Turk J Math
27 (2003) , 481 – 507.

On Positive Solutions of Boundary Value Problems for Nonlinear Second Order Difference Equations

N. Aykut and G. Sh. Guseinov

Abstract

In this paper we study nonlinear second order difference equations subject to separated linear boundary conditions. Sign properties of the associated Green's functions are investigated and existence results for positive solutions of the nonlinear boundary value problem are established. Upper and lower bounds for these positive solutions also are given.

Key Words: Difference equations, boundary value problems, positive solutions, Green's function, fixed point theorem in cones.

1. Introduction

Positive solutions of operator equations in Banach spaces are investigated in the monographs by Krasnosel'skii [11] and Guo and Lakshmikantham [9] making use of the theory of operators acting in Banach spaces with a cone and leaving this cone invariant. The significance of this investigation is due to the fact that in analysing nonlinear phenomena many mathematical models give rise to problems for which only nonnegative solutions make sense. In [11] and subsequent studies (see, [9, 2]) the idea of the method was used to prove the existence of positive solutions of nonlinear ordinary and partial differential equations, integral and integro-differential equations, and difference equations. At the beginning, a main example investigated by the cone method was the nonlinear boundary value problem (BVP)

AMS No.: 39A10.

$$-y'' = f(x, y), \quad x \in [a, b], \tag{1.1}$$

$$y(a) = y(b) = 0. \tag{1.2}$$

Later in [7, 8] instead of the simple boundary conditions of (1.2), general separated linear boundary conditions

$$\alpha y(a) - \beta y'(a) = 0, \quad \gamma y(b) + \delta y'(b) = 0 \tag{1.3}$$

were explored and the existence of positive solutions for the BVP (1.1), (1.3) was studied by a similar method. In the sequel, a discrete analogue of the BVP (1.1), (1.3) was considered. To formulate it, let a, b ($a < b$) be integers and let $[a, b]$ denote the discrete segment being the set $\{a, a+1, ..., b\}$. In [12] the existence of positive solutions is studied for the discrete BVP

$$-\triangle^2 y(t-1) = f(t, y(t)), \quad t \in [a, b], \tag{1.4}$$

$$\alpha y(a-1) - \beta \triangle y(a-1) = 0, \quad \gamma y(b) + \delta \triangle y(b) = 0, \tag{1.5}$$

where $\{y(t)\}_{t=a-1}^{b+1}$ is a desired solution and $\triangle$ denotes the forward difference operator defined by

$$\triangle y(t) = y(t+1) - y(t). \tag{1.6}$$

In [13, 6], in the BVP (1.4), (1.5) the equation

$$-\triangle[p(t-1)\triangle y(t-1)] = f(t, y(t)), \quad t \in [a, b], \tag{1.7}$$

is used. Further, in [4], the existence of positive solutions is examined for the more general equation

$$-\triangle[p(t-1)\triangle y(t-1)] + q(t)y(t) = f(t, y(t)), \quad t \in [a, b], \tag{1.8}$$

under the periodic boundary conditions

$$y(a-1) = y(b), \quad p(a-1)\triangle y(a-1) = p(b)\triangle y(b). \tag{1.9}$$

Notice that (1.9) is an important representative of nonseparable linear boundary conditions.

In this paper we investigate the existence of positive solutions for Equation (1.8) subject to the boundary conditions (1.5). So our problem is more general than both the problem (1.4), (1.5) and the problem (1.7), (1.5) as studied in [12, 13, 6]. Besides, what is more essential, our results in this paper are different from those in [12, 13, 6].

The paper is organized as follows. In Section 2 we give some needed facts about second order linear difference equations. Here, a uniqueness and existence theorem is presented, and a variation of constants formula for the nonhomogeneous equation is given.

In Section 3 the Green's function of the linear boundary value problem is constructed and in some particular cases the Green's function is explicitly calculated.

In Section 4 sign properties of the Green's function are investigated and some inequalities for it are proved.

In the last Section 5 existence results and upper and lower bounds for positive solutions of the BVP (1.8), (1.5) are established.

Finally, for easy reference, we state here the fixed-point theorem [11, p. 148; 9, p. 94] which is employed in this paper.

Theorem 1.1 (Krasnosel'skii Fixed Point Theorem) *Let $\mathcal{B}$ be a Banach space, and let $\wp \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$A : \wp \bigcap (\overline{\Omega}_2 \backslash \Omega_1) \to \wp$$

be a completely continuous operator such that either

- *(i)* $\|Ay\| \leq \|y\|, \quad y \in \wp \cap \partial\Omega_1,$ *and* $\|Ay\| \geq \|y\|, \quad y \in \wp \cap \partial\Omega_2;$

  or

- *(ii)* $\|Ay\| \geq \|y\|, \quad y \in \wp \cap \partial\Omega_1,$ *and* $\|Ay\| \leq \|y\|, \ y \in \wp \cap \partial\Omega_2.$

Then A has at least one fixed point in $\wp \bigcap (\overline{\Omega}_2 \backslash \Omega_1)$.

2. Second Order Linear Difference Equations

The theorems given in this section either are in the references [1, 3, 10] or are not difficult to verify.

Let $\mathbf{Z}$ denote the set of all integers. Consider on $\mathbf{Z}$ the second order linear homogeneous difference equation given by

$$-\triangle[p(t-1)\triangle y(t-1)] + q(t)y(t) = 0, \tag{2.1}$$

where $\triangle$ denotes the forward difference operator defined by (1.6). In equation (2.1) the coefficients $p(t)$, $q(t)$ are complex-valued functions defined on $\mathbf{Z}$ and $p(t) \neq 0$ for all $t \in \mathbf{Z}$; $y(t)$ is a desired solution.

For brevity let us set

$$y^{[\triangle]}(t) = p(t)\triangle y(t),$$

that is called the quasi $\triangle$-derivative of $y(t)$.

Theorem 2.1 *Let t_0 be a fixed point in $\mathbf{Z}$ and c_0, c_1 be given constants. Then equation* (2.1) *has a unique solution y such that*

$$y(t_0) = c_0, \quad y^{[\triangle]}(t_0) = c_1.$$

For two functions $y, z : \mathbf{Z} \to \mathbf{C}$ we define their $Wronskian$ by

$$W_t(y, z) = y(t)z^{[\triangle]}(t) - y^{[\triangle]}(t)z(t)$$

for $t \in \mathbf{Z}$.

