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An Algorithm to Recognise Small Seifert Fiber Spaces

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Abstract

The homeomorphism problem is, given two compact n-manifolds, is there an algorithm to decide if the manifolds are homeomorphic or not. The homeomorphism problem has been solved for many important classes of 3-manifolds - especially those with embedded 2-sided incompressible surfaces (cf [12], [15],[16]), which are called Haken manifolds. It is also well-known that the homeomorphism problem is easily solvable for two 3-manifolds which admit geometries in the sense of Thurston [36], [31]. Hence the recognition problem, to decide if a 3-manifold has a geometric structure, is a significant problem. The recognition problem has been solved for all geometric classes, except for the class of small Seifert fibered spaces, which either have finite fundamental group or have fundamental groups which are extensions of $\mathbb{Z}$ by a triangle group and have finite abelianisation. Our aim in this paper is to give an algorithm to recognise these last classes of 3-manifolds, i.e to decide if a given 3-manifold is homeomorphic to one in this class. A completely different solution has been announced recently by Tao Li [22]. Also Perelman's announcement of a solution of the geometrisation conjecture would enable a complete solution of the homeomorphism problem; by identifying which geometric structure a given manifold admits. However it is worth noting that practical algorithms for the homeomorphism and recogntion problems, which can be implemented via software, are very useful for experimentation in 3-manifold topology. (See for example [5], [39]).

1. Introduction

A basic problem in the topology of 3-manifolds is to decide if a given closed orientable 3-manifold has one of the eight geometries of Thurston [31],[36],[37]. We will refer to this as the recognition problem. Six of these geometries have all examples which admit Seifert fibre structures, namely $S^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, PSL(2, \mathbb{R}), \mathbb{R}^3, Solv$, while all examples admitting Nil geometric structures are Haken. The methods of Haken, as extended by

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Jaco and Oertel [11], [12],[13],[19], (cf also [21]) give an algorithm to recognise any 3-manifold in these seven geometries, which is Haken or reducible, i.e contains a 2-sided embedded incompressible surface or an embedded 2-sphere which does not bound a 3-cell. The aim of the current paper is to complete the recognition problem for all 3-manifolds which admit one of the first six geometries and are non Haken and irreducible. Such examples occur only for S^3 and $PSL(2,\mathbb{R})$ geometries. Note that as known previously, the results of [26],[27], [35],[33],[34],[21], [18], solve the recognition problem for the 3-sphere, all lens spaces and prism manifolds. (Note all the 3-manifolds in the latter class contain embedded incompressible Klein bottles. So one can either use methods of [21] to find such surfaces and then check the complements are solid tori, or show that the manifolds have double covers which are lens spaces, by [34] or [18]. Then by [25] the original manifolds must be lens spaces or prism manifolds.) Consequently, the remaining examples are all Seifert fibre spaces with three exceptional fibres, with S^2 as orbit surface. Finding an algorithm to decide if a given 3-manifold is homeomorphic to one of these 'small' Seifert fibre spaces will be the subject of this paper.

Since a key part of the algorithm is to decide if a 3-manifold M has a Heegaard splitting of genus 2, we investigate the details of finding genus 2 splittings in the final section of the paper. In particular, assuming that a 0-efficient triangulation of M has been found ([20]), we show that in a suitable almost normal projective solution space, if there is a genus 2 almost normal Heegaard surface, then there is one of bounded weight. So this gives a computable bound on the complexity of finding such surfaces, if they exist in the 3-manifold. For the issue pf practicality of the algorithm in this paper, finding genus 2 Heegaard surfaces is one of the more computationally expensive parts of the process.

In a talk at the Technion in January 1999, Casson spoke about the computational complexity of finding the connected sum decomposition of 3-manifolds. In [18], the complexity of determining the characteristic variety decomposition of a 3-manifold is investigated. The algorithm here is needed to complete the technique in [18] and is the slowest part of the method. So finding a more practical procedure is of considerable interest. In [26] a theoretical algorithm is given for finding genus 2 Heegaard splittings, but it is clearly not useful for implementation. In the final section, we address this algorithmic problem and give an improved and simplified method, using the theory of 0-efficient triangulations [JR].

