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# On near-rings with two-sided $\alpha$-derivations

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**Abstract**

In this paper, we introduce the notion of two-sided $\alpha$-derivation of a near-ring and give some generalizations of [1]. Let $N$ be a near ring. An additive mapping $f : N \to N$ is called an *$(\alpha, \beta)$-derivation* if there exist functions $\alpha, \beta : N \to N$ such that $f(xy) = f(x)\alpha(y) + \beta(x)f(y)$ for all $x, y \in N$. An additive mapping $d : N \to N$ is called a *two-sided $\alpha$-derivation* if $d$ is an $(\alpha, 1)$-derivation as well as a $(1, \alpha)$-derivation. The purpose of this paper is to prove the following two assertions: (i) Let $N$ be a semiprime near-ring, $I$ be a subset of $N$ such that $0 \in I$, $IN \subseteq I$ and $d$ be a two-sided $\alpha$-derivation of $N$. If $d$ acts as a homomorphism on $I$ or as an anti-homomorphism on $I$ under certain conditions on $\alpha$, then $d(I) = \{0\}$. (ii) Let $N$ be a prime near-ring, $I$ be a nonzero semigroup ideal of $N$, and $d$ be a $(\alpha, 1)$-derivation on N. If $d + d$ is additive on $I$, then (N,+) is abelian.

**Key words and phrases:** Prime near-ring, semiprime near-ring, $(\alpha, 1)$-derivation, $(1, \alpha)$-derivation, two-sided $\alpha$-derivation

## 1. Introduction

Throughout this paper $N$ stands for a right near-ring. An additive map $d : N \to N$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ - or equivalently (cf. [8]) that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [4], but thus for only a few papers on this subject in near-rings have been published (see [1], [2], [5] and [7]). According to [4], a near ring $N$ is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$, and semiprime

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if $xNx = \{0\}$ for $x \in N$ implies $x = 0$. A non empty subset $I$ of $N$ will be called a semigroup ideal if $IN \subseteq I$ and $NI \subseteq I$.

Let $S$ be a nonempty subset of $N$ and $d$ be a derivation of $N$. If $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in S$, then $d$ is said to act as a homomorphism or anti-homomorphism on $S$, respectively. Bell and Kappe proved [3] that if $d$ is a derivation of a semiprime ring $R$ which is either an endomorphism or anti-endomorphism, then $d = 0$. They also showed that if $d$ is a derivation of a prime ring $R$ which acts as a homomorphism on $I$, where $I$ is a nonzero right ideal, then $d = 0$ on $R$ these results were proved for near-rings in [1].

Now we introduce the notion of two-sided $\alpha$-derivation of a near-ring $N$ as follows.

An additive mapping $f : N \to N$ is called a $(\alpha, \beta)$-*derivation* if there exist functions $\alpha, \beta : N \to N$ such that $f(xy) = f(x)\alpha(y) + \beta(x)f(y)$ for all $x, y \in N$. An additive mapping $d : N \to N$ is called a *two-sided* $\alpha$-*derivation* if $d$ is an $(\alpha, 1)$-derivation as well as $(1, \alpha)$-derivation.

For $\alpha = 1$, a two-sided $\alpha$-derivation is of course just a derivation. In case $N$ is a prime ring and $d \neq 0$, Chang ([6, Theorem 1]) has shown that $\alpha$ must necessarily be a ring endomorphism. Now we give an example of a two-sided $\alpha$-derivation on a near-ring.

**Example.** Let $N = N_1 \oplus N_2$, where $N_1$ is a zero-symmetric near-ring and $N_2$ is a ring. Let $d_1$ be any map on $N_1$ and $d_2$ be a right and left $N_2$-module map on $N_2$ which is not a derivation. Define $d : N \to N$ by $d((n_1, n_2)) = (0, d_2((n_2))$ and $\alpha : N \to N$ by $\alpha((n_1, n_2)) = (d_1(n_1), 0)$. Then $d$ is a two-sided $\alpha$-derivation on $N$ but not a derivation.

## 2. The Results

We need the following lemmas.

