Turk J Math
28 (2004) , 1 – 10.

The Cross Curvature Flow of 3-Manifolds with Negative Sectional Curvature

*Bennett Chow**, *Richard S. Hamilton*

Abstract

We consider the cross curvature flow, an evolution equation of metrics on 3-manifolds. We establish short time existence when the sectional curvature has a sign. In the case of negative sectional curvature, we obtain some monotonicity formulas which support the conjecture that after normalization, for initial metrics on closed 3-manifolds with negative sectional curvature, the solution exists for all time and converges to a hyperbolic metric. This conjecture is still open at the present time.

1. The evolution equation

When $n = 3$, it is an old conjecture, which is also a consequence of the Geometrization Conjecture, that any closed 3-manifold with negative sectional curvature admits a hyperbolic metric. In this article, we introduce an evolution equation which deforms metrics on 3-manifolds with sectional curvature of one sign. Given a closed 3-manifold with an initial metric with negative sectional curvature, we conjecture that this flow will exist for all time and converge to a hyperbolic metric after a normalization. We shall establish some results, including monotonicity formulae, in support of this conjecture.

Note that in contrast to negative sectional curvature, every closed n-manifold admits a metric with negative Ricci curvature by the work of Gao and Yau [6], [7] for $n = 3$ and Lohkamp [12] for all $n \geq 3$. When $n \geq 4$, Gromov and Thurston [8] have shown that there exist closed manifolds with arbitrarily pinched negative sectional curvature which do not admit metrics with constant negative sectional curvature. It is unknown whether such manifolds admit Einstein metrics. In particular, the stability result for Ricci flow of Ye [15] assumes more than just curvature pinching depending only on dimension.[1]

Let (M, g) be a 3-dimensional Riemannian manifold with negative sectional curvature. The Einstein tensor is $P_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$. We find it convenient to raise the indices:

Key words and phrases. Cross curvature flow, geometric evolution equation, negative sectional curvature, monotonicity formula, hyperbolic metric.

*Partially supported by NSF Grant DMS-0203926.

[1] We learned from Rugang Ye that according to an example of Farrell and Ontaneda [4], it is not possible in general to deform negatively pinched metrics to Einstein metrics of negative sectional curvature in a continuous fashion. In other words, in general there is no continuous map from the space of negatively pinched metrics to the space of Einstein metrics of negative sectional curvature.

$P^{ij} = g^{ik} g^{j\ell} R_{k\ell} - \frac{1}{2} R g^{ij}$. When P^{ij} has an inverse V_{ij}, the *cross curvature tensor* is

$$h_{ij} = (\det P)\, V_{ij} \qquad \text{where} \qquad \det P = \left(\frac{\det P^{k\ell}}{\det g^{k\ell}} \right).$$

Since the tensor P_{ij} is symmetric, we can, at each point, choose an orthonormal basis in which it is diagonal and $g_{ij} = \delta_{ij}$. In this basis, R_{ij} and h_{ij} are also diagonal, and one sees that if the eigenvalues of P_{ij} are $a = -R_{2323}$, $b = -R_{1313}$, $c = -R_{1212}$, then the eigenvalues of R_{ij} are $-(b+c)$, $-(a+c)$, $-(a+b)$ and the eigenvalues of h_{ij} are bc, ac, ab. Note that our sign convention is such that R_{ijij}, $i \neq j$, are the sectional curvatures, that is, $R_{ijk\ell} = R_{ij\ell}^{h} g_{hk}$. Hence if (M^3, g) has negative sectional curvature, then both P_{ij} and h_{ij} are positive definite.

Lemma 1. *We have the following identities*

(a) $\nabla_i P^{ij} = 0$
(b) $\left(h^{-1}\right)^{ij} \nabla_i h_{jk} = \frac{1}{2} \left(h^{-1}\right)^{ij} \nabla_k h_{ij}$.