Theorem 2.2 *The Wronskian of any two solutions of equation* (2.1) *is independent of t.*

Corollary 2.1 *If y and z are both solutions of Eq.*(2.1)*, then either $W_t(y, z) = 0$ for all $t \in \mathbf{Z}$, or $W_t(y, z) \neq 0$ for all $t \in \mathbf{Z}$.*

Theorem 2.3 *Any two solutions of Eq.*(2.1) *are linearly independent if and only if their Wronskian is nonzero.*

Theorem 2.4 *Eq.* (2.1) *has two linearly independent solutions and every solution of Eq.* (2.1) *is a linear combination of these solutions.*

We say that y_1 and y_2 form *a fundamental set* (or a *fundamental system*) of solutions for Eq. (2.1) provided their Wronskian is nonzero.

Let us consider the nonhomogeneous equation

$$-\triangle[p(t-1)\triangle y(t-1)]+q(t)y(t)=h(t), \tag{2.2}$$

where $h:\mathbf{Z}\to\mathbf{C}$ is a given complex-valued function.

Theorem 2.5 *Suppose that y_1 and y_2 form a fundamental set of solutions of the homogeneous Equation* (2.1) *and $\omega=W_t(y_1,y_2)$. Then the general solution of the nonhomogeneous Equation* (2.2) *is given by*

$$y(t)=c_1y_1(t)+c_2y_2(t)+\frac{1}{\omega}\sum_{s=t_0}^{t-1}[y_1(t)y_2(s)-y_1(s)y_2(t)]h(s), \tag{2.3}$$

where t_0 is a fixed point in $\mathbf{Z}$, and c_1 and c_2 are arbitrary constants.

3. Boundary Value Problems and Green's Functions

Let $a<b$ be fixed points in $\mathbf{Z}$ and let $[a,b]$ denote the discrete interval being the set $\{a,a+1,...,b\}$. Consider the following boundary value problem (BVP):

$$-\triangle[p(t-1)\triangle y(t-1)]+q(t)y(t)=h(t),\quad t\in[a,b], \tag{3.1}$$

$$\alpha y(a-1)-\beta y^{[\triangle]}(a-1)=0,\quad \gamma y(b)+\delta y^{[\triangle]}(b)=0, \tag{3.2}$$

where $\alpha,\beta,\gamma,\delta$ are complex constants such that $\mid\alpha\mid+\mid\beta\mid\neq0$ and $\mid\gamma\mid+\mid\delta\mid\neq0$. The functions $p(t),q(t)$ and $h(t)$ are as in Section 2.

Note that each solution $y(t)$ of Eq. (3.1) must be a function defined on $[a-1, b+1]$. Since

$$y(b) = y(b+1) - \triangle y(b),$$

the boundary conditions (3.2) (in particular), when

$$\beta = 0, \quad \delta p(b) = \gamma,$$

will take the form

$$y(a-1) = 0, \quad y(b+1) = 0,$$

and are called the conjugate (or Dirichlet) boundary conditions.

Turning to the BVP (3.1), (3.2), denote by $\varphi(t)$ and $\psi(t)$ the solutions of the corresponding homogeneous equation

$$-\triangle[p(t-1)\triangle y(t-1)] + q(t)y(t) = 0, \quad t \in [a, b], \tag{3.3}$$

under the initial conditions

$$\varphi(a-1) = \beta, \quad \varphi^{[\triangle]}(a-1) = \alpha; \tag{3.4}$$

$$\psi(b) = \delta, \quad \psi^{[\triangle]}(b) = -\gamma; \tag{3.5}$$

so that φ and ψ satisfy the first and the second conditions of (3.2), respectively. Let us set

$$D = -W_t(\varphi, \psi) = \varphi^{[\triangle]}(t)\psi(t) - \varphi(t)\psi^{[\triangle]}(t). \tag{3.6}$$

Since the Wronskian of any two solutions of Eq. (3.3) is independent of $t \in [a-1, b]$, taking in (3.6), $t = a-1$ and $t = b$ we find , because of (3.4) and (3.5),

$$D = \alpha\psi(a-1) - \beta\psi^{[\triangle]}(a-1) = \gamma\varphi(b) + \delta\varphi^{[\triangle]}(b). \tag{3.7}$$

According to Theorem 2.3, $D \neq 0$ if and only if φ and ψ are linearly independent. The following theorem describes the condition $D \neq 0$ from another point of view.

Theorem 3.1 *$D \neq 0$ if and only if the homogeneous Equation* (3.3) *has only trivial solution satisfying the boundary conditions* (3.2).

Proof. If $D = 0$, then by virtue of (3.4) and (3.7), $\varphi(t)$ will be a nontrivial solution of the BVP (3.3), (3.2). Let us now assume that $D \neq 0$. Then φ and ψ will form a fundamental set of solutions of Eq.(3.3) and therefore any solution of the BVP (3.3), (3.2) will have the form

$$y(t) = c_1\varphi(t) + c_2\psi(t),$$

where c_1, c_2 are constants. Substituting this expression of $y(t)$ into the boundary conditions (3.2) and taking into account that by (3.4), (3.5) $\varphi(t)$ satisfies the first and $\psi(t)$ the second conditions of (3.2), we get

$$c_2[\alpha\psi(a-1) - \beta\psi^{[\triangle]}(a-1)] = 0, \quad c_1[\gamma\varphi(b) + \delta\varphi^{[\triangle]}(b)] = 0,$$

or by (3.7) $c_2 D = 0$, $c_1 D = 0$. Hence $c_1 = c_2 = 0$, that is, the solution $y(t)$ is trivial. This completes the proof. □

Theorem 3.2 *If $D \neq 0$, then the nonhomogeneous BVP* (3.1), (3.2) *has a unique solution $y(t)$ for which the formula*

$$y(t) = \sum_{s=a}^{b} G(t,s)h(s), \quad t \in [a-1, b+1] \tag{3.8}$$

holds, where

$$G(t,s) = \frac{1}{D}\begin{cases} \varphi(t)\psi(s), & a-1 \le t \le s \le b+1, \\ \\ \varphi(s)\psi(t), & a-1 \le s \le t \le b+1 \end{cases} \tag{3.9}$$

and $G(t,s)$ is called the Green's function of the BVP (3.1), (3.2).

Proof. Under the condition $D \neq 0$, the solutions $\varphi(t)$ and $\psi(t)$ of the homogeneous Eq. (3.3) are linearly independent and, therefore by Theorem 2.5, the general solution of the nonhomogeneous Eq. (3.1) has the form

$$y(t) = c_1\varphi(t) + c_2\psi(t) + \frac{1}{D}\sum_{s=a}^{t}[\varphi(s)\psi(t) - \varphi(t)\psi(s)]h(s), \quad t \in [a-1, b+1], \tag{3.10}$$

where c_1 and c_2 are arbitrary constants. Now we try to choose the constants c_1 and c_2 so that the function $y(t)$ satisfies also the boundary conditions (3.2).

