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On Orthogonal Generalized Derivations of Semiprime Rings

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Abstract

In this paper, we present some results concerning two generalized derivations on a semiprime ring. These results are a generalization of results of M. Brešar and J. Vukman in [2], which are related to a theorem of E. Posner for the product of derivations on a prime ring.

Key words and phrases: derivation, orthogonal derivations, generalized derivation, orthogonal generalized derivations, prime ring, semiprime ring.

1. Introduction

Throughout R will represent an associative ring. R is said to be *2-torsion free* if $2x = 0$, $x \in R$ implies $x = 0$. Recall that R is *prime* if $xRy = 0$ implies $x = 0$ or $y = 0$, and R is *semiprime* if $xRx = 0$ implies $x = 0$. An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In [1], Brešar defined the following notion. An additive mapping $D : R \to R$ is said to be a *generalized derivation* if there exists a derivation $d : R \to R$ such that

$$D(xy) = D(x)y + xd(y) \qquad \text{for all } x, y \in R.$$

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Hence the concept of a generalized derivation covers both the concepts of a derivation and of a left multiplier (i.e., an additive map f satisfying $f(xy) = f(x)y$ for all $x, y \in R$). This notion is found in P. Ribenboim [8], where some module structure of these higher generalized derivations was treated. Other properties of generalized derivations were given by B. Hvala [3], T.K. Lee [4] and the second author [5], [6] and [7]. We note that for a semiprime ring R, if D is a function from R to R and $d : R \to R$ is an additive mapping such that $D(xy) = D(x)y + xd(y)$ for all $x, y \in R$, then D is uniquely determined by d and moreover d must be a derivation by [[1], Remark 1].

Let $d : R \to R$ be a derivation and a an element of R. We denote a generalized derivation $D : R \to R$ determined by a derivation d of R by (D, d) and a_l the left multiplication by a. Then it is easy to see that $(d + a_l, d)$ is a generalized derivation, and so there are many examples generalized derivations.

Two additive maps $d, g : R \to R$ are called *orthogonal* if

$$d(x)Rg(y) = 0 = g(y)Rd(x) \qquad \text{for all } x, y \in R.$$

In [2] Brešar and Vukman introduced the notion of orthogonality for a pair d, g of derivations on a semiprime ring, and they gave several necessary and sufficient conditions for d and g to be orthogonal. In this paper, our aim is to extend their results to orthogonal generalized derivations.

For a semiprime ring R and an ideal U of R, it is well-known that the left and right annihilators of U in R coincide. We denote the annihilator of U by $Ann(U)$. Note that $U \cap Ann(U) = 0$ and $U \oplus Ann(U)$ is an essential ideal of R.

Throughout this paper we assume that R is a 2-torsion free semiprime ring unless stated otherwise.

2. Orthogonal generalized derivations

Now we introduce the notation of orthogonal generalized derivations as follows.

Definition. Two generalized derivations (D, d) and (G, g) of R are called *orthogonal* if

$$D(x)RG(y) = 0 = G(y)RD(x) \quad \text{for all } x, y \in R.$$

Example. Let d and g be two derivations of R. We set $S = R \oplus R$. Then the maps d_1 and g_2 from S to S which are defined by

$$d_1((x,y)) = (d(x),0) \quad \text{and} \quad g_2((x,y)) = (0,g(y)) \text{ for all } x,y \in R$$

are derivations of S. Moreover, if (D,d) and (G,g) are generalized derivations of R, and we define

$$D_1((x,y)) = (D(x),0) \quad \text{and} \quad G_2((x,y)) = (0,G(y)) \text{for all } x,y \in R,$$

then (D_1,d_1) and (G_2,g_2) are generalized derivations of S such that D_1 and G_2 are orthogonal. Therefore there are many pairs of generalized derivations which are orthogonal.

The main purpose of this paper is to prove the following result, which corresponds to [[2], Theorem 1].

Theorem 1 *Let (D,d) and (G,g) be generalized derivations of R. Then the following conditions are equivalent.*

(i) (D,d) and (G,g) are orthogonal.

(ii) For all $x,y \in R$, the following relations hold.

(a) $D(x)G(y) + G(x)D(y) = 0$.

(b) $d(x)G(y) + g(x)D(y) = 0$.

(iii) $D(x)G(y) = d(x)G(y) = 0$ for all $x,y \in R$.

(iv) $D(x)G(y) = 0$ for all $x,y \in R$ and $dG = dg = 0$.

(v)$(DG,dg)$is a generalized derivation and $D(x)G(y) = 0$ for all $x,y \in R$.

