ containing N. Furthermore, we give a complete description of conjugacy classes of finite subgroups of $\mathcal{M}_{0,r}$. As a corollary we obtain a description of conjugacy classes of maximal finite subgroups of the hyperelliptic mapping class group.

2. Maximal finite subgroups of $\mathcal{M}_{0,r}$

This section is devoted to the proof of the following theorem.

Theorem 3 *Finite subgroup N of $\mathcal{M}_{0,r}$ is a maximal finite subgroup of $\mathcal{M}_{0,r}$ if and only if N is isomorphic to one of the following:*

1. *the cyclic group $\mathbb{Z}_{r-1}$ if $r \neq 4$,*
2. *the dihedral group D_r*
3. *the dihedral group D_{r-2} if $r = 5$ or $r \geq 7$,*
4. *the alternating group A_4 if $r \equiv 4$ or $10 \pmod{12}$*
5. *the symmetric group S_4 if $r \equiv 0, 2, 6, 8, 12, 14, 18$ or $20 \pmod{24}$*
6. *the alternating group A_5 if $r \equiv 0, 2, 12, 20, 30, 32, 42$ or $50 \pmod{60}$*

Let N be a finite subgroup of $\mathcal{M}_{0,r}$. By the positive solution to the Nielsen realisation problem [4], we could assume that N is a group of automorphisms of the Riemann sphere $\mathbb{P}^1$. From the classification of such groups we have the following five possibilities:

1. $N \cong \mathbb{Z}_n$ is generated by a rotation of order n,
2. $N \cong D_n$ is generated by two rotations of order n and 2 respectively,
3. $N \cong S_4$ is the symmetry group of a cube (octahedron),
4. $N \cong A_5$ is the symmetry group of a dodecahedron (icosahedron), and

5. $N \cong A_4$ is the symmetry group of a tetrahedron.

In each case we shall find a maximal finite subgroup M of $\mathcal{M}_{0,r}$ containing N. What is more, from the below analysis it will be clear how to construct each of maximal subgroups listed in the above theorem.

1. As a fundamental set for the action of a cyclic group N we can choose the interior of the bigon F with vertices in the south and north poles together with one of its edges (Figure 1).

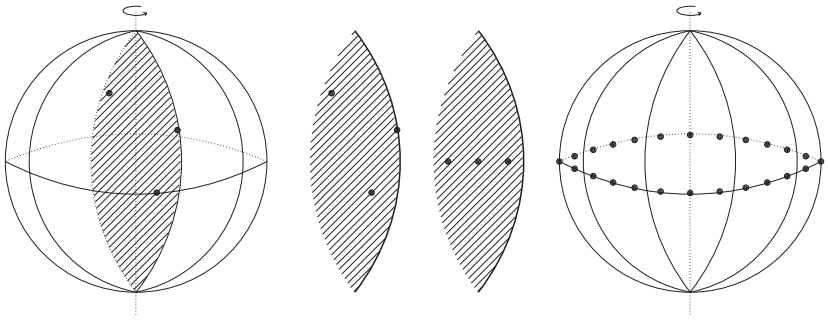

Figure 1. Fundamental set for the action of $\mathbb{Z}_n$

Let $\mathcal{P}$ be the set of these marked points P_i, which have trivial stabiliser in N and say it has s elements. Without loss of generality, we can assume that N acts on the unit sphere S^2 in $\mathbb{R}^3$ by orthogonal rotations having the pair of poles as fixed points.

We claim that we can also assume that $\mathcal{P}$ is a set of points on the equator which are vertices of a regular s-gon. To prove this, we will show how to "move" points of $\mathcal{P}$ to such symmetric position without changing the geometry of the action of N.