From (3.10) we have

$$y^{[\triangle]}(t) = c_1\varphi^{[\triangle]}(t) + c_2\psi^{[\triangle]}(t) + \frac{1}{D}\sum_{s=a}^{t}[\varphi(s)\psi^{[\triangle]}(t) - \varphi^{[\triangle]}(t)\psi(s)]h(s). \tag{3.11}$$

Consequently

$$y(a-1) = c_1\varphi(a-1) + c_2\psi(a-1) = c_1\beta + c_2\psi(a-1),$$

$$y^{[\triangle]}(a-1) = c_1\varphi^{[\triangle]}(a-1) + c_2\psi^{[\triangle]}(a-1) = c_1\alpha + c_2\psi^{[\triangle]}(a-1).$$

Substituting these values of $y(a-1)$ and $y^{[\triangle]}(a-1)$ in the first condition of (3.2) we get

$$c_2[\alpha\psi(a-1) - \beta\psi^{[\triangle]}(a-1)] = 0.$$

On the other hand, by (3.7)

$$\alpha\psi(a-1) - \beta\psi^{[\triangle]}(a-1) = D \neq 0.$$

Consequently $c_2 = 0$ and (3.10), (3.11) take the forms

$$y(t) = c_1\varphi(t) + \frac{1}{D}\sum_{s=a}^{t}[\varphi(s)\psi(t) - \varphi(t)\psi(s)]h(s), \tag{3.12}$$

$$y^{[\triangle]}(t) = c_1\varphi^{[\triangle]}(t) + \frac{1}{D}\sum_{s=a}^{t}[\varphi(s)\psi^{[\triangle]}(t) - \varphi^{[\triangle]}(t)\psi(s)]h(s).$$

Hence

$$y(b) = c_1\varphi(b) + \frac{1}{D}\sum_{s=a}^{b}[\delta\varphi(s) - \varphi(b)\psi(s)]h(s),$$

$$y^{[\triangle]}(b) = c_1\varphi^{[\triangle]}(b) + \frac{1}{D}\sum_{s=a}^{b}[-\gamma\varphi(s) - \varphi^{[\triangle]}(b)\psi(s)]h(s),$$

and substituting these values of $y(b)$ and $y^{[\triangle]}(b)$ in the second condition of (3.2) we get

$$c_1[\gamma\varphi(b) + \delta\varphi^{[\triangle]}(b)] - \frac{\gamma\varphi(b) + \delta\varphi^{[\triangle]}(b)}{D}\sum_{s=a}^{b}\psi(s)h(s) = 0.$$

Since by (3.7)

$$\gamma\varphi(b) + \delta\varphi^{[\triangle]}(b) = D \neq 0,$$

hence

$$c_1 = \frac{1}{D}\sum_{s=a}^{b}\psi(s)h(s).$$

Putting this value of c_1 in (3.12) we obtain

$$y(t) = \frac{1}{D}\sum_{s=a}^{t}\varphi(s)\psi(t)h(s) + \frac{1}{D}\sum_{s=t+1}^{b}\varphi(t)\psi(s)h(s),$$

that is, formulas (3.8), (3.9) hold. □

It can be verified without difficulty that for the solution $y(t)$ of the nonhomogeneous equation (3.1) under the nonhomogeneous boundary conditions

$$\alpha y(a-1) - \beta y^{[\triangle]}(a-1) = d_1, \quad \gamma y(b) + \delta y^{[\triangle]}(b) = d_2$$

the formula

$$y(t) = w(t) + \sum_{s=a}^{b} G(t,s)h(s)$$

holds, where the function $G(t,s)$ is defined by (3.9) and

$$w(t) = \frac{d_2}{D}\varphi(t) - \frac{d_1}{D}\psi(t).$$

Consider the following two particular cases when the Green's function $G(t,s)$ of the BVP (3.1), (3.2) can be calculated explicitly.

1. In the BVP (3.1), (3.2) let $p(t) \equiv 1$ and $q(t) \equiv 0$. Then we have

$$\varphi(t) = \beta + \alpha(t-a+1), \quad \psi(t) = \delta + \gamma(b-t),$$

$$D = \alpha\beta + \gamma\delta + \alpha\gamma(b-a+1),$$

$$G(t,s) = \frac{1}{D}\begin{cases} [\beta + \alpha(t-a+1)]\,[\delta + \gamma(b-s)]\,, & a-1 \le t \le s \le b+1, \\ \\ [\beta + \alpha(s-a+1)]\,[\delta + \gamma(b-t)]\,, & a-1 \le s \le t \le b+1. \end{cases}$$

2. Let now $p(t) \neq 0$ be arbitrary and $q(t) \equiv 0$. Then we have

$$\varphi(t) = \beta + \alpha \sum_{k=a-1}^{t-1} \frac{1}{p(k)}, \qquad \psi(t) = \delta + \gamma \sum_{k=t}^{b-1} \frac{1}{p(k)},$$

$$D = \alpha\beta + \gamma\delta + \alpha\gamma \sum_{k=a-1}^{b-1} \frac{1}{p(k)},$$

$$G(t,s)=\frac{1}{D}\begin{cases}\left[\beta+\alpha\sum_{k=a-1}^{t-1}\frac{1}{p(k)}\right]\left[\delta+\gamma\sum_{k=s}^{b-1}\frac{1}{p(k)}\right], & a-1\leq t\leq s\leq b+1,\\ \left[\beta+\alpha\sum_{k=a-1}^{s-1}\frac{1}{p(k)}\right]\left[\delta+\gamma\sum_{k=t}^{b-1}\frac{1}{p(k)}\right], & a-1\leq s\leq t\leq b+1.\end{cases}$$

4. Sign Properties of the Green's Function

Consider the BVP (3.1), (3.2). In this section we assume that

$$p(t)>0,\quad q(t)\geq 0; \tag{4.1}$$

$$\alpha,\beta,\gamma,\delta\geq 0,\quad \alpha+\beta>0,\quad \gamma+\delta>0. \tag{4.2}$$

Let $\varphi(t)$ and $\psi(t)$ be the solutions of the homogeneous Equation (3.3) satisfying initial conditions (3.4) and (3.5), respectively.