Proof. The first identity is the contracted second Bianchi identity $g^{ij} \nabla_i R_{jk} = \frac{1}{2} \nabla_k R$. Using (a), we have

$$\begin{aligned} \left(h^{-1}\right)^{ij} \nabla_i h_{jk} &= (\det P)^{-1} P^{ij} \nabla_i \left(V_{jk} \det P\right) \\ &= (\det P)^{-1} \nabla_k \det P = \frac{1}{2} \nabla_k \log \det h \end{aligned}$$

where $\det h = \det h_{ij} / \det g_{ij}$. That implies (b). □

The above identities imply that the cross curvature tensor is dual to the Ricci tensor in the following sense.

Lemma 2. *Let (M^n, g) be Riemannian manifold.*

(a) *If the Ricci curvature is positive, then the identity map $\iota : (M, g_{ij}) \to (M, R_{ij})$ is harmonic, and if the Ricci curvature is negative, then $\iota : (M, g_{ij}) \to (M, -R_{ij})$ is harmonic.*
(b) *If $n = 3$ and the sectional curvature is negative (or positive), then $\iota : (M, h_{ij}) \to (M, g_{ij})$ is harmonic.*

Proof. Given two Riemannian metrics γ and $\hat{\gamma}$ on a manifold M, the Laplacian of the identity map $\iota : (M, \gamma) \to (M, \hat{\gamma})$ is given by

$$\begin{aligned} (\Delta \iota)^k &= \gamma^{ij} \left(\hat{\Gamma}_{ij}^k - \Gamma_{ij}^k \right) \\ &= \gamma^{ij} \hat{\gamma}^{k\ell} \left(\nabla_i \hat{\gamma}_{j\ell} + \nabla_j \hat{\gamma}_{i\ell} - \nabla_\ell \hat{\gamma}_{ij} \right) \qquad (1) \\ &= -\gamma^{ij} \gamma^{k\ell} \left(\hat{\nabla}_i \gamma_{j\ell} + \hat{\nabla}_j \gamma_{i\ell} - \hat{\nabla}_\ell \gamma_{ij} \right) \qquad (2) \end{aligned}$$

where Γ_{ij}^k, $\hat{\Gamma}_{ij}^k$, ∇ and $\hat{\nabla}$ denote the Christoffel symbols and covariant derivatives for the metrics γ and $\hat{\gamma}$ respectively. Statement (a) follows from (1) with $\gamma_{ij} = g_{ij}$ and $\hat{\gamma}_{ij} = R_{ij}$. Statement (b) follows from (2) with $\gamma_{ij} = h_{ij}$ and $\hat{\gamma}_{ij} = g_{ij}$. □

With this duality in mind we define the *cross curvature flow* (XCF)[2] on a 3-manifold as the flow on the space of Riemannian metrics whose flow lines $g(t)$ satisfy

$$\frac{\partial}{\partial t} g_{ij} = 2h_{ij}$$

if the sectional curvature is negative and by $\frac{\partial}{\partial t} g_{ij} = -2h_{ij}$ if the sectional curvature is positive. This is similar to the much more general Ricci flow equation $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, which has been extensively studied (see for example [11] and [13]). However, as we will see in section 2, the XCF is fully nonlinear, whereas the Ricci flow is quasi-linear.

2. Short time existence

Let μ_{ijk} denote the volume form and raise indices by $\mu^{ijk} = g^{ip} g^{jq} g^{kr} \mu_{pqr}$.

Lemma 3. *We have the following identities*

(a) $\mu^{pqk} R_{kjrs} \mu^{rs\ell} = -2P^{m\ell} \left(\delta_j^p \delta_m^q - \delta_m^p \delta_j^q \right)$, *and*

(b) $h_{ij} = \frac{1}{8} R_{i\ell pq} \mu^{pqk} R_{kjrs} \mu^{rs\ell}$.