(vi)There exist ideals U and V of R such that:

(a) $U \cap V = 0$ and $U \oplus V$ is an essential ideal of R.

(b) $D(R), d(R) \subset U$ and $G(R), g(R) \subseteq V$.

(c) $D(V) = d(V) = 0$ and $G(U) = g(U) = 0$.

For the proof of Theorem 1 we need some lemmas.

Lemma 1 ([2], Lemma 1) *Let R be a 2-torsion free semiprime ring and a,b elements of R. Then the following conditions are equivalent.*

(i) $axb = 0$ for all $x \in R$.

(ii) $bxa = 0$ *for all* $x \in R$.

(iii) $axb + bxa = 0$ *for all* $x \in R$.

If one of the three conditions is fulfilled, then $ab = ba = 0$.

Lemma 2 *If* (D, d) *and* (G, g) *are orthogonal generalized derivations of* R*, then the following relations hold.*

(i) $D(x)G(y) = G(x)D(y) = 0$*, hence* $D(x)G(y) + G(x)D(y) = 0$ *for all* $x, y \in R$.

(ii) d *and* G *are orthogonal, and* $d(x)G(y) = G(y)d(x) = 0$ *for all* $x, y \in R$.

(iii) g *and* D *are orthogonal, and* $g(x)D(y) = D(y)g(x) = 0$ *for all* $x, y \in R$.

(iv) d *and* g *are orthogonal derivations.*

(v) $dG = Gd = 0$ *and* $gD = Dg = 0$.

(vi) $DG = GD = 0$.

Proof. (i) By the hypothesis we have $D(x)zG(y) = 0$ for all $x, y, z \in R$. Hence we get $D(x)G(y) = G(x)D(y) = 0$ for all $x, y \in R$ by Lemma 1.

(ii), (iii) By $D(x)G(y) = 0$ and $D(x)zG(y) = 0$ for all $x, y, z \in R$, we get

$$0 = D(rx)G(y) = (D(r)x + rd(x))G(y) = rd(x))G(y) \text{ for all } r, x, y \in R.$$

Since R is semiprime, $d(x)G(y) = 0$ for all $x, y \in R$. Then we have

$$d(xr)G(y) = (d(x)r + xd(r))G(y) = d(x)rG(y) = 0 \text{ for all } r, x, y \in R.$$

Therefore by Lemma 1, we obtain $G(y)d(x) = 0$ for all $x, y \in R$, which shows (ii). The proof of (iii) is similar.

(iv) We have

$$0 = D(xz)G(yw) = (D(x)z + xd(z))(G(y)w + yg(w)) \text{ for all } x, y, z, w \in R$$

by (i). Thus we get $xd(z)yg(w) = 0$ for all $x, y, z, w \in R$ by (ii) and (iii). Since R is semiprime, we see that $d(z)yg(w) = 0$ for all $y, z, w \in R$, which shows that d and g are orthogonal.

(v), (vi) Using (ii) and (iv), we have

$$0 = G(d(x)zG(y)) = Gd(x)zG(y) + d(x)g(zG(y)) = Gd(x)zG(y)$$

for all $x, y, z \in R$. Replacing y by $d(x)$ in the above relation, we get $Gd = 0$ by the semiprimeness of R. Similarly, since each of $d(G(x)zd(y)) = 0, D(g(x)zD(y)) = 0, g(D(x)zg(y)) = 0$ and $G(D(x)zG(y)) = 0$ holds for all $x, y, z \in R$, we have $dG = Dg = gD = DG = GD = 0$, respectively. □

By Lemma 2, [[2], Theorem 1, (v)] is partially extended as follows.

Corollary 1 *If (D, d) and (G, g) are orthogonal generalized derivations of R, then dg is a derivation and $(DG, dg) = (0, 0)$ is a generalized derivation.*

In [2] it was shown that if d, g are derivations and dg is a derivation, then d and g are orthogonal. This result does not extend to generalized derivations; we give an example later.

Lemma 3 *Let R be a semiprime ring. Let U be an ideal of R and $V = Ann(U)$. If (D, d) is a generalized derivation of R such that $D(R), d(R) \subset U$, then $D(V) = d(V) = 0$.*

Proof. If $x \in V$, then $xU = 0$. By the hypothesis we have $d(U) \subset U$. Hence $0 = D(x)y + xd(y) = D(x)y$ for all $y \in U$. Since $D(x) \in U \cap V$ and R is semiprime, we get $D(x) = 0$ for all $x \in V$. Similarly, we obtain $d(V) = 0$. □

We now have enough information to prove Theorem 1.

Proof of Theorem 1 $(i) \Rightarrow (ii), (iii), (iv)$ and (v) are proved by Lemma 2 and Corollary 1.