Lemma 4.1 *Under the conditions* (4.1), (4.2) *the solutions* $\varphi(t)$ *and* $\psi(t)$ *possess the following properties:*

$$\varphi(t)\geq 0,\quad t\in[a-1,b+1];\quad \psi(t)\geq 0,\quad t\in[a-1,b]; \tag{4.3}$$

$$\varphi(t)>0,\quad t\in[a,b+1];\quad \psi(t)>0,\quad t\in[a-1,b-1]; \tag{4.4}$$

$$\varphi^{[\triangle]}(t)\geq 0,\quad t\in[a-1,b];\quad \psi^{[\triangle]}(t)\leq 0,\quad t\in[a-1,b]. \tag{4.5}$$

Proof. To investigate the properties of the solutions $\varphi(t)$, $\psi(t)$ and their derivatives we deduce the "discrete integral equations " for these functions (see [4, 5]). Summing the equation

$$\triangle[p(\tau-1)\triangle y(\tau-1)]=q(\tau)y(\tau)$$

from a to k, and taking into account the initial condition $\varphi^{[\triangle]}(a-1)=\alpha$, we get

$$\varphi^{[\triangle]}(k)=\alpha+\sum_{\tau=a}^{k}q(\tau)\varphi(\tau), \tag{4.6}$$

or

$$\triangle\varphi(k) = \frac{\alpha}{p(k)} + \frac{1}{p(k)}\sum_{\tau=a}^{k} q(\tau)\varphi(\tau).$$

Hence summing from $a-1$ to $t-1$, and using the initial condition $\varphi(a-1) = \beta$ and the formula

$$\sum_{k=a}^{m} u(k)\sum_{\tau=a}^{k}\vartheta(\tau) = \sum_{\tau=a}^{m}\left[\sum_{k=\tau}^{m} u(k)\right]\vartheta(\tau),$$

we find, for all $t \geq a-1$,

$$\varphi(t) = \beta + \alpha\sum_{k=a-1}^{t-1}\frac{1}{p(k)} + \sum_{\tau=a}^{t-1}\left[\sum_{k=\tau}^{t-1}\frac{1}{p(k)}\right] q(\tau)\varphi(\tau). \tag{4.7}$$

For the solution $\psi(t)$ we can, in a similar way, obtain the equations

$$\psi^{[\triangle]}(k) = -\gamma - \sum_{\tau=k+1}^{b} q(\tau)\psi(\tau), \tag{4.8}$$

$$\psi(t) = \delta + \gamma\sum_{k=t}^{b-1}\frac{1}{p(k)} + \sum_{\tau=t+1}^{b}\left[\sum_{k=t}^{\tau-1}\frac{1}{p(k)}\right] q(\tau)\psi(\tau). \tag{4.9}$$

From (4.7) we have

$$\varphi(a-1) = \beta \geq 0, \quad \varphi(a) = \beta + \frac{\alpha}{p(a-1)} > 0.$$

Therefore from (4.7), step by step, we get

$$\varphi(t) > 0 \text{ for } t = a+1, ..., b+1.$$

Thus, $\varphi(t) \geq 0$ for $t \in [a-1, b+1]$, and $\varphi(t) > 0$ for $t \in [a, b+1]$. Consequently, from (4.6) we get also $\varphi^{[\triangle]}(k) \geq 0$ for $k \in [a-1, b]$. So, the statements of the lemma for φ are proved. The statements of the lemma for ψ can be proved in a similar way using (4.9) and (4.8). Hence the lemma is proved. □

Let us now investigate the sign of the number D defined by (3.7). Setting (4.6), (4.7) and (4.8), (4.9) in (3.7), we find

$$D = \alpha\delta + \beta\gamma + \alpha\gamma \sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \delta q(b)\varphi(b) + \sum_{\tau=a}^{b-1} \left[\delta + \gamma \sum_{k=\tau}^{b-1} \frac{1}{p(k)} \right] q(\tau)\varphi(\tau), \quad (4.10)$$

and

$$D = \alpha\delta + \beta\gamma + \alpha\gamma \sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \sum_{\tau=a}^{b} \left[\beta + \alpha \sum_{k=a-1}^{\tau-1} \frac{1}{p(k)} \right] q(\tau)\psi(\tau). \quad (4.11)$$

Hence, in the case $q(t) \equiv 0$ $(a \leq t \leq b)$, we have

$$D = \alpha\delta + \beta\gamma + \alpha\gamma \sum_{k=a-1}^{b-1} \frac{1}{p(k)} \; . \quad (4.12)$$

From formulas (4.10)–(4.12) the following result follows.

Lemma 4.2 *Under the conditions* (4.1) *and* (4.2)*:*

- *(i) If* $q(t) \neq 0$ *for some* t $(a \leq t \leq b)$ *, then* $D > 0$.
- *(ii) If* $q(t) \equiv 0$ $(a \leq t \leq b)$*, then* $D > 0$ *if and only if* $\alpha + \gamma > 0$.

Now from the formula (3.9) for $G(t, s)$, by Lemmas 4.1 and 4.2, we get the following theorem.

Theorem 4.1 *Let conditions* (4.1) *and* (4.2) *hold. In the case* $q(t) \equiv 0$ $(a \leq t \leq b)$*, let* $\alpha + \gamma > 0$*. Then:*

- *(i)* $G(t, s) \geq 0$ *for* $t, s \in [a-1, b]$.

- *(ii)* $G(t,s) > 0$ *for* $t, s \in [a, b-1]$.
- *(iii) If* $\delta > 0$, *then* $G(t,s) > 0$ *for* $t, s \in [a, b]$.

Theorem 4.2 *Let conditions* (4.1) *and* (4.2) *hold. Besides, in the case* $q(t) \equiv 0 \quad (a \leq t \leq b)$ *let* $\alpha + \gamma > 0$. *Then :*

- *(i)* $0 \leq G(t,s) \leq G(s,s)$ *for* $t, s \in [a-1, b]$ *;*
- *(ii)* $G(t,s) \geq \sigma G(s,s)$ *for* $t \in [a, b-1]$ *and* $s \in [a-1, b]$,

where

$$\sigma = \min\{I_1, I_2\} \tag{4.13}$$

in which

$$I_1 = \left\{\beta + \frac{\alpha}{p(a-1)}\right\}\left\{\beta + \alpha \sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \sum_{\tau=a}^{b-1}\left[\sum_{k=\tau}^{b-1}\frac{1}{p(k)}\right] q(\tau)\varphi(\tau)\right\}^{-1},$$

$$I_2 = \left\{\delta + \frac{\gamma + \delta q(b)}{p(b-1)}\right\}\left\{\delta + \gamma \sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \sum_{\tau=a}^{b}\left[\sum_{k=a-1}^{\tau-1}\frac{1}{p(k)}\right] q(\tau)\psi(\tau)\right\}^{-1}.$$

Proof. By Lemma 4.1, $\varphi(t)$ is nondecreasing and $\psi(t)$ is nonincreasing for $t \in [a-1, b+1]$. Besides $\varphi(t) \geq 0$ for $t \in [a-1, b+1]$, and $\psi(t) \geq 0$ for $t \in [a-1, b]$. Therefore we have, for $a - 1 \leq t \leq s \leq b$,

$$G(t,s) = \frac{1}{D}\varphi(t)\psi(s) \leq \frac{1}{D}\varphi(s)\psi(s) = G(s,s),$$

and we have, for $a - 1 \leq s \leq t \leq b$,

$$G(t,s) = \frac{1}{D}\varphi(s)\psi(t) \leq \frac{1}{D}\varphi(s)\psi(s) = G(s,s).$$

So the statement (i) of the theorem is proved.