Proof. It is straightforward to verify these formulas using the orthonormal basis described before Lemma 1, noting that $\mu_{123} = \mu^{123} = 1$ in that basis. □

Lemma 4. *If (M, g) is a closed 3-manifold with negative (or positive) sectional curvature, then for any smooth initial metric a solution to the XCF exists for a short time.*

Proof. We consider the case of negative sectional curvature since the case of positive sectional curvature is similar. Let $\tilde{g}_{ij}$ denote a variation of the metric g_{ij} and let tildes also denote the variations of various curvature tensors. We have

$$\tilde{R}_{ijk\ell} = \frac{1}{2} \left(\frac{\partial^2 \tilde{g}_{jk}}{\partial x^i \partial x^\ell} + \frac{\partial^2 \tilde{g}_{i\ell}}{\partial x^j \partial x^k} - \frac{\partial^2 \tilde{g}_{j\ell}}{\partial x^i \partial x^k} - \frac{\partial^2 \tilde{g}_{ik}}{\partial x^j \partial x^\ell} \right) + \cdots$$

where the dots denote terms with 1 or less derivatives of the metric. Applying the equality in Lemma 3 (a) yields

$$\begin{aligned} \tilde{h}_{ij} = & -\frac{1}{8} \left(\frac{\partial^2 \tilde{g}_{\ell p}}{\partial x^i \partial x^q} + \frac{\partial^2 \tilde{g}_{iq}}{\partial x^\ell \partial x^p} - \frac{\partial^2 \tilde{g}_{\ell q}}{\partial x^i \partial x^p} - \frac{\partial^2 \tilde{g}_{ip}}{\partial x^\ell \partial x^q} \right) P^{m\ell} \left(\delta_j^p \delta_m^q - \delta_m^p \delta_j^q \right) \\ & - \frac{1}{8} \left(\frac{\partial^2 \tilde{g}_{\ell p}}{\partial x^j \partial x^q} + \frac{\partial^2 \tilde{g}_{jq}}{\partial x^\ell \partial x^p} - \frac{\partial^2 \tilde{g}_{\ell q}}{\partial x^j \partial x^p} - \frac{\partial^2 \tilde{g}_{jp}}{\partial x^\ell \partial x^q} \right) P^{m\ell} \left(\delta_i^p \delta_m^q - \delta_m^p \delta_i^q \right) \\ & + \cdots . \end{aligned}$$

[2] We owe this nice abbreviation to Ben Andrews.

Thus the map E which takes g to $2h$ is a second-order non-linear operator. Its symbol is obtained from $\tilde{h}_{ij}$ by replacing $\frac{\partial}{\partial x^i}$ by a cotangent vector ζ_i in the highest (second) order terms

$$\sigma DE\,(g)\,(\zeta)\,\tilde{g}_{ij} = -P^{m\ell}\,(\zeta_i\zeta_m\tilde{g}_{\ell j} + \zeta_\ell\zeta_j\tilde{g}_{im} - \zeta_i\zeta_j\tilde{g}_{\ell m} - \zeta_\ell\zeta_m\tilde{g}_{ij})\,,$$

where DE denotes the linearization of E. Since the sectional curvature is negative, $P^{m\ell}$ is positive and the eigenvalues of the symbol are nonnegative.

There are no solutions of $E\,(g) = 2h$ unless a certain integrability condition holds. One checks that the integrability condition is $L\,(h_{ij}) = 0$, where

$$L\,(T)_k \doteqdot \left(h^{-1}\right)^{ij}\nabla_i T_{jk} - \frac{1}{2}\left(h^{-1}\right)^{ij}\nabla_k T_{ij}.$$

Thus Lemma 1b insures that the integrability condition holds. By Theorem 5.1 of [9], a solution to the XCF exists for short time. □

The cross curvature flow equation is fully nonlinear. In fact, Ben Andrews has observed that, when the 3-manifold is embeddable into euclidean space or Minkowski space, the XCF is equivalent to the Gauss curvature flow, which is a parabolic Monge-Ampere equation.