$(ii) \Rightarrow (i)$ If we take xz instead of x in (a), then by (b) we have

$$0 = D(x)zG(y) + G(x)zD(y) \text{ for all } x, y, z \in R.$$

Thus by Lemma 1 we arrive at $D(x)RG(y) = G(y)RD(x) = 0$ for all $x, y \in R$.

$(iii) \Rightarrow (i)$ Since $0 = (D(x)z + xd(z))G(y) = D(x)zG(y)$ for all $x, y, z \in R$, we get the result by Lemma 1.

$(iv) \Rightarrow (i)$ Since $dg = 0$, we have $0 = dG(xy) = dG(x)y + G(x)d(y) + d(x)g(y) + xdg(y) = G(x)d(y)$ for all $x, y \in R$ by [[2], Theorem 1]. Thus $0 = G(x)zd(y)+xg(z)d(y) = G(x)zd(y)$ for all $x, y, z \in R$. Hence we get $d(y)G(x) = 0$ for all $x, y \in R$ by Lemma 1. Then (i) follows from (iii).

$(v) \Rightarrow (i)$ Since (DG, dg) is a generalized derivation, dg is a derivation. Then we obtain

$$DG(xy) = DG(x)y + xdg(y) \text{ for all } x, y \in R,$$

and we have

$$DG(xy) = D(G(x)y + xg(y)) = DG(x)y + G(x)d(y) + D(x)g(y) + xdg(y)$$

for all $x, y \in R$. Comparing the last two relations, we get $G(x)d(y) + D(x)g(y) = 0$ for all $x, y \in R$. Since $D(x)G(y) = 0$ for all $x, y \in R$, we get

$$0 = D(x)G(yz) = D(x)G(y)z + D(x)yg(z) = D(x)yg(z) \text{ for all } x, y, z \in R.$$

Using (v), we have

$$0 = D(x)G(yz) = D(x)G(y)z + D(x)yg(z) = D(x)yg(z) \text{ for all } x, y, z \in R.$$

Hence we obtain $g(z)D(x) = 0$ for all $x, z \in R$. Replacing z by yz in the last relation we get $g(y)zD(x) = 0$ for all $x, y, z \in R$. Thus we have $D(x)g(y) = 0$ for all $x, y \in R$. This implies that $G(x)d(y) = 0$ for all $x, y \in R$, which shows that $d(y)G(x) = 0$ for all $x, y \in R$. Therefore by (iii), we get the result.

$(i) \Rightarrow (vi)$ Let U_0 be the ideal of R generated by $d(R) \cup D(R)$, let $Ann(U_0) = V$ and $Ann(V) = U$. By Lemma 2, we see that $D(x)G(y) = G(x)D(y) = 0, d(x)G(y) = g(x)D(y) = 0$ and $d(x)g(y) = g(y)d(x) = 0$ for all $x, y \in R$. Since $D(R), d(R) \subset U_0$ we obtain $G(R), g(R) \subset V$. Moreover by Lemma 3 and $U_0 \subset U$ we have $D(V) = d(V) = 0$ and $G(U) = g(U) = 0$. Since R is semiprime, $U \oplus V$ is an essential ideal of R, which shows (vi). □

Now we give an example of generalized derivations (D, d) and (G, g) which are not orthogonal, such that (DG, dg) is a generalized derivation.

Example. Let a and b be non-zero elements of R. Let $D = a_l$ and $G = b_l$ be left multiplications of a and b, respectively. We assume that $ab = 0$. Then $(D, 0)$ and $(G, 0)$ are non-zero generalized derivations such that $DG = 0$. If $(D, 0)$ and $(G, 0)$ are orthogonal, then by Theorem 1(iii) $aRbR = 0$. If R is a prime ring, then $a = 0$ or $b = 0$, a contradiction. If R is semiprime, then taking $a = b$, we also have a contradiction. Thus for a 2-torsion free prime ring or semiprime ring, there exist non-orthogonal generalized derivations (D, d) and (G, g) such that (DG, dg) is a generalized derivation.

3. Products of generalized derivations

In this section, we give some results for the product of two generalized derivations as follows.

Theorem 2 *Let (D, d) and (G, g) be generalized derivations of R. Then the following conditions are equivalent.*

(i) (DG, dg) is a generalized derivation.

(ii) (GD, gd) is a generalized derivation.

(iii) D and g are orthogonal, and G and d are orthogonal.