The canonical projection $\pi\colon S' \to S'/N$, where S' is obtained from S^2 by removing the poles, is a covering. Since $\pi(S')$ is homeomorphic to a two–punctured sphere, we could find a homeomorphism $f\colon \pi(S') \to \pi(S')$ isotopic to the identity which maps $\pi(\mathcal{P})$ onto $\pi(\mathcal{P}')$, where $\mathcal{P}'$ is a set of s points on the equator which are vertices of a regular s-gon. Now f has a lifting and if we extend it to $\widetilde{f}\colon S^2 \to S^2$ we see that $\widetilde{f}N\widetilde{f}^{-1} = N$ acts on $\widetilde{f}(S^2) = S^2$ in such a way that $\mathcal{P}'$ is the set of marked points (Figure 1). Call this model for the action of N *regular*.

We would like to point out that although two topological models for the action of N (with the set of marked points with trivial stabiliser equal to $\mathcal{P}$ and $\mathcal{P}'$ respectively) are models for the action of the same subgroup N of $\mathcal{M}_{0,r}$, they could correspond to non isomorphic realisations of N as an automorphism group of punctured Riemann sphere.

Now, using regular model for the action of N, it is obvious that N is contained in the cyclic group N' of order s.

If $s = r - 2$ or $s = r$ (i.e. both poles are marked or not), N' is contained in the dihedral group N'' generated by N' and a half turn leaving the equator invariant; we fall into the second case. Therefore we could assume that $s = r - 1$. To complete the case it is enough to prove that for $r \neq 4$, N' is a maximal subgroup of $\mathcal{M}_{0,r}$.

If a group $\mathbb{Z}_n$ acts on the sphere with r marked points then at least one of them has trivial stabiliser (since $r \geq 3$) and its orbit has length n, so $n \leq r$. This proves that the action of $\mathbb{Z}_{r-1}$ cannot be extended to the action of any cyclic group. Similarly, if we have the action of a dihedral group D_n then the stabiliser of at least one of the marked points has order one or two, so $n \leq r$. What is more, there is no action by the group D_{r-1}.

The only cyclic subgroups of A_4, S_4, A_5 are, respectively, $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$, so we could restrict ourselves to the case $r \leq 6$. The maximal order of a stabiliser in A_4 is 3, so the minimal length of an orbit is 4 and we see that the action of A_4 is possible only for $r \geq 4$. Similarly, actions of S_4 and A_5 are possible only for $r \geq 6$ and $r \geq 12$, respectively. Therefore the possible extension of the action of $\mathbb{Z}_{r-1}$ could exits only for $r = 4$. Conversely if $r = 4$, using the above technique of obtaining regular action, we could extend action of $\mathbb{Z}_3$ to the action of A_4.

2. If N is a dihedral group, then using similar methods such as in the case of cyclic groups we could construct a regular model for the action of N, i.e. such that the action of the canonical cyclic subgroup K of N is regular and N is generated by K and a half turn, leaving the equator invariant.

Now, it is clear that, depending on whether the poles are marked or not, this group is contained in D_{r-2} or D_r. Using similar arguments like in the case of a cyclic group, we prove that D_r for $r \geq 3$ and D_{r-2} for $r = 5$ or $r \geq 7$ are maximal. If $r = 3$ then D_{r-2} is a cyclic group and if $r = 4$ or $r = 6$ then we could extend the action of D_{r-2} to the action of D_4 and S_4 respectively.

3. If N is the group of orientation preserving symmetries of a cube, then we have three orbits of points with nontrivial stabiliser in N: the centres of faces, the centres of edges

Table 1. Possible values of k

	F	E	V	cube	dodecahedron	tetrahedron
(a)	−	−	−	$r/24$	$r/60$	$r/12$
(b)	+	−	−	$(r-6)/24$	$(r-12)/60$	$(r-4)/12$
(c)	−	+	−	$(r-12)/24$	$(r-30)/60$	$(r-6)/12$
(d)	+	+	−	$(r-18)/24$	$(r-42)/60$	$(r-10)/12$
(e)	−	−	+	$(r-8)/24$	$(r-20)/60$	$(r-4)/12$
(f)	+	−	+	$(r-14)/24$	$(r-32)/60$	$(r-8)/12$
(g)	−	+	+	$(r-20)/24$	$(r-50)/60$	$(r-10)/12$
(h)	+	+	+	$(r-26)/24$	$(r-62)/60$	$(r-14)/12$

and the vertices with stabilisers of order $4, 2$ and 3, respectively. Since the length of an orbit is equal to the index of the stabiliser, these orbits have lengths $6, 12$ and 8, respectively.