Since given an initial metric with negative sectional curvature, a solution exists for short time, the next question is whether negative sectional curvature is preserved. This requires one to show that there are negative upper and lower bounds for the sectional curvatures. The estimates in the remainder of the paper assume the existence of the solution on a given time interval. Hopefully they are steps along the way to prove bounds for the sectional curvatures and convergence to hyperbolic. However these conjectures remain open.

3. Evolution of the Einstein tensor

Since Lemma 4 establishes short-time existence for the XCF, we can begin examining the long-time behavior of the flow and the geometry of the evolving metric $g(t)$. One important technique is to find functions constructed from $g(t)$ which are decreasing in t. Such functions are often integrals of curvature quantities; the fact that they are decreasing gives bounds, valid for all time, on those particular curvature integrals.

In Sections 4 and 5 we will construct two monotone functions for the cross curvature flow. In preparation for that we derive the formula for the evolution of the Einstein tensor under the XCF. It is convenient to express the Einstein tensor as

$$P^{mn} = -\frac{1}{4}\mu^{ijm}\mu^{k\ell n}R_{ijk\ell}, \tag{3}$$

which is a special case of Lemma 3a.

Lemma 5. *The evolution of the Einstein tensor is given by*

$$\frac{\partial}{\partial t}P^{ij} = \nabla_k\nabla_\ell\left(P^{k\ell}P^{ij} - P^{ik}P^{j\ell}\right) - \det P\,g^{ij} - H\,P^{ij}$$

where H is the trace $g^{ij}h_{ij}$.

Proof. The evolution of the Riemann curvature tensor is given by the standard formula

$$\frac{\partial}{\partial t}R_{ijk\ell} = \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_i\nabla_k h_{j\ell} - \nabla_j\nabla_\ell h_{ik} + g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right).$$

Since the evolution of the volume form is given by $\frac{\partial}{\partial t}\mu_{ijk} = H\mu_{ijk}$ and $\frac{\partial}{\partial t}\mu^{ijk} = -H\mu^{ijk}$, we may compute using (3) that

$$\begin{aligned}\frac{\partial}{\partial t}P^{mn} &= -\frac{1}{4}\mu^{ijm}\mu^{k\ell n}\left(\nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_i\nabla_k h_{j\ell} - \nabla_j\nabla_\ell h_{ik}\right)\\ &\quad - \frac{1}{4}\mu^{ijm}\mu^{k\ell n}g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right) - 2HP^{mn}\\ &= \mu^{ijm}\mu^{k\ell n}\nabla_i\nabla_k h_{j\ell} - \frac{1}{2}\mu^{ijm}\mu^{k\ell n}g^{pq}R_{ijp\ell}h_{qk} - 2HP^{mn}.\end{aligned}$$

The lemma follows from the identity

$$\frac{1}{2}\mu^{ijm}\mu^{k\ell n}g^{pq}R_{ijp\ell}h_{qk} + HP^{mn} = \det P\, g^{mn}.$$

We can verify this last identity by choosing a basis where $g_{ij} = \delta_{ij}$, P^{ij} and h_{ij} are diagonal, $\mu^{123} = 1$, and $R_{ijk\ell} \neq 0$ only if $(i,j) = (k,\ell)$ as unordered pairs. For example:

$$\begin{aligned}\frac{1}{2}\mu^{ij1}\mu^{k\ell 1}g^{pq}R_{ijp\ell}h_{qk} &= \mu^{k\ell 1}g^{pq}R_{23p\ell}h_{qk}\\ &= R_{2323}h_{22} - R_{2332}h_{33}\\ &= -P^{11}\left(h_{22} + h_{33}\right)\end{aligned}$$

so that $\frac{1}{2}\mu^{ij1}\mu^{k\ell 1}g^{pq}R_{ijp\ell}h_{qk} + HP^{11} = P^{11}h_{11} = \det P$. One can similarly check that the off-diagonal components are zero. □

4. Monotonicity of the volume of the Einstein tensor

We will show the monotonicity of

$$vol\,(P) = \int_M \sqrt{\det P}\, d\mu$$

where $d\mu$ is the volume form of g_{ij}. Note that $vol(P)$ is scale-invariant.