**Lemma 1** . *Let $N$ be a prime near-ring and $I$ a nonzero semigroup ideal of $N$. If $u + v = v + u$ for all $u, v \in I$, then $(N, +)$ is abelian.*

**Proof.** By the hypothesis, we have $xu + yu = yu + xu$ for all $u \in I$ and $x, y \in N$. Then we get $(x + y - x + y)u = 0$ for all $u \in I$ and $x, y \in N$. It means that $(x + y - x - y)I = (x - y - x - y)NI = 0$. Since $I$ is a nonzero semigroup ideal we have $x + y - x - y = 0$ for all $x, y \in N$ by the primeness of $N$. Thus $(N, +)$ is abelian. $\square$

**Lemma 2** *Let $N$ be a right near-ring, $d$ a $(\alpha,1)$-derivation of $N$ and $I$ a multiplicative semigroup of $N$ which contains $0$. If $d$ acts as an anti-homomorphism on $I$ and $\alpha(0)=0$, then $x0=0$ for all $x\in I$.*

**Proof.** Since $0x=0$ for all $x\in I$ and $d$ acts as an anti-homomorphism on $I$ it is clear that $d(x)0=0$ for all $x\in I$. Taking $x0$ instead of $x$, one can obtain $d(x)\alpha(0)+x0=0$ for all $x\in I$. Thus we have $x0=0$ for all $x\in I$. □

**Lemma 3** *Let $N$ be a near-ring and $I$ be a multiplicative subsemigroup of $N$. If $d$ is a two-sided $\alpha$-derivation of $N$ such that $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y\in I$, then*

$$n(d(x)\alpha(y)+xd(y))=nd(x)\alpha(y)+nxd(y) \text{ for all } n,x,y\in I.$$

*Furthermore, if $\alpha(I)=I$, then*

$$n(d(x)y+\alpha(x)d(y))=nd(x)y+n\alpha(x)d(y) \text{ for all } n,x,y\in I.$$

A proof can be given by using a similar approach to that in the proof of [ 8, Lemma 1].

**Lemma 4** . *Let $N$ be a prime near-ring and $I$ a nonzero semigroup ideal of $N$. Let $d$ be a nonzero $(\alpha,1)$-derivation on $N$ such that $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y\in I$. If $x\in N$ and $xd(I)=\{0\}$, then $x=0$.*

**Proof.** Assume that $xd(I)=0$. Then $xd(uy)=0$ for all $y\in N,u\in I$. Hence $0=x(d(u)\alpha(y)+ud(y))=xud(y)$ for all $y\in N,u\in I$. Since I is a nonzero semigroup ideal and $d$ is nonzero, it is clear that $x=0$ by the primeness of $N$. □

**Lemma 5** *Let $N$ be a prime near-ring and $I$ a nonzero semigroup ideal of $N$ and $d$ a nonzero $(\alpha,1)$-derivation on $N$. If $d(x+y-x-y)=0$ for all $x,y\in I$, then $(x+y-x-y)d(z)=0$ for all $x,y,z\in I$.*

**Proof.** Assume that $d(x+y-x-y)=0$ for all $x,y\in I$. Let us take $yz$ and $xz$ instead of $y$ and $x$, where $z\in I$ respectively. Then $0=d((x+y-x-y)z)=d(x+y-x-y)\alpha(z)+(x+y-x-y)d(z)=(x+y-x-y)d(z)$ for all $x,y,z\in I$. □

**Lemma 6** *Let $N$ be a near-ring and $I$ a multiplicative subsemigroup of $N$. Let $d$ be a $(\alpha,1)$- derivation of $N$ such that $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y\in I$ and $\alpha(I)=I$.*

*(i) If $d$ acts as a homomorphism on $I$, then*

*$d(y)xd(y)=yxd(y)=d(y)x\alpha(y)$ for all $x,y\in I$.*

*(ii) If $d$ acts as an anti-homomorphism on $I$, then*

*$d(y)xd(y)=xyd(y)=d(y)\alpha(y)x$ for all $x,y\in I$.*

**Proof.** **(i)** Let $d$ act as a homomorphism on $I$. Then

$$d(xy)=d(x)\alpha(y)+xd(y)=d(x)d(y)\qquad \text{for all } x,y\in I. \tag{1}$$

Substituting $yx$ for $x$ in (1), we infer that

$$d(yx)\alpha(y)+yxd(y)=d(yx)d(y)=d(y)d(xy)\qquad \text{for all } x,y\in I. \tag{2}$$

By Lemma 3, $d(y)d(xy)=d(y)d(x)\alpha(y)+d(y)xd(y)=d(yx)\alpha(y)+d(y)xd(y)$. Using this relation in (2), we get $yxd(y)=d(y)xd(y)$.