Deciding if a 3-manifold is a small Seifert fibre space is also of interest in determining the range of boundary slopes giving exceptional Dehn surgeries of knots in the 3-sphere (cf [8]).

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2. Involutions and genus two Heegaard splittings of small Seifert fibre spaces

In this section, we collect together some well-known facts about genus two Heegaard splittings, small Seifert fibre spaces and involutions of 3-manifolds. Note first that a Seifert fibre space has a foliation by embedded circles, with the property that every circle fibre has a small solid torus regular neighbourhood which is filled by loops in the foliation.

Moreover the foliation on such a solid torus is either the product foliation (in which case we call the original fibre *regular* or else is formed by gluing the ends of a cylinder $D^2 \times I$ by a map $(z, 0) \to (e^{\frac{2\pi iq}{p}} z, 1)$ where p, q are relatively prime positive integers with $p \geq 2$. Here D is the unit disk in the complex plane. The circle fibres near the original are formed by gluing together p intervals of the form $\{(z, t); 0 \leq t \leq 1\}$ where $z \neq 0$. The original circle fibre consists of all points of the form $\{(0, t); 0 \leq t \leq 1\}$ and is exceptional with multiplicity p. The orbit surface is formed by projecting each fibre to a point and has a cone point corresponding to each exceptional fibre.

Definition 2.1. An embedded or immersed torus $T \to M$ is called incompressible if the induced map $\pi_1(T) \to \pi_1(M)$ is one-to-one.

Definition 2.2. A vertical torus T in a Seifert fibre space is a union of regular fibres. Consequently, under the projection map from the Seifert fibre space to its orbit space, given by projecting each Seifert fibre to a point, T is the inverse image of an immersed or embedded loop in the orbit space.

A closed orientable manifold is called atoroidal if it has no immersed or embedded incompressible tori. We note the following useful result due to Waldhausen. We only give a special case as needed below.

Theorem 2.1. *Suppose that M is a closed orientable Seifert fibre space with S^2 as orbit space and at least 4 exceptional fibres. Then M is Haken, i.e contains an embedded incompressible vertical torus. In fact, any vertical torus which projects to a simple closed curve in the orbit surface, with at least two exceptional fibres on each side, is incompressible.*

Another important result on Seifert fibre spaces was proved by Scott in [32]

Theorem 2.2. *Suppose that M is a closed 3-manifold with $\pi_1(M)$ infinite and M is finitely covered by a Seifert fibred space. Then M is Seifert fibred.*

Definition 2.3. A handlebody is a small closed regular neighbourhood of a bouquet of circles embedded in $\mathbb{R}^3$ or in any orientable 3-manifold. A Heegaard splitting of a closed orientable 3-manifold M is obtained by expressing M as a union of two handlebodies H_+, H_- which have a common boundary surface S. Then S is called a Heegaard surface for M.

Definition 2.4. A vertical Heegaard splitting of a Seifert fibre space with three exceptional fibres is formed by joining small fibred regular neighbourhoods of two of the exceptional fibres by a small horizontal tube.

Note that a hozizontal tube is a small regular neighbourhood of an arc which projects one-to-one to an embedded arc in the orbit surface, connecting two cone points and missing the third one.

See [3] for a discussion of Heegaard splittings of small Seifert fibred spaces. In particular, in [3] and [23], it is shown that there are finitely many splittings of genus 2 up to isotopy. See also [30] for a comprehensive discussion about finiteness of genus 2 Heegaard splttings of 3-manifolds. As well as possibly several non isotopic vertical splittings, which can arise by different choices of exceptional fibres, there are sometimes other splittings, called horizontal, depending on the choice of small Seifert fibre space. However we will not have to consider the latter class of splittings in the method below.