Proposition 6. *If (M, g) is a 3-manifold with negative sectional curvature, then $vol(P_{ij})$ is nondecreasing under the XCF.*

This follows from the more general computation

Lemma 7. *For any* $\eta \in \mathbb{R}$

$$\frac{d}{dt}\int_M (\det P)^\eta \, d\mu = \eta \int_M \left(\frac{1}{2}\left|T^{ijk} - T^{jik}\right|_V^2 - \eta \left|T^i\right|_V^2\right)(\det P)^\eta \, d\mu + (1-2\eta)\int_M (\det P)^\eta H \, d\mu$$

where $T^{ijk} = P^{i\ell}\nabla_\ell P^{jk}$, $T^i = V_{jk}T^{ijk} = P^{ij}\nabla_j \log\det P$, *and the norms are with respect to the metric* V_{ij}.

Proof. We compute using the evolution of P^{ij} that

$$\begin{aligned}\frac{d}{dt}\int_M (\det P)^\eta \, d\mu &= \int_M (\det P)^\eta \left(\eta\left(V_{ij}\partial_t P^{ij} - g_{ij}\left(\partial_t g^{ij}\right)\right) + H\right) d\mu \\ &= \eta \int_M (\det P)^\eta \, V_{ij}\nabla_k \nabla_\ell \left(P^{k\ell}P^{ij} - P^{ik}P^{j\ell}\right) d\mu \\ &\quad + (1-2\eta)\int_M (\det P)^\eta H \, d\mu \\ &= -\eta \int_M \nabla_k \left[(\det P)^\eta \, V_{ij}\right]\left(P^{k\ell}\nabla_\ell P^{ij} - P^{j\ell}\nabla_\ell P^{ik}\right) d\mu \\ &\quad + (1-2\eta)\int_M (\det P)^\eta H \, d\mu\end{aligned}$$

The lemma follows after rewriting the term $\nabla_k\left[(\det P)^\eta \, V_{ij}\right]$ in the integrand as $(\det P)^\eta \left(\eta V_{pq}\left(\nabla_k P^{pq}\right)V_{ij} - V_{ip}V_{jq}\nabla_k P^{pq}\right)$. □

Decompose T^{ijk} into its irreducible components (the orthogonal group $O(3)$ for the metric V acts on the bundle of 3-tensors which are symmetric in the last two components)

$$T^{ijk} = E^{ijk} - \frac{1}{10}\left(P^{ij}T^k + P^{ik}T^j\right) + \frac{2}{5}P^{jk}T^i,$$

where the coefficients $-\frac{1}{10}$ and $\frac{2}{5}$ are chosen so that $V_{ij}E^{ijk} = V_{ik}E^{ijk} = V_{jk}E^{ijk} = 0$ (recall that T^{ijk} is symmetric in j and k). Using this we find that

$$\left|T^{ijk} - T^{jik}\right|_V^2 = \left|E^{ijk} - E^{jik}\right|_V^2 + \left|T^i\right|_V^2. \tag{4}$$

Taking $\eta = 1/2$ in the lemma, we have

$$\frac{d}{dt}\int_M (\det P)^{1/2} \, d\mu = \frac{1}{4}\int_M \left|E^{ijk} - E^{jik}\right|_V^2 (\det P)^{1/2} \, d\mu \geq 0$$

and Proposition 6 follows.