If $G(s,s) = 0$ for a given $s \in [a-1,b]$, then the statement (ii) of the theorem is obvious for such values. Let now $s \in [a-1,b]$ and $G(s,s) \neq 0$. Consequently, $G(s,s) > 0$ for all such s. Let us take any $t \in [a, b-1]$. Then we have, for $s \in [a-1,t]$,

$$\frac{G(t,s)}{G(s,s)} = \frac{\psi(t)}{\psi(s)} \geq \frac{\psi(b-1)}{\psi(a-1)} = \frac{\delta + \dfrac{\gamma + \delta q(b)}{p(b-1)}}{\delta + \gamma \displaystyle\sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \sum_{\tau=a}^{b} \left[\sum_{k=a-1}^{\tau-1} \frac{1}{p(k)} \right] q(\tau)\psi(\tau)},$$

and we have for, $s \in [t,b]$,

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(t)}{\varphi(s)} \geq \frac{\varphi(a)}{\varphi(b)} = \frac{\beta + \dfrac{\alpha}{p(a-1)}}{\beta + \alpha \displaystyle\sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \sum_{\tau=a}^{b-1} \left[\sum_{k=\tau}^{b-1} \frac{1}{p(k)} \right] q(\tau)\varphi(\tau)}.$$

The theorem is proved. □

Note that the number σ defined by (4.13) satisfies the inequality $0 < \sigma < 1$.

5. Existence of Positive Solutions

In this section we consider the nonlinear BVP

$$-\triangle[p(t-1)\triangle y(t-1)] + q(t)y(t) = f(t, y(t)), \quad t \in [a,b], \tag{5.1}$$

$$\alpha y(a-1) - \beta y^{[\triangle]}(a-1) = 0, \quad \gamma y(b) + \delta y^{[\triangle]}(b) = 0. \tag{5.2}$$

We will assume that the following conditions are satisfied :

- **(H1)** $p(t) > 0, \quad q(t) \geq 0.$
- **(H2)** $\alpha, \beta, \gamma, \delta \geq 0, \quad \alpha + \beta > 0, \quad \gamma + \delta > 0;$

  if $q(t) \equiv 0$ $(a \leq t \leq b)$, then $\alpha + \gamma > 0$.
- **(H3)** $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ is continuous with respect to ξ and $f(t,\xi) \geq 0$ for

$\xi \in \mathbf{R}^+$, where $\mathbf{R}^+$ denotes the set of nonnegative real numbers.

Regarding BVP (5.1), (5.2) denote by $G(t,s)$ the Green's function of the problem (3.1), (3.2). By the Theorems 4.1 and 4.2 the inequalities

$$0 \le G(t,s) \le G(s,s) \text{ for } t,s \in [a-1,b]; \tag{5.3}$$

$$G(t,s) \ge \sigma G(s,s) \text{ for } t \in [a,b-1] \text{ and } s \in [a-1,b] \tag{5.4}$$

hold, where the number σ is defined by (4.13).

By Theorem 3.2, finding a solution $y(t),\ t \in [a-1,b+1]$ of the BVP (5.1), (5.2) is equivalent to finding a solution $y(t)$ of the summation equation

$$y(t) = \sum_{s=a}^{b} G(t,s) f(s,y(s)), \quad t \in [a-1,b+1].$$

Hence for the solution $y(t),\ t \in [a-1,b+1]$ of the BVP (5.1), (5.2) the equation

$$y(t) = \sum_{s=a}^{b} G(t,s) f(s,y(s)), \quad t \in [a,b] \tag{5.5}$$

holds. Conversely, if a function $y(t),\ t \in [a,b]$ is a solution of Eq. (5.5), then the extension $y(t),\ t \in [a-1,b+1]$ of this function, where

$$y(a-1) = \sum_{s=a}^{b} G(a-1,s) f(s,y(s)), \tag{5.6}$$

$$y(b+1) = \sum_{s=a}^{b} G(b+1,s) f(s,y(s)), \tag{5.7}$$

will be the solution of the BVP (5.1), (5.2).

Thus between solutions of the BVP (5.1), (5.2) and the Equation (5.5), there is a one-to-one correspondence. Consequently, the existence and uniqueness of solution of the

BVP (5.1), (5.2) is equivalent to that for Eq. (5.5).

We investigate Eq. (5.5) in the $b-a+1$ dimensional real Banach space $\mathcal{B}$ of real-valued functions $y(t)$ defined on [a,b] with the norm

$$\| y \| = \max_{a \le t \le b} | y(t) |.$$

Solving Eq. (5.5) in $\mathcal{B}$ is equivalent to finding fixed points of the operator $A : \mathcal{B} \to \mathcal{B}$ defined by

$$Ay(t) = \sum_{s=a}^{b} G(t,s) f(s, y(s)), \quad t \in [a,b] \tag{5.8}$$

From the continuity of $f(t,\xi)$ with respect to ξ it follows that the operator A defined by (5.8) is completely continuous in $\mathcal{B}$.
Let us set

$$\wp = \{ y \in \mathcal{B} \mid y(t) \ge 0 \quad \text{for} \quad t \in [a,b] \}. \tag{5.9}$$

Evidently $\wp$ is a cone in $\mathcal{B}$. Moreover, for all $y \in \wp$, by (5.3) and (H3), we have from (5.8), $Ay(t) \ge 0$ for all $t \in [a,b]$. Therefore the operator A leaves the cone $\wp$ invariant, i.e. $A(\wp) \subset \wp$.