Next, as is well-known, for any Heegaard splitting S of genus two of a closed orientable 3-manifold M, there is a canonical involution g of M preserving S and the pair of handlebodies H_+, H_- which are the closures of the components of $M \setminus S$ (see [2] and [38]). The quotient of M via the action of g is the 3-sphere, with S, H_+, H_- mapping respectively to a 2-sphere and a pair of complementary 3-balls, so that the fixed set Γ of g projects to a knot or link Γ' in the 3-sphere. Moerover Γ' is 3-bridge, i.e there is a projection of S^3 to $\mathbb{R}$ so that the knot or link has precisely 3 maxima at value 1 and 3 minima at value -1 and no other critical points with respect to this projection. It is quite easy to check that the involution g can be found algorithmically, once the Heegaard splitting has been constructed. We will give the details of this algorithm in the final section.

Our approach is to use the fact that the 3-bridge knots or links and their symmetries, that are associated with genus 2 splittings of small Seifert fibre spaces are completely understood by work of [3], [6], [23]. These knots or links are often called Montesinos.

We finish this section with a summary of the method, with the details in the next two sections, assuming that a given vertical Heegaard splitting of genus 2 and its canonical involution g have been found. We will distinguish three cases, depending on whether two or more, one, or zero of the exceptional fibres of the Seifert fibring are of even multiplicity. The first case is extremely easy, since then there is a 2-fold cover which is Haken, in nearly all instances. In the second case, we use the fact that rotation r halfway along the Seifert fibring gives a second involution which commutes with g. So there is an induced involution on the 3-sphere, denoted by r', which has a circle of fixed points C, meeting Γ' in two points. This is a strong inversion of the 3-bridge knot or link and can be found readily by standard techniques, as in for instance [21]. Once we have found such a circle C, the lift to M of C must be the exceptional fibre of even multiplicity. It is easy to check if the complement of this loop is Seifert fibred and so this case is then finished.

The most difficult case is the third one, where each exceptional fibre has odd multiplicity. Consequently, rotation r along the Seifert fibring by π induces an involution r' on the 3-sphere leaving Γ' invariant, with fixed circle C disjoint from Γ'. So the 3-bridge knot or link is periodic (with period 2) and again this involution r' acting on the knot complement can be found by standard techniques. (Note that symmetries of such knots have been studied in [**?**]). Our idea is to keep iterating this process. Essentially this means we are finding smaller and smaller rotations r_k along the circles of the Seifert fibering by an angle of $\pi \times 2^{1-k}$, for $k = 1, \dots$. The period of r_k is 2^k. The final step is to show that for some k sufficiently large, r_k is isotopic to the identity, since the translate of S by r_k becomes isotopic to S. In this case it is follows by [10] or [7] that the isotopy is essentially the Seifert fibring and we are done.

3. Exceptional fibres of even multiplicity

Case 1. Two or more exceptional fibres of even multiplicity

Let M denote a Seifert fibre space with S^2 as orbit surface and three exceptional fibres F_1, F_2, F_3 with $PSL(2, \mathbf{R})$ or S^3 geometry. We assume in this case that F_1 and F_2 have even multiplicities m_1 and m_2, with the multiplicity of F_3 being some integer $m_3 > 1$. Then clearly there is a non-zero element of $H^1(M, Z_2)$ so that the induced map $\pi_1(M) \to Z_2$ takes the classes of F_1 and F_2 to non zero images, while the class of F_3 is mapped to zero. (Note that since we are mapping to an Abelian group, this is independent of the choice of paths joining the exceptiona fibres to the base point of $\pi_1(M)$.) Let M' denote the induced 2-fold covering space of M. Then there are at most four exceptional fibres in the lifted Seifert fibring of M', namely the lifts F'_1 and F'_2 of F_1 and F_2 respectively, with two lifts $\tilde{F}_3$ and $\overline{F}_3$ of F_3. Clearly the multiplicities of the these lifted fibres are $\frac{m_1}{2}$, $\frac{m_2}{2}$, m_3 and m_3 respectively.