5. Approach to hyperbolic in an integral sense

Recall that the goal of the cross curvature flow is to deform a metric with negative sectional curvature on a 3-manifold to a hyperbolic metric. We shall show that an integral measure of the difference of the metric from hyperbolic is monotone decreasing. Let

$$J = \int_M \left(\frac{P}{3} - (\det P)^{1/3} \right) d\mu$$

where $P = g_{ij}P^{ij}$. By the arithmetic-geometric mean inequality (applied in a basis in which P_{ij} is diagonal and $g_{ij} = \delta_{ij}$), the integrand is nonnegative, and identically zero if and only if $P_{ij} = \frac{1}{3}Pg_{ij}$, i.e., g_{ij} has constant curvature.

Theorem 8. *Under the cross curvature flow* $\frac{dJ}{dt} \leq 0$.

Proof. We compute

$$\begin{aligned}
\frac{d}{dt}\int_M P\, d\mu &= \int_M \left[(\partial_t g_{ij})\, P^{ij} + g_{ij}\partial_t P^{ij} + PH \right] d\mu \\
&= \int_M \left[2h_{ij}P^{ij} + g_{ij}\left(-\det P\, g^{ij} - H\, P^{ij} \right) + PH \right] d\mu \\
&= 3\int_M \det P\, d\mu.
\end{aligned}$$

By the definition of h_{ij} we can replace $\det P$ by $(\det h)^{1/3} (\det P)^{1/3}$. Combining this with the previous lemma with $\eta = 1/3$ and formula (4), we find that

$$\begin{aligned}
\frac{dJ}{dt} = &-\frac{1}{6}\int_M \left(\left| E^{ijk} - E^{jik} \right|^2 + \frac{1}{3}\left| T^i \right|^2 \right) (\det P)^{1/3}\, d\mu \\
&- \int_M \left(\frac{H}{3} - (\det h)^{1/3} \right) (\det P)^{1/3}\, d\mu.
\end{aligned}$$

This is nonpositive (and 0 if and only if g_{ij} has constant negative sectional curvature). □

6. A maximum principle estimate

We can also obtain information about the long-time behavior of geometric flows like the XCF by using the maximum principle for parabolic equations. That typically involves finding a function $f(x,t)$, constructed tensorially from the metric and its derivatives, which satisfies an inequality of the form $\partial_t f \geq \Delta f$. Since the higher order terms in the evolution of P_{ij} are of divergence form with both first and second order terms, in comparison to the Ricci flow, it is much more difficult to obtain good maximum principle estimates. But, as we show next, there is at least one function which, under the XCF, satisfies an equation for which the maximum principle can be applied. At the end of this section we use the maximum principle to show that the XCF preserves the set of metrics of negative sectional curvature unless singularities arise in finite time.

Before starting, note that the formula $\Box f = P^{ij}\nabla_i\nabla_j f$ defines an elliptic operator acting on functions f; by Lemma 1 this is the same as $\nabla_i\nabla_j\left(P^{ij}f\right)$.

Proposition 9.

$$\frac{\partial}{\partial t}\log\det P = \Box\log\det P + \frac{1}{2}\left|T^{ijk} - T^{jik}\right|^2 - 2H.$$

Proof. This follows from the computations

$$\begin{aligned}
\frac{\partial}{\partial t}\log\det P &= V_{ij}\partial_t P^{ij} - g_{ij}\partial_t g^{ij} \\
&= V_{ij}\nabla_k\nabla_\ell\left(P^{k\ell}P^{ij} - P^{ik}P^{j\ell}\right) - 2H \\
&= \nabla_k\left[V_{ij}\left(P^{k\ell}\nabla_\ell P^{ij} - P^{j\ell}\nabla_\ell P^{ik}\right)\right] \\
&\quad - \left(\nabla_k V_{ij}\right)\nabla_\ell\left(P^{k\ell}P^{ij} - P^{ik}P^{j\ell}\right) - 2H \\
&= P^{k\ell}\nabla_k\left(V_{ij}\nabla_\ell P^{ij}\right) - 2H \\
&\quad + V_{ip}V_{jq}\nabla_k P^{pq}\left(P^{k\ell}\nabla_\ell P^{ij} - P^{j\ell}\nabla_\ell P^{ik}\right)
\end{aligned}$$

and

$$\Box\log\det P = P^{k\ell}\nabla_k\left(V_{ij}\nabla_\ell P^{ij}\right).$$

□

The maximum principle cannot be directly applied to the equation of Proposition 9 because the last term has the wrong sign. But we can proceed as follows. Let $f = \log\det P$ and introduce the functions

$$m(t) = \min_M f(x,t) \qquad \text{and} \qquad X(t) = \min\left\{H(x,t) : f(x,t) = m(t)\right\}.$$