In the next theorem we will use the following well-known Contraction Mapping Theorem named also as the Banach Fixed Point Theorem: *Let E be a Banach space and K be a nonemty closed subset of E. Assume $A : K \to K$ is a contraction, i.e. there is a λ, $0 < \lambda < 1$, such that $\|Ax - Ay\| \le \lambda \|x - y\|$ for all x, y in K. Then A has a unique fixed point in K, that is, a unique point $x_0 \in K$ such that $Ax_0 = x_0$.*

Theorem 5.1 *Assume that conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied. Assume also that the function $f(t,\xi)$ satisfies with respect to ξ the Lipschitz condition*

$$| f(t,\xi_1) - f(t,\xi_2) | \le \pounds \mid \xi_1 - \xi_2 |, \quad \xi_1, \xi_2 \in \mathbf{R},$$

where $\pounds$ is a constant not depending on t, ξ_1, ξ_2. If

$$£\left(\max_{a\le t\le b}\sum_{s=a}^{b}G(t,s)\right)<1,$$

then the BVP (5.1), (5.2) has a unique solution $y(t)$, $t\in[a-1,b+1]$ such that

$$y(t)\ge 0 \text{ for } t\in[a,b].$$

Proof. For all $y,z\in\mathcal{B}$ and $t\in[a,b]$ we have

$$\mid Ay(t)-Az(t)\mid\le\sum_{s=a}^{b}G(t,s)\mid f(s,y(s))-f(s,z(s))\mid$$

$$\le £\sum_{s=a}^{b}G(t,s)\mid y(s)-z(s)\mid\le £\parallel y-z\parallel\sum_{s=a}^{b}G(t,s)$$

$$\le £\parallel y-z\parallel\max_{a\le t\le b}\sum_{s=a}^{b}G(t,s).$$

Hence

$$\parallel Ay-Az\parallel\le\left(£\max_{a\le t\le b}\sum_{s=a}^{b}G(t,s)\right)\parallel y-z\parallel.$$

Therefore applying the contraction mapping theorem to the operator $A:\wp\to\wp$ we get the statement of the theorem. □

To get an existence theorem without uniqueness of solution, we will apply in the next theorem the following Brouwer Fixed Point Theorem: *Let E be a finite dimensional linear normed space and K be a nonempty bounded, closed, and convex subset of E. Assume $A:E\to E$ is a continious (nonlinear, in general) operator. If the operator A leaves the set K invariant, i.e. if $A(K)\subset K$, then A has at least one fixed point in K.*

Theorem 5.2 *Assume that conditions (H1), (H2), and (H3) are satisfied. Assume also that there is a number $R>0$ such that*

$$\left(\max_{a\le t\le b}\sum_{s=a}^{b}G(t,s)\right).\max_{a\le t\le b,0\le\xi\le R}f(t,\xi)\le R.$$

Then the BVP (5.1), (5.2) *has at least one solution* $y(t)$, $t\in[a-1,b+1]$ *such that*

$$0\le y(t)\le R,\quad t\in[a,b].$$

Proof. Let us set

$$\wp_R=\{y\in\mathcal{B}\mid\parallel y\parallel\le R, y(t)\ge 0\quad\text{for all}\quad t\in[a,b]\}.$$

The set $\wp_R$ is bounded, closed, and convex. Moreover the operator A leaves the set $\wp_R$ invariant. Indeed for each $y\in\wp_R$ we have $Ay(t)\ge 0$ and

$$Ay(t)=\sum_{s=a}^{b}G(t,s)f(s,y(s))\le\left(\max_{a\le s\le b,0\le\xi\le R}f(s,\xi)\right)\sum_{s=a}^{b}G(t,s)$$

$$\le\left(\max_{a\le s\le b,0\le\xi\le R}f(s,\xi)\right)\max_{a\le t\le b}\sum_{s=a}^{b}G(t,s)\le R.$$

Hence $\parallel Ay\parallel\le R$ and therefore $Ay\in\wp_R$. Besides the operator A is continuous. Consequently, applying the Brouwer fixed point theorem to the operator A we get the statement of the theorem. □

Notice that the latter condition of Theorem 5.2 is satisfied for sufficiently large R if $\mid f(t,\xi)\mid\le c_1+c_2\mid\xi\mid^{\lambda}$, where c_1,c_2, λ are positive constants and $\lambda<1$.

Now together with the cone $\wp$ defined by (5.9) we define the second cone $\wp_0$ in $\mathcal{B}$ by

$$\wp_0=\{y\in\wp\mid\min_{a\le t\le b-1}y(t)\ge\sigma\parallel y\parallel\},$$

where the number σ is defined by (4.13).

Lemma 5.1 *$Ay\in\wp_0$ for all $y\in\wp$. In particular, the operator A leaves the cone $\wp_0$ invariant, i.e., $A(\wp_0)\subset\wp_0$.*

Proof. For all $y \in \wp$ taking into account (5.3) and (H3) we have from (5.8), $Ay(t) \geq 0$ for all $t \in [a, b]$. Further, making use of (5.4) and (5.3) we have from (5.8),

$$\min_{a \leq t \leq b-1} Ay(t) \geq \sigma \sum_{s=a}^{b} G(s,s) f(s, y(s)) \geq \sigma \max_{a \leq t \leq b} \sum_{s=a}^{b} G(t,s) f(s, y(s)) = \sigma \parallel Ay \parallel.$$

Therefore $Ay \in \wp_0$. The lemma is proved. □

Let us set

$$g = \sum_{s=a}^{b} G(s,s), \qquad g_1 = \sum_{s=a}^{b-1} G(t_0, s), \tag{5.10}$$

where t_0 is any fixed point in $[a, b-1]$.

In the next theorem we also assume the following condition on $f(t, \xi)$.

- **(H4)** There exist numbers $0 < r < R < \infty$ such that for all $t \in [a, b]$:

$$f(t, \xi) \leq \frac{1}{g} r \quad \text{if} \quad 0 \leq \xi \leq r; \qquad f(t, \xi) \geq \frac{1}{\sigma g_1} R \quad \text{if} \quad R \leq \xi < \infty.$$

Theorem 5.3 *Assume that conditions* $(H1)$*–*$(H4)$ *are satisfied. Then the* BVP (5.1), (5.2) *has at least one solution* $y(t)$, $t \in [a-1, b+1]$ *such that*

$$\sigma r \leq y(t) \leq \frac{R}{\sigma} \qquad \textit{for} \qquad t \in [a, b-1]. \tag{5.11}$$

$$0 \leq y(b) \leq \frac{R}{\sigma}. \tag{5.12}$$

Proof. For $y \in \wp_0$ with $\parallel y \parallel = r$ (hence $0 \leq y(s) \leq r$ for $s \in [a, b]$) we have for all $t \in [a, b]$

$$Ay(t) \leq \sum_{s=a}^{b} G(s,s) f(s, y(s)) \leq \frac{r}{g} \sum_{s=a}^{b} G(s,s) = r = \parallel y \parallel. \tag{5.13}$$

Now if we let $\Omega_1 = \{y \in \mathcal{B} \mid \| y \| < r\}$, then (5.13) shows that $\| Ay \| \leq \| y \|$, $y \in \wp_0 \bigcap \partial\Omega_1$. Further, let

$$R_1 = \frac{1}{\sigma} R \text{ and } \Omega_2 = \{y \in \mathcal{B} \mid \| y \| < R_1\}.$$

Then $y \in \wp_0$ and $\| y \| = R_1$ implies

$$\min_{a \leq s \leq b-1} y(s) \geq \sigma \| y \| = \sigma R_1 = R,$$

hence $y(s) \geq R$ for all $s \in [a, b-1]$. Therefore,

$$Ay(t_0) = \sum_{s=a}^{b} G(t_0, s) f(s, y(s)) \geq \sum_{s=a}^{b-1} G(t_0, s) f(s, y(s))$$

$$\geq \frac{1}{\sigma g_1} R \sum_{s=a}^{b-1} G(t_0, s) = \frac{1}{\sigma} R = R_1 = \| y \| .$$

Hence $\| Ay \| \geq \| y \|$ for all $y \in \wp_0 \bigcap \partial\Omega_2$.