If neither m_1 nor m_2 is equal to 2, then it follows immediately that M' is a Haken 3-manifold with an embedded incompressible vertical torus, by Theorem 1. We can clearly test to see if $H^1(M, Z_2)$ is non zero, find all such possible double coverings M' and decide if any are a Haken Seifert fibre space, using [19] or [21]. We summarise the argument for the benefit of the reader. In these papers, there is an algorithm to find an embedded incompressible (vertical) torus, if one exists. Splitting M along such a torus, when one is found, we want to check if the two pieces are Seifert fibred with boundary fibrings which match up. The methods of [19] or [21] can be used to find properly embedded (vertical) incompressible annuli in the two pieces, which are not isotopic into the boundary torus, if such annuli exist. If we eventually end up with (vertical) solid tori, after a finite number of steps findiing such annuli and splitting along them, then we can verify if the original manifold admits a Seifert fibring. An obstruction is if two Seifert fibred spaces are glued along a boundary annulus or torus so that the Seifert fibrings do not match. Consequently, we can decide if any of the double coverings is a Seifert fibre space or none of them are.

If some double covering of M is Seifert fibred, then it follows by Theorem 2 that M must be a Seifert fibre space, as required. So this completes the argument if neither of m_1 nor m_2 is equal to 2.

If say m_1 is equal to 2 but $m_2 > 2$, then although M' has three exceptional fibres, $H^1(M', Z_{m_3})$ is non zero. So we can find a cyclic covering M^* of index m_3 of M' which has m_3 exceptional fibres. So again we are done unless M is a $(2, 3, 4)$ Seifert fibre space. This follows since $m_3 > 3$ implies that M^* will be a Haken Seifert fibre space and we can test all double coverings M' to see if any have non zero $H^1(M', Z_{m_3})$. We are again using Theorem 2.

Finally we must deal with the case of finite fundamental group, namely the binary octahedral spaces with $(2, 3, 4)$ multiplicities of exceptional fibres. (The case of multiplicities $(2, 2, n)$ is that of prism manifolds and these have been discussed previously). Here we can proceed using the result of [28], that there is a regular dihedral cover of order 6 by a prism manifold, for such Seifert fibre spaces. We can clearly use a similar approach to the previous paragraph, to test for such a covering space. Prism manifolds have been discussed before. Moreover by [28], once such a prism manifold covering has been found, then it follows that the original manifold is a binary octahedral space. This completes the discussion of Case 1.

Case 2. One exceptional fibre of even multiplicity

We use the previous notation, namely M has three exceptional fibres F_1, F_2, F_3 with multiplicities m_1, m_2, m_3 with m_1 even and m_2, m_3 odd. Let r denote rotation by π along the Seifert fibring (where an ordinary fibre has length 2π and an exceptional fibre of multiplicity p has length $\frac{2\pi}{p}$.) Let g denote the canonical involution associated to a genus two vertical Heegaard splitting of M. Let Γ be the fixed set of g and let Γ' be the 3-bridge knot or link given by the projection of Γ to the 3-sphere, viewed as the orbit space of g. We suppose that the knot or link Γ' has been found, as has also the Heegaard surface S and the involution g.

It is well known that r and g must commute. In fact, a convenient way of showing this is to view the exceptional fibres as projecting to points on the equator of the orbit 2-sphere. We can then describe g by a map which interchanges the Northern and Southern hemispheres of the 2-sphere, i.e. regular fibres projecting to one hemisphere are mapped to regular fibres projecting to the other one, by g. All fibres projecting to the equator are then mapped to themselves under g but with orientations reversed. So g has exactly two fixed points on each such fibre. It is now obvious that r and g do commute. We can now let r' denote the involution given by the induced action of r on $S^3 = M/g$, with C and C' being the fixed sets of r and r' respectively.