Given t_0, let $x_0 \in M$ be any point so that $f(x_0,t_0) = m(t_0)$, Then $m(t_0) - m(t_0 - \Delta t) \geq f(x_0,t_0) - f(x_0, t_0 - \Delta t)$. Now divide by Δt and let $\Delta t \to 0$. Using Proposition 9 and the fact that $\Box f \geq 0$ at a minimum point, we obtain

$$m'(t_0) \geq \frac{\partial f}{\partial t}(x_0,t_0) \geq (\Box f - 2H)(x_0,t_0) \geq -2H(x_0,t_0),$$

where the time derivative of the Lipschitz function is the lim inf of backwards difference quotients. Thus

$$\frac{d}{dt}\min_M \log\det P(t) \geq -2X(t).$$

Suppose for some $T < \infty$ we have $\inf_{M\times[0,T)}\det P = 0$. It is not difficult to show that there then exists a sequence of times $t_i \to T$ such that $\min_M \log\det P(t_i) \to \infty$ and $X(t_i) \to \infty$. Hence there exists $x_i \in M$ such that $\det P(x_i,t_i) \to 0$ and $H(x_i,t_i) \to \infty$. Thus, if $0 < a \leq b \leq c$ denote the absolute values of the principal sectional curvatures, then there exists a sequence of points and times such that $bc \to \infty$ and $abc \to 0$; in particular, one of the sectional curvatures tends to zero and another tends to minus infinity.

Proposition 10. *Let* $\left(M^3, g(t)\right)$, $t \in [0,T)$, *be a solution to the XCF on a closed 3-manifold starting from a metric* $g(0)$ *with negative sectional curvature. If* $T < \infty$ *and* $\inf_{M\times[0,T)} \det P = 0$, *then* $g(t)$ *has negative sectional curvature for all* $t \in [0,t)$ *and there exists a sequence of points and times* (x_i, t_i) *with* $t_i \to T$ *such that* $c(x_i, t_i) \to \infty$ *and* $a(x_i, t_i) \to 0$.

We are curious if the proposition may be used to prove that $\inf_{M\times[0,T)} \det P > 0$ for all $T < \infty$. One further goal is to show that for any $T < \infty$, there exist negative upper and lower bounds for the sectional curvatures on $M \times [0,T)$.

7. Conclusion

We have obtained estimates, in the form of monotonicity formulae, for the cross curvature flow which lead us to hope and expect that it should exist for all time and converge after normalization to a hyperbolic metric on closed 3-manifolds with initial metric of negative sectional curvature. In view of the well-developed theory of Ricci flow [11] and the ground breaking work of Perelman [13] on the second author's program for Ricci flow as an approach to the Thurston Geometrization and Poincaré conjectures, it is hopeful that further progress can be made on the cross curvature flow.

Recently, Ben Andrews has obtained new estimates for the cross curvature flow[1] . In the case when the universal cover of the initial 3-manifold is isometrically embedded as a hypersurface in Euclidean or Minkowski 4-space, the Gauss curvature flow (see [5], [14], [3], [10], [2] for earlier works on the GCF) induces the XCF for the metric. In that case Andrews has proved convergence results. In general, he expects long time existence and convergence of the XCF to reduce to proving local in time regularity (higher derivative estimates). It would be interesting to obtain a Li-Yau-Hamilton Harnack type inequality for the XCF. Such gradient estimates of Harnack type enable one to compare the curvatures at different points and times and often allow one to convert integral estimates to pointwise estimates. Harnack type estimates exist for many geometric evolution equations so it is hopeful that one exists for the XCF.