Consequently, by the first part of Theorem 1.1, it follows that A has a fixed point y in $\wp_0 \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$. We have $r \leq \| y \| \leq R_1$. Hence, since for $y \in \wp_0$ we have $y(t) \geq \sigma \| y \|$, $t \in [a, b-1]$, it follows that (5.11) and (5.12) hold. □

Remark 5.1 *The signs of $y(a-1)$ and $y(b+1)$ can be determined from the boundary conditions* (5.2).

Remark 5.2 *If*

$$\lim_{\xi \to 0^+} \frac{f(t,\xi)}{\xi} = 0 \text{ and } \lim_{\xi \to \infty} \frac{f(t,\xi)}{\xi} = \infty$$

for all $t \in [a, b]$, then the condition $(H4)$ will be satisfied for $r > 0$ sufficiently small and $R > 0$ sufficiently large.

Below in Theorem 5.4 we assume the following condition on $f(t, \xi)$.

- **(H5)** There exist numbers $0 < r < R < \infty$ such that $gr < \sigma g_1 R$ and for all $t \in [a, b]$:

$$f(t, \xi) \geq \frac{1}{\sigma g_1}\xi \quad \text{if} \quad 0 \leq \xi \leq r; \quad f(t, \xi) \leq \frac{1}{g}R \quad \text{if} \quad 0 \leq \xi \leq R.$$

Theorem 5.4 *Assume that conditions* $(H1)$*–*$(H3)$ *and* $(H5)$ *are satisfied. Then the* BVP (5.1), (5.2) *has at least one solution* $y(t)$*,* $t \in [a-1, b+1]$ *such that*

$$\sigma r \leq y(t) \leq R \quad \textit{for} \quad t \in [a, b-1]; \quad 0 \leq y(b) \leq R.$$

Proof. For $y \in \wp_0$ with $\| y \| = r$ (hence $0 \leq y(s) \leq r$ for $s \in [a, b]$), we have

$$Ay(t_0) = \sum_{s=a}^{b} G(t_0, s) f(s, y(s)) \geq \sum_{s=a}^{b-1} G(t_0, s) f(s, y(s)) \geq \frac{1}{\sigma g_1} \sum_{s=a}^{b-1} G(t_0, s) y(s)$$

$$\geq \frac{1}{\sigma g_1} \sigma \| y \| \sum_{s=a}^{b-1} G(t_0, s) = \| y \|. \tag{5.14}$$

Now if we let $\Omega_1 = \{y \in \mathcal{B} \mid \| y \| < r\}$, then (5.14) shows that $\| Ay \| \geq \| y \|$, $y \in \wp_0 \bigcap \partial\Omega_1$. Further, let

$$\Omega_2 = \{y \in \mathcal{B} \mid \| y \| < R\}.$$

Then for $y \in \wp_0$ with $\| y \| = R$ (hence $0 \leq y(s) \leq R$ for $s \in [a, b]$), we have for all $t \in [a, b]$,

$$Ay(t) \leq \sum_{s=a}^{b} G(s, s) f(s, y(s)) \leq \frac{1}{g} R \sum_{s=a}^{b} G(s, s) = R = \| y \| .$$

Therefore $\| Ay \| \leq \| y \|$ for all $y \in \wp_0 \bigcap \partial\Omega_2$.

Consequently, by the second part of Theorem 1.1, it follows that A has a fixed point y in $\wp_0 \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$. We have $r \leq \| y \| \leq R$. Hence, $y(t) \leq R$ for $t \in [a, b]$, and since for $y \in \wp_0$ we have $y(t) \geq \sigma \parallel y \parallel$ for $t \in [a, b-1]$, it follows that $y(t) \geq \sigma r$ for $t \in [a, b-1]$. The theorem is proved. □

Next we will assume that instead of (H2) the following more strong condition holds:

- **(H2′)** $\alpha, \beta, \gamma \geq 0, \ \delta > 0, \quad \alpha + \beta > 0;$

  if $q(t) \equiv 0$ $(a \leq t \leq b)$, then $\alpha + \gamma > 0$.

In this case by Theorem 4.1 we will have

$$G(t, s) > 0 \quad \text{for} \quad t, s \in [a, b]. \tag{5.15}$$

Let us set

$$m = \min G(t, s), \quad M = \max G(t, s), \quad t, s \in [a, b] \tag{5.16}$$

and form the cone

$$\wp_1 = \{y \in \wp \mid \min_{a \leq t \leq b} y(t) \geq \frac{m}{M} \parallel y \parallel\},$$

where the cone $\wp$ is defined by (5.9).

Lemma 5.2 *Assume that conditions $(H1)$, $(H2')$ and $(H3)$ are satisfied and let A be the operator defined by (5.8). Then $Ay \in \wp_1$ for all $y \in \wp$. In particular, the operator A leaves the cone $\wp_1$ invariant, i.e.,$A(\wp_1) \subset \wp_1$.*

Proof. For all $y \in \wp$ obviously we have $Ay(t) \geq 0$ for all $t \in [a, b]$. Further,

$$\min_{a \leq t \leq b} Ay(t) \geq m \sum_{s=a}^{b} f(s, y(s)) \geq \frac{m}{M} \sum_{s=a}^{b} \left\{ \max_{a \leq t \leq b} G(t, s) \right\} f(s, y(s))$$

$$\geq \frac{m}{M} \max_{a \leq t \leq b} \sum_{s=a}^{b} G(t, s) f(s, y(s)) = \frac{m}{M} \parallel Ay \parallel.$$

Therefore, $Ay \in \wp_0$. □

In the next theorem we also assume the following condition on $f(t, \xi)$.