The key to this case is to describe an algorithm which finds r' and C' from g and Γ'. Notice that here $C = F_1$, since rotation by π along the regular fibres, will fix precisely the points of any exceptional fibre of even multiplicity. So, once r' and C' have been found, we can lift them to r and C. The latter is one of the exceptional fibres of M. We can then test if M is a Seifert fibre space by checking if $M \setminus C$ is Seifert fibred, which can be done exactly as in Case 1 above. Note that Case 1 could have also been done by this method, in that we would have found two or all three of the exceptional fibres as fixed set of r. However, the method given for Case 1 is simpler to implement and more efficient computationally.

Now the isotopy class of r' acting on $Q = S^3 \setminus \Gamma'$ can be found by using the hyperbolic geometry library of algorithms SNAPPEA, of J. Weeks ([39]). By Thurston [36], either there is a complete hyperbolic metric of finite volume or a Seifert structure on Q. The reason is that a Montesinos knot or link complement is either atoroidal or is a torus knot or link complement. In the first case there is a hyperbolic structure and in the second, we can proceed as in Case 1 to find the Seifert structure. In the latter case, it follows that Q is obtained by gluing two solid tori together and r' acts as a standard rotation of each solid torus with two unknotted arcs of fixed points. So it is easy to find C' as required.

Finally, in the hyperbolic case, by Mostow rigidity, there is a hyperbolic isometry of order 2 in the isotopy class of r'. Now this involution will preserve the Ford domain $\mathbb{D}$ of the cusps of Q and using SNAPPEA, the ideal polyhedron $\mathbb{D}$ can be constructed. The involution must induce a permutation of the faces of $\mathbb{D}$ and so can be found. Note that Epstein and Penner [9] prove the existence and uniqueness of this canonical ideal cell decomposition of Q.

4. All exceptional fibres of odd multiplicity

If M has no exceptional fibres of even multiplicity, the rotation r by π along the Seifert fibring has no fixed points and commutes with the involution g associated with a vertical Heegaard splitting, as explained in Case 2 above. So there is an induced involution r' on $S^3 = M/g$ as before. Consequently r' has an unknotted circle C' of fixed points, which is disjoint from the knot or link Γ', using our previous notation. In fact, if C' intersected Γ', then r would have fixed points in M. Moreover, it is well known by an old result of Livesay that if there is an involution on S^3 with a circle of fixed points, then the circle must be unknotted, i.e bounds an embedded disk. So Γ' is called a periodic knot or link.

We can use exactly the same procedure as in Case 2 of the previous section to decide if any given genus 2 Heegaard splitting S of M has an associated involution g, so that the projection of the fixed set of g to S^3 is a periodic knot or link. Lifting the symmetry r' of the projected branch set will give a free involution r of M. We can then quotient M by this involution to give a new manifold $M_1 = M/r$. Repeating, it can be checked if M_1 has Heegaard genus 2 and the procedure can be iterated. We produce a sequence $M_1, ..., M_k$ of manifolds all covered by M with respective degrees $2, ..., 2^k$.

If at any stage, the process stops, then we can conclude that M is not a small Seifert fibre space with all exceptional fibres having odd multiplicity. On the other hand, if M is such a space, then the regular covering projection from M to M_k will have a cyclic covering transformation group of order 2^k. On the other hand, from the next section, it follows that there is an explicit bound n to the number of inequivalent genus 2 splittings of M. Hence once $2^k > n$, then some non trivial covering translation r_i will have the property that $r_i(S)$ is isotopic to S, where S is a fixed genus 2 splitting of M.

Now we can iterate this process and find a sequence $r_{i_1}, ..., r_{i_j}$ of covering translations of M (of increasing orders) which all have the property that they are isotopic to maps $\phi_1, ..., \phi_j$ which map S to itself. Now as is well known, g is central in the mapping class group of S. Therefore, the maps ϕ_m restricted to S can be assumed to commute with g. Moreover then the same can be assumed for the maps ϕ_m on the handlebodies H_+, H_- and so ϕ_m induces a map ${\phi_m}'$ on $Q = S^3 \setminus \Gamma'$. since the latter manifold can be assumed hyperbolic, as discussed previously, it has a finite mapping class group which can be determined by SNAPPEA, using the uniqueness of the Ford domain.