Let k be the number of marked points in a fixed fundamental set F, which have trivial stabiliser. Suppose that centres of faces, edges and vertices are marked points. Then counting lengths of orbits, we have $r = 24k + 8 + 12 + 6$, i.e. $k = \frac{r-26}{24}$. In similar way, according to whether centres of faces, edges or vertices are marked points or not, we obtain other possible values of k (Table 1).

Observe that for the fixed number r of distinguished points, only one of the above cases could occur. Since there is no monomorphism of S_4 into $\mathbb{Z}_n, D_n, A_4, A_5$, every such group is maximal.

4. The case of the group of orientation preserving symmetries of a dodecahedron is fully analogous to the case of a cube and we obtain also eight possibilities (Table 1).

5. If N is the group of rotations of a regular tetrahedron, then by the same reasons as before, we have eight possibilities (Table 1).

Observe that, since the centres of faces are vertices of the dual tetrahedron, it is not difficult to show that groups in cases (b), (e) and (d), (g) are conjugate.

However the problem of maximality here is more involved due to the fact that there are natural monomorphisms of A_4 into S_4 and A_5 induced by embeddings of a tetrahedron into a cube and dodecahedron (Figure 2).

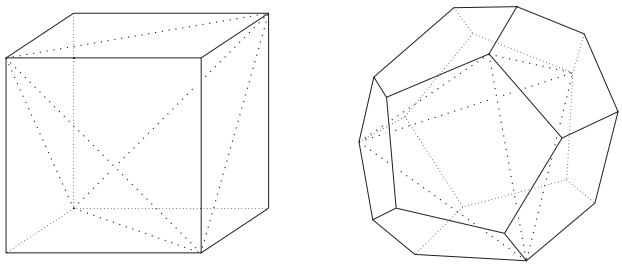

Figure 2. Embeddings of a tetrahedron

Suppose that N has an extension $N' \cong S_4$. Since S_4 contains only one subgroup isomorphic to A_4, if we realise N' as the group of rotations of a cube, we see that such extension could exist only when vertices and centres of faces of the tetrahedron (defined as points of S^2 with stabiliser in N of order 3) are of the same type (marked or not), because after extension they fall into one orbit. In particular, in cases (b), (d), (e), (g) such extension could not exist.

The same conclusion is true for the extension of N to the action of A_5; however now one has to be more careful, because A_5 contains five subgroups isomorphic to A_4.

Conversely, using the same method as in the case of a cyclic group, it is easy to obtain in cases (a), (c), (f), (h) extension of N to the action of S_4.

3. Conjugacy classes of finite subgroups of $\mathcal{M}_{0,r}$

Let N be a finite subgroup of $\mathcal{M}_{0,r}$. From the previous section, it follows that N is isomorphic either to the cyclic group $\mathbb{Z}_n$ or to the dihedral group D_n or to the symmetry group of a cube (octahedron) S_4 or the symmetry group of a dodecahedron (icosahedron) A_5 or else the symmetry group of a tetrahedron A_4. We are now ready to extend this description to conjugacy classes.

Theorem 4 *The set of conjugacy classes of subgroups isomorphic to N has exactly two elements if $N \cong \mathbb{Z}_2$ with r even or $N \cong D_n$ with $2n|r$ or $2n|r-2$ and one element for the remaining cases.*

A necessary condition for two subgroups N_1, N_2 of $\mathcal{M}_{0,r}$ to be conjugate is that corresponding orbits of points with nontrivial stabiliser are of the same type, i.e. they are marked or not. Counting for each possible type of the action the number of points of a fundamental set with trivial stabiliser one can see that for, most cases, different actions live on spheres with different numbers of distinguished points. The only exceptions are those of the first part of the theorem.