Now suppose that one starts the XCF at a metric of negative sectional curvature. Then $\det P = abc$ is positive, and stays positive as long as the flow remains in the set of metrics of negative sectional curvature. Conversely, as long as $\det P > 0$, the evolving metric $g(t)$ has negative sectional curvature, and all the results of the previous sections apply. It is natural, then, to ask whether $\det P$ can limit to zero in finite time. Another interesting question is to characterize solutions to the XCF where $\partial_t \mathrm{vol}(P) = 0$. Note by the formula in Theorem 8, $\partial_t J = 0$ if and only if g has constant sectional curvature.

Acknowledgement

We are especially grateful to the referee for the many suggestions which we have followed, often verbatim, to greatly improve the exposition of this paper.

References

[1] Andrews, Ben, personal communication.

[2] Andrews, Ben, *Gauss curvature flow: the fate of the rolling stones,* Invent. Math. **138** (1999), no. 1, 151–161; *Motion of hypersurfaces by Gauss curvature,* Pacific J. Math. **195** (2000), no. 1, 1–34.

[3] Chow, Bennett, *On Harnack's inequality and entropy for the Gaussian curvature flow,* Comm. Pure Appl. Math. **44** (1991), no. 4, 469–483.

[4] Farrell, F.T.; Ontaneda, P., *A caveat on the convergence of the Ricci flow for pinched negatively curved manifolds,* math.DG/0311176.

[5] Firey, William J., *Shapes of worn stones,* Mathematika **21** (1974), 1–11.

[6] Gao, L. Zhiyong, *The construction of negatively Ricci curved manifolds,* Math. Ann. **271** (1985), no. 2, 185–208.

[7] Gao, L. Zhiyong; Yau, Shing-Tung, *The existence of negatively Ricci curved metrics on three-manifolds,* Invent. Math. **85** (1986), no. 3, 637–652.

[8] Gromov, Mikhail; Thurston, William P., *Pinching constants for hyperbolic manifolds,* Invent. Math. **89** (1987), no. 1, 1–12.

[9] Hamilton, Richard S., *Three-manifolds with positive Ricci curvature,* J. Differential Geom. **17** (1982), no. 2, 255–306.

[10] Hamilton, Richard S., *Remarks on the entropy and Harnack estimates for the Gauss curvature flow,* Comm. Anal. Geom. **2** (1994), no. 1, 155–165.

[11] Hamilton, Richard S., *The formation of singularities in the Ricci flow,* Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995; *Four-manifolds with positive isotropic curvature,* Comm. Anal. Geom. **5** (1997), no. 1, 1–92; *Non-singular solutions of the Ricci flow on three-manifolds,* Comm. Anal. Geom. **7** (1999), no. 4, 695–729.

[12] Lohkamp, Joachim, *Metrics of negative Ricci curvature,* Ann. of Math. (2) **140** (1994), no. 3, 655–683.

[13] Perelman, Grisha, *The entropy formula for the Ricci flow and its geometric applications,* math.DG/0211159; *Ricci flow with surgery on three-manifolds,* math.DG/0303109; *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds,* math.DG/0307245.

[14] Tso, Kaising, *Deforming a hypersurface by its Gauss-Kronecker curvature,* Comm. Pure Appl. Math. **38** (1985), no. 6, 867–882.

[15] Ye, Rugang, *Ricci flow, Einstein metrics and space forms,* Trans. Amer. Math. Soc. **338** (1993), no. 2, 871–896.

Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, USA
E-mail address: `benchow@math.ucsd.edu`

Department of Mathematics, Columbia University, 2990 Broadway NY, NY 10027, USA
E-mail address: `hamilton@math.columbia.edu`