Similarly, taking $yx$ instead of $y$ in (1) we obtain

$$d(x)\alpha(yx)+xd(yx)=d(x)d(yx)=d(xy)d(x)\ \text{ for all } x,y\in I. \tag{3}$$

On the other hand $d(xy)d(x)=(d(x)\alpha(y)+xd(y))d(x)=d(x)\alpha(y)d(x)+xd(y)d(x)=d(x)\alpha(y)d(x)+xd(yx)$. Using this relation in (3) we get $d(x)\alpha(yx)=d(x)\alpha(y)\alpha(x)=$

$d(x)\alpha(y)d(x)$. Since $\alpha(I)=I$ it is clear that $d(x)wd(x)=d(x)w\alpha(x)$ for all $x,w\in I$.

**(ii)** Since $d$ acts as an anti-homomorphism on $I$, we have

$$d(xy)=d(x)\alpha(y)+xd(y)=d(y)d(x) \qquad \text{for all } x,y\in I. \tag{4}$$

Taking $xy$ for $y$ in (4), we get

$$\begin{aligned} d(x)\alpha(xy)+xd(xy) &= d(xy)d(x) \\ &= (d(x)\alpha(y)+xd(y))d(x) \\ &= d(x)\alpha(y)d(x)+xd(y)d(x) \\ &= d(x)\alpha(y)d(x)+xd(xy) \qquad \text{for all } x,y\in I. \end{aligned}$$

From this relation we get $d(x)\alpha(xy)=d(x)\alpha(y)d(x)$. Since $\alpha(I)=I$, we get $d(x)\alpha(x)y=d(x)yd(x)$ for all $x,y\in I$. Similarly, taking $xy$ instead of $x$ in (4), one can prove the relation $d(y)xd(y)=xyd(y)$.

□

The following theorem is a generalization of [1, Theorem].

**Theorem 1** *Let $N$ be a semiprime near-ring and $I$ be a subset of $N$ such that $0\in I$ and $IN\subseteq I$. Let $d$ be a two-sided $\alpha$-derivation on $N$ such that $\alpha(I)=I$ and $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y\in I$.*
*(i) If $d$ acts as a homomorphism on $I$, then $d(I)=\{0\}$.*
*(ii) If $d$ acts as an anti-homomorphism on $I$ and $\alpha(0)=0$, then $d(I)=\{0\}$.*

**Proof.** **(i)** Suppose that $d$ acts as a homomorphism on $I$. By Lemma 6 we have

$$d(y)xd(y)=d(y)x\alpha(y) \qquad \text{for all } x,y\in I. \tag{5}$$

Right multiplying (5) by $d(z)$, where $z\in I$, and using the hypothesis that $d$ acts as a homomorphism on $I$ together with Lemma 3, we obtain $d(y)xd(y)z=0$ for all $x,y,z\in I$.

Taking $xn$ instead of $x$, where $n \in N$, we get $d(y)xnd(y)z = 0$ for all $x, y, z \in I$ and $n \in N$. In particular, $d(y)xNd(y)x = \{0\}$. By the semiprimeness of $N$ we conclude that $d(y)x = 0$. Since $\alpha(I) = I$, it is clear that

$$d(y)\alpha(x) = 0 \qquad \text{for all } x, y \in I. \tag{6}$$

Substituting $yn$ for $y$ in (6) and left-multiplying (6) by $d(z)$, where $z \in I$, we get $d(z)d(y)n\alpha(x) + d(z)\alpha(y)d(n)\alpha(x) = 0$. Since the second summand is zero by (6) we get $0 = d(z)d(y)n\alpha(x) = d(zy)n\alpha(x) = d(z)\alpha(y)n\alpha(x) + zd(y)n\alpha(x) = zd(y)n\alpha(x)$, that is $zd(y)nx = 0$ for all $x, y, z \in I, n \in N$. Since $N$ is semiprime, we have

$$zd(y) = 0 \qquad \text{for all } y, z \in I. \tag{7}$$

Combining (6) and (7) shows that $d(yz) = 0$ for all $y, z \in I$. In particular, $d(xnx) = 0$ for all $x \in I, n \in N$; and since $d$ acts as a homomorphism on $I$, we have

$$0 = d(xn)d(x) = d(x)nd(x) + \alpha(x)d(n)d(x).$$

Since $\alpha(I) = I$, the second summand is zero by (7). Hence $d(x) = 0$ for all $x \in I$.