If r is even, then according to whether poles are marked or not, we have two non-conjugate actions of $\mathbb{Z}_2$. Now suppose $2n|r$. If we have an action of D_n then there are two possibilities to distribute r marked points: the first is that all marked points form one orbit of length $2n$, and the second when marked points form two orbits of order n. This two possibilities give two non-conjugate actions of D_n. Similar situation holds for $2n|r-2$.

Conversely, let N_1 and N_2 be two isomorphic subgroups of $\mathcal{M}_{0,r}$ with the same structure of orbits of points with nontrivial stabiliser. From the previous section it follows that there exist homeomorphisms $f_i\colon X \to X_i$ for $i = 1, 2$, such that $N_i' = f_i N_i f_i^{-1}$ acts on the sphere X_i in a "regular" way (i.e. $X_1 = X_2 = S^2$, sets of marked points coincide, both groups act by orthogonal rotations and as a groups of isometries of S^2, $N_1 = N_2$). Let $\varphi\colon X_1 \to X_2$ be the identity map. Observe that φ is the identity as a map of S^2, though as an element of $\mathcal{M}_{0,r}$ it could permute marked points. So for $\psi = f_1^{-1}\varphi^{-1}f_2$ we have

$$\psi N_2 \psi^{-1} = f_1^{-1}\varphi^{-1}f_2 N_2 f_2^{-1}\varphi f_1 = f_1^{-1}\varphi^{-1}N_2'\varphi f_1 = f_1^{-1}N_1' f_1 = N_1.$$

Cororally 5 *Two maximal finite subgroups of $\mathcal{M}_{0,r}$ are conjugate if and only if they are isomorphic.*

4. Conjugacy classes of maximal finite subgroups of $\mathcal{M}_g^h$

Suppose that a closed, connected orientable surface T_g of genus $g \geq 2$ is embedded in $\mathbb{R}^3$ in such a manner that it is invariant under the half turn ρ about the y-axis (figure 3).

The hyperelliptic mapping class group $\mathcal{M}_g^h$ is defined to be the centraliser of ρ in the mapping class group of T_g. By [1] the quotient $\mathcal{M}_g^h/\langle\rho\rangle$ is isomorphic to the mapping class group $\mathcal{M}_{0,2g+2}$ of a sphere with $2g+2$ marked points $P_1, \ldots, P_{2g+2}$. For simplicity we will identify $\mathcal{M}_g^h/\langle\rho\rangle$ with $\mathcal{M}_{0,2g+2}$. Let $\pi\colon \mathcal{M}_g^h \to \mathcal{M}_{0,2g+2}$ be the canonical projection

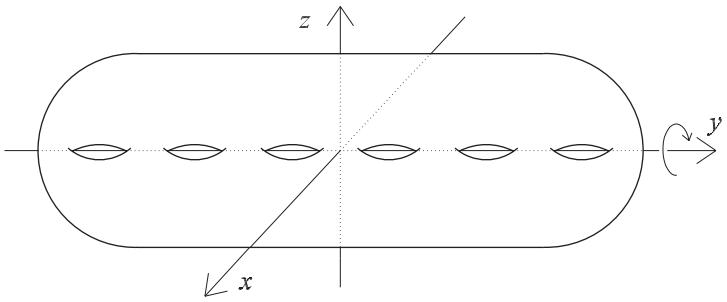

Figure 3. Hyperelliptic involution ρ

and $\mathcal{K}, \mathcal{L}$ be sets of maximal finite subgroups of $\mathcal{M}_g^h$ and $\mathcal{M}_{0,r}$ respectively.

