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Shape Operator A_H for Slant Submanifolds in Generalized Complex Space Forms

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Abstract

In this article, we establish an inequality between the sectional curvature function K and the shape operator A_H at the mean curvature vector for slant submanifolds in generalized complex space forms. Also a sharp relationship between the k-Ricci curvature and the shape operator A_H is proved.

Key Words: Shape operator, slant submanifolds, generalized complex space form, k-Ricci curvature.

1. Preliminaries

In the introduction of [2], B. Y. Chen recalls as one of the basic problems in submanifold theory:

"Find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold".

In the above mentioned paper, B. Y. Chen establishes a relationship between sectional curvature function K and the shape operator A_H for submanifolds in real space forms.

Also, in [3], B. Y. Chen proves a sharp inequality between the k-Ricci curvature and the shape operator A_H.

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In [6], we establish a relationship between the sectional curvature function K and the shape operator A_H and a sharp relationship between the k-Ricci curvature and the shape operator A_H, respectively, for slant submanifolds in complex space forms.

Let $\widetilde{M}$ be an almost Hermitian manifold with almost complex structure J and Riemannian metric g. One denotes by $\widetilde{\nabla}$ the operator of covariant differentiation with respect to g in $\widetilde{M}$.

Definition. If the almost complex structure J satisfies

$$(\widetilde{\nabla}_X J)Y + (\widetilde{\nabla}_Y J)X = 0,$$

for any vector fields X and Y on $\widetilde{M}$, then the manifold $\widetilde{M}$ is called a *nearly-Kaehler manifold* [5], [11].

Remark. The above condition is equivalent to

$$(\widetilde{\nabla}_X J)X = 0, \quad \forall X \in \Gamma T\widetilde{M}.$$

For an almost complex structure J on the manifold $\widetilde{M}$, the *Nijenhuis tensor field* is defined by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for any vector fields X, Y tangent to $\widetilde{M}$, where $[,]$ is the Lie bracket.

A necessary and sufficient condition for a nearly-Kaehler manifold to be Kaehler is the vanishing of the Nijenhuis tensor N_J.

Any 4-dimensional nearly-Kaehler manifold is a Kaehler manifold.

Example. Let S^6 be the 6-dimensional unit sphere defined as follows:

Let $\mathbf{E}^7$ be the set of all purely imaginary Cayley numbers. Then $\mathbf{E}^7$ is a 7-dimensional subspace of the Cayley algebra C.

Let $\{1, e_0, e_1, ..., e_6\}$ be a basis of the Cayley algebra, 1 being the unit element of C.

If $X = \sum_{i=0}^{6} x^i e_i$ and $Y = \sum_{i=0}^{6} y^i e_i$ are two elements of $\mathbf{E}^7$, one defines the *scalar product* in $\mathbf{E}^7$ by

$$< X, Y >= \sum_{i=0}^{6} x^i y^i,$$

and the *vector product* by

$$X \times Y = \sum_{i \neq j} x^i y^j e_i * e_j,$$

$*$ being the multiplication operation of C.

Consider the 6-dimensional unit sphere S^6 in $\mathbf{E}^7$:

$$S^6 = \{X \in \mathbf{E}^7 \mid \ < X, X >= 1\}.$$

The scalar product in $\mathbf{E}^7$ induces the natural metric tensor field g on S^6.

The tangent space $T_X S^6$ at $X \in S^6$ can naturally be identified with the subspace of $\mathbf{E}^7$ orthogonal to X.

Define the endomorphism J_X on $T_X S^6$ by

$$J_X Y = X \times Y, \text{ for } Y \in T_X S^6.$$

It is easy to see that

$$g(J_X Y, J_X Z) = g(Y, Z), \ Y, Z \in T_X S^6.$$

The correspondence $X \mapsto J_X$ defines a tensor field J such that $J^2 = -I$.

Consequently, S^6 admits an almost Hermitian structure (J, g).

This structure is a non-Kaehlerian nearly-Kaehlerian structure (its Betti numbers of even order are 0).

We will consider a class of almost Hermitian manifolds, called *RK-manifolds*, which contains nearly-Kaehler manifolds.