- **(H6)** There exist numbers $0 < r < R < \infty$ such that for all $t \in [a, b]$:

$$f(t, \xi) \le \frac{1}{(b-a+1)M} r \quad \text{if} \quad 0 \le \xi \le r;$$

$$f(t, \xi) \ge \frac{M}{(b-a+1)m^2} R \quad \text{if} \quad R \le \xi < \infty,$$

where m and M are defined by (5.16).

Theorem 5.5 *Assume that conditions* $(H1)$, $(H2')$, $(H3)$, *and* $(H6)$ *are satisfied. Then the* BVP (5.1), (5.2) *has at least one solution* $y(t)$, $t \in [a-1, b+1]$ *such that*

$$\frac{m}{M} r \le y(t) \le \frac{M}{m} R, \quad t \in [a, b] \tag{5.17}$$

Proof. For $y \in \wp_1$ with $\| y \| = r$ (hence $0 \le y(s) \le r$ for $s \in [a, b]$), we have for all $t \in [a, b]$,

$$Ay(t) \le M \sum_{s=a}^{b} f(s, y(s)) \le M \frac{1}{(b-a+1)M} r \sum_{s=a}^{b} 1 = r = \| y \|. \tag{5.18}$$

Now if we let $\Omega_1 = \{y \in \mathcal{B} \mid \| y \| < r\}$, then (5.18) shows that $\| Ay \| \le \| y \|$, $y \in \wp_1 \bigcap \partial\Omega_1$. Further, let

$$R_1 = \frac{M}{m} R \text{ and } \Omega_2 = \{y \in \mathcal{B} \mid \| y \| < R_1\}.$$

Then $y \in \wp_1$ and $\| y \| = R_1$ implies

$$\min_{t \in [a,b]} y(t) \ge \frac{m}{M} \| y \| = \frac{m}{M} R_1 = R,$$

hence $y(s) \ge R$ for all $s \in [a, b]$. Therefore, for all $t \in [a, b]$,

$$Ay(t) \geq m\sum_{s=a}^{b} f(s, y(s)) \geq m\frac{M}{(b-a+1)m^2}R\sum_{s=a}^{b} 1 = \frac{M}{m}R = R_1 = \parallel y \parallel.$$

Hence $\parallel Ay \parallel \geq \parallel y \parallel$ for all $y \in \wp_1 \bigcap \partial\Omega_2$.

Consequently, by the first part of Theorem 1.1, it follows that A has a fixed point y in $\wp_1 \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$. We have $r \leq \parallel y \parallel \leq R_1$. Hence, since for $y \in \wp_1$ we have $y(t) \geq \frac{m}{M} \parallel y \parallel$, $t \in [a, b]$, it follows that (5.17) holds. $\square$

Remark 5.3 *From* (5.6) *and Theorem* 4.1 *it follows that together with (5.17) we also have* $y(a-1) \geq 0$.

Remark 5.4 *If*

$$\lim_{\xi \to 0^+} \frac{f(t, \xi)}{\xi} = 0 \text{ and } \lim_{\xi \to \infty} \frac{f(t, \xi)}{\xi} = \infty$$

for all $t \in [a, b]$*, then the condition* $(H6)$ *will be satisfied for* $r > 0$ *sufficiently small and* $R > 0$ *sufficiently large.*

Below in Theorem 5.6 we assume the following condition on $f(t, \xi)$.

- **(H7)** There exist numbers $0 < r < R < \infty$ such that for all $t \in [a, b]$:

$$f(t, \xi) \geq \frac{M}{(b-a+1)m^2}\xi \quad \text{if} \quad 0 \leq \xi \leq r;$$

$$f(t, \xi) \leq \frac{1}{(b-a+1)M}\xi \quad \text{if} \quad R \leq \xi < \infty.$$

Theorem 5.6 *Assume that conditions* $(H1)$*,* $(H2')$*,* $(H3)$*, and* $(H7)$ *are satisfied. Then the BVP* (5.1), (5.2) *has at least one solution* $y(t)$*,* $t \in [a-1, b+1]$ *with property* (5.17).

The proof is analogous to that of Theorem 5.5 and uses the second part of Theorem 1.1.

Remark 5.5 *If*

$$\lim_{\xi\to 0^+}\frac{f(t,\xi)}{\xi}=\infty \text{ and } \lim_{\xi\to\infty}\frac{f(t,\xi)}{\xi}=0$$

for all $t\in[a,b]$, then the condition $(H7)$ will be satisfied for $r>0$ sufficiently small and $R>0$ sufficiently large.

References

[1] Agarwal, R.P., Difference Equations and Inequalities, *Marcel Dekker, New York*, 1992.

[2] Agarwal, R.P., O'Regan, D., and Wong, P.J.Y., Positive Solutions of Differential, Difference and Integral Equations, *Kluwer, Dordrecht*, 1999.

[3] Agarwal, R.P. and Wong, P.J.Y., Advanced Topics in Difference Equations, *Kluwer, Dordrecht*, 1997.

[4] Atici, F.M. and Guseinov, G.Sh., Positive periodic solutions for nonlinear difference equations with periodic coefficients, *J.Math.Anal.Appl.* **232** (1999), 166-182.

[5] Atici, F. and Guseinov, G.Sh., Criteria for the stability of second order difference equations with periodic coefficients, *Communications in Appl.Anal.* **3** (1999), 503-515.

[6] Atici, F. and Peterson, A., Bounds for positive solutions for a focal boundary value problem, *Computers Math.Appl.* **36** (1998), 99-107.

[7] Erbe, L.H., Hu, S., and Wang, H., Multiple positive solutions of some boundary value problems, *J. Math.Anal.Appl.* **184** (1994), 640-648.

[8] Erbe, L.H. and Wang, H., On the existence of positive solutions of ordinary differential equations, *Proc.Amer.Math.Soc.* **120** (1994), 743-748.

[9] Guo, D. and Lakshmikantham, V., Nonlinear Problems in Abstract Cones, *Academic Press, San Diego*, 1988.

[10] Kelley, W.G., and Peterson, A.C., Difference Equations: An Introduction with Applications, *Academic Press, New York*, 1991.

[11] Krasnosel'skii, M.A., Positive Solutions of Operator Equations, *Noordhoff, Groningen*, 1964.

[12] Merdivenci, F., Two positive solutions of a boundary value problem for difference equations, *J.Difference Equations Appl.* **1** (1995), 262-270.

[13] Merdivenci, F., Green's matrices and positive solutions of a discrete boundary value problem, *PanAmerican Math.J.* **5** (1995), 25-42.

N. AYKUT
Department of Mathematics
Ege University
35100 Bornova, İzmir-TURKEY
G. Sh. GUSEINOV
Department of Mathematics
Atilim University
06836 Incek, Ankara-TURKEY

Received 06.07.2002