Lemma 6 *The map $\varphi\colon \mathcal{K} \to \mathcal{L}$ given by $\varphi(K) = \pi(K) = K/\langle\rho\rangle$ is a bijection.*

Proof. Observe that since ρ is central in $\mathcal{M}_g^h$, it is contained in each maximal finite subgroup of $\mathcal{M}_g^h$, so φ is well defined. Clearly φ is surjective, so let us prove that it is also injective. If $\pi(K) = L$ for $K \in \mathcal{K}$ and $L \in \mathcal{L}$ then $K \leq \pi^{-1}(L)$. Comparing orders, we obtain $K = \pi^{-1}(L)$. □

Lemma 7 *Under the above assumptions, φ induces a bijection between conjugacy classes of subgroups in $\mathcal{K}$ and $\mathcal{L}$.*

Proof. Obviously conjugate elements of $\mathcal{K}$ are mapped onto conjugated elements of $\mathcal{L}$. The converse statement, follows from bijectivity of φ. □

Therefore, using above lemmas and known description of liftings of finite subgroups of $\mathcal{M}_{0,2g+2}$ to $\mathcal{M}_g^h$ [2], we could reformulate theorems of previous sections in terms of the group $\mathcal{M}_g^h$.

Theorem 8 *Finite subgroup N of $\mathcal{M}_g^h$ is a maximal finite subgroup of $\mathcal{M}_g^h$ if and only if N is isomorphic to one of the following:*

1. $\mathbb{Z}_{4g+2}$

2. $V_{2g+2} = \langle x, y \mid x^4, y^{2g+2}, (xy)^2, (x^{-1}y)^2 \rangle$

3. $U_{2g} = \langle x, y \mid x^2, y^{4g}, xyxy^{2g+1} \rangle$ *if* $g \geq 3$

4. $\mathbb{Z}_2 \times A_4$ *if* $g \equiv 1 \pmod 6$

5. $SL(2,3) = \langle x, y \mid x^4, y^3, (xy)^3, yx^2y^{-1}x^2 \rangle$ *if* $g \equiv 4 \pmod 6$

6. $\mathbb{Z}_2 \times S_4$ *if* $g \equiv 3$ *or* $11 \pmod{12}$

7. $W_1 = \langle x, y \mid x^2, y^3, (xy)^4(yx)^4, (xy)^8 \rangle$ *if* $g \equiv 2$ *or* $6 \pmod{12}$

8. $W_2 = \langle x, y \mid x^4, y^3, yx^2y^{-1}x^2, (xy)^4 \rangle$ *if* $g \equiv 5$ *or* $9 \pmod{12}$

9. $W_3 = \langle x, y \mid x^4, y^3, (xy)^8, x^2(xy)^4 \rangle$ *if* $g \equiv 0$ *or* $8 \pmod{12}$

10. $\mathbb{Z}_2 \times A_5$ *if* $g \equiv 5, 9, 15$ *or* $29 \pmod{30}$

11. $SL(2,5) = \langle x, y \mid x^4, y^3, (xy)^5, yx^2y^{-1}x^2 \rangle$ *if* $g \equiv 0, 14, 20$ *or* $24 \pmod{30}$.

Two maximal finite subgroups of $\mathcal{M}_g^h$ *are conjugate within* $\mathcal{M}_g^h$ *if and only if they are isomorphic.*

Groups in the first part of the theorem are respectively liftings of (1) $\mathbb{Z}_{r-1}$, (2) D_r, (3) D_{r-2}, (4)–(5) A_4, (6)–(9) S_4, (10)–(11) A_5, where $r = 2g + 2$. The reason for the assumption $g \geq 3$ in case (3) is that for $g = 2$ the underlying group D_{r-2} is not maximal.

Cororally 9 *If* $g \equiv 0, 5, 9, 14, 15, 20, 24$ *or* $29 \pmod{30}$ *then* $\mathcal{M}_g^h$ *contains exactly five conjugacy classes of maximal finite subgroups. For* $g = 2$ *there are three and for other values of* g *there are exactly four such classes.*

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