We conclude that by choosing a product $r_{i_a} r_{i_b}$ or $r_{i_a} {r_{i_b}}^{-1}$ if necessary, we can find a covering transformation r^* which is isotopic to the identity of M by a map $F : M \times I \to M$. The induced isotopy $F' : M/ < r^* > \times I \to M/ < r^* >$ is an isotopy from the identity to the identity. Moreover the basepoint of $M/ < r^* >$ must traverse a non contractible loop α under the action of F', since this loop lifts to a path in M. We conclude that the homotopy class of α is a non trivial central element in $\pi_1(M/ < r^* >)$ and so the quotient manifold and hence M are Seifert fibred spaces as required, by [7] or [10].

5. Finding genus two Heegaard splittings

In this section, we describe an improved algorithm to find genus two Heegaard splittings of an arbitrary closed orientable triangulated 3-manifold M. The first step is to summarise facts about normal and almost normal surfaces and 0-efficient triangulations. Then we show how to express the least weight almost normal class of such a Heegaard surface S as a bounded weight solution of a suitable projective solution space.

Definition 5.1. A normal surface L is a closed surface mapped into M so that L is a finite union of elementary disks. Each such disk is properly embedded in a tetrahedron of the triangulation and has either 3 or 4 boundary arcs, i.e is a triangle or quadrilateral. FInally each boundary arc is embedded in a different face of the tetrahedron.

Note that tetrahedra need not be embedded, i.e there can be self-identifications of vertices, edges or faces. The same is true for elementary disks and moreover normal surfaces may be embedded, immersed or can have branch points. We will mainly be interested in embedded surfaces. Also elementary disks can be moved around by proper isotopies and we call such isotopy classes, normal disk-types. A very simple example is a vertex linking embedded normal 2-sphere S_v, which is obtained by the union of all the triangular elementary disks cutting off a given vertex v of the triangulation.

Definition 5.2. A closed orientable 3-manifold M has a 0-efficient triangulation, if there are no embedded normal projective planes and there is a single embedded normal 2-sphere, which is vertex linking for the unique vertex in the triangulation.

The main theorem of [20] is then an algorithmic procedure to convert any given triangulation of a closed orientable irreducible 3-manifold into a 0-efficient triangulation.

There is a natural way of adding normal surfaces.

Definition 5.3. If L_1, L_2 are two normal surface, their Haken sum is the surface $L_1 + L_2 = L_1 \cup L_2$. A curve C of $L_1 \cap L_2$ is called regular if none of the elementary disk pairs meeting along C are pairs of incompatible quadrilaterals, i.e quadrilaterals with edges on different faces of a tetrahedron. If C is a regular curve, then a regular exchange is uniquely defined along C and is given by a standard cut-and-paste between the elementary disks, so as to give new elementary disks.

Definition 5.4. A normal surface L is called fundamental if it cannot be written as a sum $L = L_1 + L_2$, for non-empty normal surfaces L_1, L_2.

Normal surfaces can be conveniently described by their coordinates, i.e a $7t$ vector of non-negative integers, where each coordinate gives the number of a particular elementary disk-type and t is the number of tetrahedra. For there are clearly $4t$ elementary triangular disks and $3t$ elementary quadrilaterals. Haken sum is then just usual vector addition. The weight of a normal surface is just the sum of all the coordinates.

An important observation of Haken is that normal surfaces form a cone in the integer lattice $\mathbb{Z}^{7t}$. The reason is that normal surfaces are defined by a homogeneous system of $6t$ linear equations, called the compatibility equations. For on one side of a face, if one picks out a given isotopy class of arc joining two given edges, then the total number of triangular and quadrilateral disks on one side of this face with the given arc in their boundaries, must equal the total number of elementary disks on the other side of the face with the same arc-type in their boundaries. (If there is self-identification of faces, then possibly some elementary disks might have two copies of the same arc-type in their boundary, in which case each such disk will contribute two to the sum).