Proof. $(i) \Rightarrow (iii)$ Assume that (DG, dg) is a generalized derivation. Thus, as in the proof of the Theorem 1 $(v) \Rightarrow (i)$ we obtain

$$G(x)d(y) + D(x)g(y) = 0 \qquad \text{for all } x, y \in R.$$

Replacing y by yz in the above relation, where $z \in R$, we get

$$G(x)yd(z) + D(x)yg(z) = 0 \quad \text{for all } x, y, z \in R. \tag{$*$}$$

Since (DG, dg) is a generalized derivation, dg is a derivation. Therefore d and g are orthogonal by [[2], Theorem 1]. Thus we have

$$0 = G(x)g(z)yd(z) + D(x)g(z)yg(z) = D(x)g(z)yg(z) \text{ for all } x, y, z \in R.$$

Hence we get $D(x)g(z)RD(x)g(z) = 0$ for all $x, z \in R$. By the semiprimeness of R, we obtain $D(x)g(z) = 0$ for all $x, z \in R$. Thus $D(x)yg(z) = 0$ for all $x, y, z \in R$ and, by $(*)$ we have $G(x)yd(z) = 0$ for all $x, y, z \in R$.

(iii) $\Rightarrow$ *(i)* Since D and g are orthogonal, we get $D(x)yg(z) = 0$ for all $x, y, z \in R$. Substituting rx for x in the last relation, we arrive at $0 = D(rx)yg(z) = D(r)xyg(z) + rd(x)yg(z) = rd(x)yg(z)$ for all $r, x, y, z \in R$. Hence $d(x)yg(z) = 0$ for all $x, y, z \in R$ by the semiprimeness of R. Thus by [[2], Theorem 1], we conclude that dg is a derivation. Moreover since $D(x)yg(z) = 0$ for all $x, y, z \in R$, we also get $D(x)(g(z)RD(x))g(z) = 0$ and so $D(x)g(z) = 0$ for all $x, z \in R$ by the semiprimeness of R. Similarly, since G and d are orthogonal, we have $G(x)d(y) = 0$ for all $x, y \in R$. Thus we obtain $DG(xy) = DG(x)y + xdg(y)$ for all $x, y \in R$, which means that (DG, dg) is a generalized derivation. □

(ii) $\Leftrightarrow$ *(iii)* is proved in a similar way.

Corollary 2 *Let (D, d) and (G, g) be generalized derivations of R. If (DG, dg) is a generalized derivation, then there exist ideals U and V of R such that the following conditions hold.*

(i) $U \cap V = 0$ and $U \oplus V$ is an essential ideal of R.

(ii) $d(R) \subset U$, $d(V) = 0$, $g(R) \subset V$, $G(R) \subset V$, and $g(U) = G(U) = 0$.

Proof. Let U_0 be the ideal of R generated by the set $d(R)$. We set $V = Ann(U_0)$ and $U = Ann(V)$. Then by Lemma 3 and Theorem 2, we get D and g, G and d, and d and g are orthogonal, respectively. Since $d(R) \subset U$ and $g(R), G(R) \subset V$, then we have $g(U) = G(U) = 0$ by Lemma 3. Moreover, if $x \in V$, then $Ux = 0$ and so $Ud(x) = 0$. Thus $d(x) = 0$, which shows that $d(V) = 0$. □

Corollary 3 *Let (D, d) be a generalized derivations of R. If (D^2, d^2) is a generalized derivation, then $d = 0$.*

Proof. Since d^2 is a derivation, d and d are orthogonal by [[2], Theorem 1]. Hence we have $d(x)yd(x) = 0$ for all $x, y \in R$. Therefore by the semiprimeness of R, we get $d(R) = 0$. □

Corollary 4 *Let (D, d) be a generalized derivation of R. If $D(x)D(y) = 0$ for all $x, y \in R$, then $D = d = 0$.*

Proof. By the hypothesis we have

$$0 = D(x)D(yz) = D(x)(D(y)z + yd(z)) = D(x)yd(z) \text{ for all } x, y, z \in R.$$

Hence we see that $d(z)D(x) = 0$ for all $x, z \in R$ by Lemma 1. Replacing x by xz in the last relation, we get

$$0 = d(z)D(x)z + d(z)xd(z) = d(z)xd(z) \text{ for all } x, z \in R.$$

By the semiprimeness of R, we obtain $d = 0$. Then we have $0 = D(xy)D(y) = D(x)yD(x)$ for all $x, y \in R$ by the hypothesis. Thus we get $D = 0$ by the semiprimeness of R. □

Corollary 5 *Let R be a 2-torsion free prime ring. If generalized derivations (D, d) and (G, g) of R satisfy one of the conditions of Theorem 2, then $D = d = 0$ or $G = g = 0$.*

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