**(ii)**. Now assume that $d$ acts as an anti-homomorphism on $I$. Note that $a0 = 0$ for all $a \in I$ by Lemma 2. According to Lemma 6 we have

$$xyd(y) = d(y)xd(y) \qquad \text{for all } x, y \in I, \tag{8}$$

$$d(y)\alpha(y)x = d(y)xd(y) \qquad \text{for all } x, y \in I. \tag{9}$$

Replacing $x$ by $xd(y)$ in (8) and using Lemma 6, we get

$$\begin{aligned} xd(y)yd(y) = d(y)xd(y^2) &= d(y)x(d(y)\alpha(y) + yd(y)) \\ &= d(y)xd(y)\alpha(y) + d(y)xyd(y). \end{aligned} \tag{10}$$

Substituting $xy$ for $x$ in (8), we have

$$xy^2d(y) = d(y)xyd(y) \qquad \text{for all } x, y \in I. \tag{11}$$

Right-multiplying (8) by $\alpha(y)$, we obtain

$$xyd(y)\alpha(y) = d(y)xd(y)\alpha(y) \qquad \text{for all } x, y \in I. \tag{12}$$

Replacing $x$ by $y$ in (8) we get $y^2d(y) = d(y)yd(y)$; and left-multiplying this relation by $x$, we have

$$xy^2d(y) = xd(y)yd(y) \qquad \text{for all } x, y \in I. \tag{13}$$

Using (11), (12) and (13) in (10), one obtains $xyd(y)\alpha(y) = 0$. In particular, $ynyd(y)\alpha(y) = 0$, where $n \in N$. Hence $yd(y)\alpha(y)Nyd(y)\alpha(y) = \{0\}$. By the semiprimeness of $N$

$$yd(y)\alpha(y) = 0 \qquad \text{for all } x, y \in I. \tag{14}$$

According to (12) we get $d(y)xd(y)\alpha(y) = 0$. Using this relation in (9), we have

$$d(y)\alpha(y)x\alpha(y) = 0 \qquad \text{for all } x, y \in I. \tag{15}$$

Replacing $x$ by $xnd(y)$ in (15), we have $d(y)\alpha(y)xd(y)\alpha(y) = d(y)\alpha(y)xnd(y)\alpha(y)x = 0$ for all $x, y \in I$,$n \in N$. Hence

$$d(y)\alpha(y)x = 0 \qquad \text{for all } x, y \in I. \tag{16}$$

Using (16) in (9), we obtain that $d(y)xd(y) = 0$, and so we have $d(y)xnd(y)x = 0$ for all $x, y \in I, n \in N$. Hence

$$d(y)x = 0 \qquad \text{for all } x, y \in I. \tag{17}$$

Therefore $xd(z)d(yn)x = 0$ for all $x, y, z \in I, n \in N$. Thus $0 = xd(z)(d(y)n + \alpha(y)d(n))x = xd(z)d(y)\alpha(y)d(n)x$ for all $x, y, z \in I, n \in N$. Since $\alpha(I) = I$ the second summand is zero by (17). Hence $xd(z)d(y)Nx = \{0\}$, and so $xd(z)d(y)Nxd(z)d(y) = \{0\}$. By the semiprimeness of $N$ we get $0 = xd(z)d(y) = xd(yz)$. Therefore $0 = xd(y)z +$