Now the solution space of the compatibility equations is a vector subspace of $\mathbb{R}^{7t}$. The normal surfaces correspond then to non-negative integer solutions and so lie in a cone in the non-negative octant of $\mathbb{R}^{7t}$. Finally Haken introduced the idea of adding an extra equation, requiring that the sum of all the coordinates of a vector should be 1. Adding this converts integer solutions into rational ones and it is simple to convert backwards. The compatibility equations plus this extra equation and the non-negativity constraints on coordinates give the so-called projective solution space $\mathbb{P}$, which is a compact convex polytope in $\mathbb{R}^{7t}$. It turns out, for a single vertex 0-efficient triangulation, that the dimension of $\mathbb{P}$ is $2t$, but we will not need this fact.

Normal surfaces can be formed by shrinking, i.e isotopy of embedded surfaces so as to decrease the number of intersections with the faces and edges of the triangulation. In particular there are well-known moves, such as a disk compression where a surface meets a face in a simple closed curve. Another important move is when there is an arc of intersection of a surface and a face, so that the arc has both ends on the same edge, forming a bigon. In this case, the arc can be pushed across the edge, reducing the number of intersections of the surface and the edge by 2. In [20], various barriers are described, which are surfaces and subcomplexes with the property that if a disjoint surface is shrunk to a normal surface, then the shrinking process can be done in the complement of the barrier.

Almost normal surfaces arise from sweepouts. A sweepout is an isotopy of a Heegaard surface S across M, starting and ending at a graph, which can be viewed as a spine of the two handlebodies H_+, H_- bounded by S. A more complex process than shrinking is done to the sweepout, resulting in an isotopy of the Heegaard surface into minimax form. Before describing this, we need to comment on the definition of strong irreducibility of Heegaard splittings, due to Casson Gordon. This is the remarkable property that any essential compressing disk for H_+ must intersect any essential compressing disk for H_-. (An essential compressing disk is one which is not isotopic into S). Then by [26], [27]. [33], if S is a strongly irreducible Heegaard splitting of M, then S is isotopic to an almost normal surface.

Definition 5.5. An almost normal surface S is a union of elementary triangular and quadrilateral disks except for a single piece which is either a properly embedded octagon or a properly embedded annulus in a tetrahedron. The octagon has every boundary arc running between two different edges (before self identification of the tetrahedron) and similarly the annulus has both boundary curves being the same as the boundary of an elementary triangular or quadrilateral disk. Finally there is a boundary compression of the annulus, i.e an embedded disk with one boundary arc on the annulus and running between the two boundary curves and the other boundary arc on the boundary of the tetrahedron.

Next we describe the algorithm to find a genus 2 Heegaard splitting S of M, with a 0-efficient triangulation. Notice we now may assume that M is irreducible, atoroidal and is not a lens space, the 3-sphere or a prism manifold. By [26] or [33], S is isotopic to an almost normal surface, since it is easy to see that S is strongly irreducible. (If S were weakly reducible, then M would be a connected sum of lens spaces and so would be reducible, contrary to assumption). This process is algorithmic and can be summarised as follows; start with any sweepout of M and decrease the number of intersections with the triangulation to achieve a least (global) maximum in the family of surfaces. The resulting minimax surface(s) will be almost normal.

The conclusion is that once S has been isotoped to be almost normal, then all the components of intersection of S and the tetrahedra of the triangulation are normal triangles, quadrilaterals or one special piece which is either an octagon or two normal disks with a boundary parallel tube attached. In the latter case, S is a normal torus with a tube. If we can bound the weight of the normal torus, then there are finitely many possible choices for it. Hence we can check all of these and all possible boundary parallel tubes and see if any give Heegaard surfaces.

In the latter case, we have an octagon and can form the normal solution space consisting of vectors of normal triangles, quadrilaterals and multiples of this particular octagon. It is easy to see (cf [18], [11],[19]) that normal surface techniques carry over into this setting. So there is a cone consisting of non negative solutions of the compatibility equations in the non negative octant. Intersecting this cone with the hyperplane of vectors whose coordinates sum to one, gives a compact convex polyhedron called the projective solution space $\mathbb{P}_A$ of almost normal surfaces. again the key issue is to find a bound on the weight of almost normal genus 2 surfaces, which are in the isotopy classes of Heegaard splittings.

Now a standard argument of Haken (cf for example [19]), shows that there are finitely many fundamental solutions in $\mathbb{P}_A$ and these can be found algorithmically and have bounded weight. The definition of fundamental is the same using almost normal surfaces as for normal surfaces. Now if S is isotopic to a fundamental solution in $\mathbb{P}_A$, then this is of bounded weight as required. So the key problem remaining is when the least weight almost normal surface in the isotopy class of S is not fundamental. The situation when S is isotopic to a normal torus T with an unknotted tube attached is siimilar. In that case, if T is fundamental in $\mathbb{P}$, then its weight is bounded as desired. So again we need to study the case when the least weight T is not fundamental.

Now to establish the claim, assume that the least weight almost normal class of S (or normal class of T) can be decomposed into a Haken sum of normal surfaces U and V. We focus on the case S; the case of T is dealt with by a very similar argument.

Notice that Euler characteristics add under Haken summation, so either;
(1) $\chi(U) = -2$ and $\chi(V) = 0$
(2) $\chi(U) = \chi(V) = -1$

(3) $\chi(U) = 1$ and $\chi(V) = -3$
(4) $\chi(U) = 2$ and $\chi(V) = -4$, without loss of generality.

In case (2), both classes U and V represent non orientable surfaces of genus 3. We claim that if a pair of normal and almost normal surfaces of this type are combined to form an embedded orientable surface S of genus 2, then there is a closed non orientable surface K disjoint from S. For the surfaces U, V must be homologous (as Z_2 cycles) and then meet along one or three orientation reversing loops. The orientable surface S constructed by all regular cut-and-pastes is disjoint from both non orientable surfaces U', V' formed by cut-and-pasting U, V along only 2-sided loops of intersection. So S is not a Heegaard surface.

In case (3), U represents a normal or almost normal real projective plane. By 0-efficiency, we see that $M = RP^3$ and this case has been excluded. (We can shrink the boundary 2-sphere of a small regular neighbourhood of U to the vertex of the triangulation. Here U is a barrier surface).

In the case (4), U represents a normal or almost normal 2-sphere. Since the triangulation is 0-efficient, there is only the single vertex linking normal 2-sphere which could be this class. But then the sum of U and V is not connected, so this case does not occur.

Finally in case (1), we need to follow the more complicated argument in [26]. Here V could be a torus or Klein bottle. The Klein bottle can be ruled out, since M is assumed to be atoroidal, then M must be a prism manifold, which has been excluded (cf [29]). So V is an almost normal or normal torus. Now in [26], it is shown that there are a bounded number of such tori or almost normal summands of a least weight almost normal representative of S. Moreover here we are assuming that every such summand is itself fundamental. Hence we see that there is indeed a bound on the weight of S as required.

The case of a normal torus T with an unknotted tube attached follows exactly the same pattern and the discussion is now complete.

Conjecture 5.1. *We conjecture that the class of a least weight genus* 2 *almost normal surface representing a Heegaard splitting is actually fundamental in the almost normal projective solution space. Similary if the least weight representative is a normal torus with an unknotted tube attached, then the torus should be fundamental. Possibly, the class of the surface might be a vertex solution of the polytope* $\mathbb{P}_A$ *or* $\mathbb{P}$ *respectively.*

Remark 5.1. We can view the above as also an algorithm to recognise if a given knot or link Γ' in S^3 is Montesinos. For we can build the branched double cover M of S^3 over Γ' and test if it is Seifert fibred or not. By results of Meeks Scott [24], all involutions on Seifert fibred spaces are shown to be fibre preserving. So once we know that M is Seifert fibred, it is easy to decide if Γ' is Montesinos or not. (There are various easy cases to exclude, such as the case of an involution coming from a horizontal Heegaard splitting.)

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