$x\alpha(y)d(z) = x\alpha(y)d(z)$. In particular $0 = \alpha(y)d(z)n\alpha(y)d(z)$. Hence $0 = \alpha(y)d(z)$. Recalling (17) we now have $0 = d(xy)$ for all $x, y \in I$, so $d(xxn) = 0$ for all $x \in I, n \in N$.Thus $0 = d(xn)d(x) = (d(x)n + \alpha(x)d(n))d(x) = d(x)nd(x) + \alpha(x)d(n)d(x) = d(x)nd(x) + \alpha(x)d(xn)$. Since the second summand is zero, we get $d(x)nd(x) = 0$. Therefore $d(x) = 0$ for all $x \in I$. □

**Corollary 1** *Let $N$ be a semiprime near-ring and $d$ a two-sided $\alpha$-derivation of $N$ such that $\alpha$ is onto and $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in N$.*
*(i)If $d$ acts as a homomorphism on $N$, then $d = 0$.*
*(ii) If $d$ acts as an anti-homomorphism on $N$ such that $\alpha(0) = 0$, then $d = 0$.*

**Corollary 2** *Let $N$ be a prime near-ring and $I$ a nonzero subset of $N$ such that $0 \in I$ and $IN \subseteq I$. Let $d$ be a two-sided $\alpha$-derivation on $N$ such that $\alpha(I) = I$ and $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in I$.*

*(i) If $d$ acts as a homomorphism on $I$, then $d = 0$.*
*(ii) If $d$ acts as an anti-homomorphism on $I$ and $\alpha(0) = 0$, then $d = 0$.*

**Proof.** By Theorem 1, we have $d(x) = 0$ for all $x \in I$. Then $0 = d(xn) = d(x)\alpha(n) + xd(n) = xd(n)$, and so $xmd(n) = 0$ for all $x \in I, n, m \in N$. By the primeness of $N$ we have $x = 0$ or $d(n) = 0$ for all $x \in I, n \in N$. Since $I$ is nonzero, we have $d(n) = 0$ for all $n \in N$. □

**Theorem 2** . *Let $N$ be a prime near-ring, $I$ a nonzero semigroup ideal of $N$ and $d$ a nonzero $(\alpha, 1)$-derivation of $N$ such that $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in I$. If $d(x + y - x - y) = 0$ for all $x, y \in I$, then $(N, +)$ is abelian.*

**Proof.** Suppose that $d(x + y - x - y) = 0$ for all $x, y \in I$. Then we have $(x + y - x - y)d(z) = 0$ for all $x, y, z \in I$ by Lemma 5. Since $d \neq 0$, it is clear that $x + y - x - y = 0$ for all $x, y \in I$ by Lemma 4. Hence $(N, +)$ is abelian by Lemma 1. □

**Corollary 3** . *Let $N$ be a prime near-ring, $I$ a nonzero semigroup ideal of $N$ and $d$ a nonzero $(\alpha, 1)$-derivation of $N$ such that $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in I$. If $d + d$ is additive on $I$, then $(N, +)$ is abelian.*

**Proof.** Assume that $d + d$ is an additive on $I$. Then

$$(d + d)(x + y) = (d + d)(x) + (d + d)(y) = d(x) + d(x) + d(y) + d(y).$$

for all $x, y \in I$. On the other hand,

$$(d + d)(x + y) = d(x + y) + d(x + y) = d(x) + d(y) + d(x) + d(y).$$

for all $x, y \in I$. The above two expressions for $(d+d)(x+y)$ yield $d(x)+d(y) = d(y)+d(x)$ for all $x, y \in I$, that is $d(x+y-x-y) = 0$. Then the proof is complete by Theorem 2. □

**Example**. Let $N = N_1 \oplus N_2$, where $N_1$ and $N_2$ are prime near-rings. Define $d : N \to N$ by $d((x, y))) = (0, y)$ and $\alpha : N \to N$ by $\alpha((x, y)) = (x, 0)$ for all $(x, y) \in N$. Then $d$ is a two-sided $\alpha$-derivation on $N$ such that $d$ acts as a homomorphism on $N$ and $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in N$. Furthermore, if $N_2$ is commutative, then $d$ acts as an anti-homomorphism on $N$ and if $N_2$ is abelian, then $d(x + y - x - y) = 0$ for all $x, y \in N$. But $d \neq 0$ and $(N, +)$ is not abelian. Therefore the primeness condition on $N$ in Corollary 2 and Theorem 2 cannot be omitted.

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