Definition [10]. A *RK-manifold* $(\widetilde{M}, J, g)$ is an almost Hermitian manifold for which the curvature tensor $\widetilde{R}$ is invariant by J, i.e.

$$\widetilde{R}(JX, JY, JZ, JW) = \widetilde{R}(X, Y, Z, W),$$

for any $X, Y, Z, W \in \Gamma T\widetilde{M}$.

An almost Hermitian manifold $\widetilde{M}$ is of *pointwise constant type* if, for any $p \in \widetilde{M}$ and $X \in T_p\widetilde{M}$, we have

$$\lambda(X, Y) = \lambda(X, Z),$$

where

$$\lambda(X,Y) = \widetilde{R}(X,Y,JX,JY) - \widetilde{R}(X,Y,X,Y)$$

and Y and Z are unit tangent vectors on $\widetilde{M}$ at p, orthogonal to X and JX, i.e.

$$g(Z,Z) = g(Y,Y) = 1,$$

$$g(X,Y) = g(JX,Y) = g(X,Z) = g(JX,Z) = 0.$$

The manifold $\widetilde{M}$ is said to be of *constant type* if for any unit $X, Y \in \Gamma T\widetilde{M}$ with $g(X,Y) = g(JX,Y) = 0$, $\lambda(X,Y)$ is a constant function.

Recall the following result [10].

Theorem. *Let $\widetilde{M}$ be a RK-manifold. Then $\widetilde{M}$ is of pointwise constant type if and only if there exists a function α on $\widetilde{M}$ such that*

$$\lambda(X,Y) = \alpha[g(X,X)g(Y,Y) - (g(X,Y))^2 - (g(X,JY))^2],$$

for any $X, Y \in \Gamma T\widetilde{M}$.

Moreover, $\widetilde{M}$ is of constant type if and only if the above equality holds good for a constant α.

In this case, α is the *constant type* of $\widetilde{M}$.

Definition. A *generalized complex space form* is a RK-manifold of constant holomorphic sectional curvature and of constant type.

We will denote a generalized complex space form by $\widetilde{M}(c,\alpha)$, where c is the constant holomorphic sectional curvature and α the constant type, respectively.

Each complex space form is a generalized complex space form. The converse statement is not true. The sphere S^6 endowed with the standard nearly-Kaehler structure is an example of generalized complex space form which is not a complex space form.

Let $\widetilde{M}(c,\alpha)$ be a generalized complex space form of constant holomorphic sectional curvature c and of constant type α. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c,\alpha)$ has the following expression [10]:

$$\widetilde{R}(X,Y)Z = \frac{c+3\alpha}{4}[g(Y,Z)X - g(X,Z)Y] + \tag{1.1}$$

$$+\frac{c-\alpha}{4}[g(X,JZ)JY-g(Y,JZ)JX+2g(X,JY)JZ].$$

Let M be an n-dimensional submanifold of an $2m$-dimensional generalized complex space form $\widetilde{M}(c,\alpha)$. We denote by $K(\pi)$ the *sectional curvature* of M associated with a plane section $\pi\subset T_pM, p\in M$. Let ∇ and h be the Levi-Civita connection of M and the second fundamental form, respectively.

Then the equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W)=R(X,Y,Z,W)+ \tag{1.2}$$

$$+g(h(X,W),h(Y,Z))-g(h(X,Z),h(Y,W)),$$

for any vectors X,Y,Z,W tangent to M, where R is the *Riemann curvature tensor* of M.

We denote by H the *mean curvature vector* at $p\in M$, i.e.

$$H(p)=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i), \tag{1.3}$$

where $\{e_1,...,e_{2m}\}$ is an orthonormal basis of the tangent space $T_p\widetilde{M}(c,\alpha)$, such that $\{e_1,...,e_n\}$ are tangent to M.

Also, we set

$$h_{ij}^r=g(h(e_i,e_j),e_r),\quad i,j=1,...,n;\quad r=n+1,...,2m, \tag{1.4}$$

and

$$\|h\|^2=\sum_{i,j=1}^{n}g(h(e_i,e_j),h(e_i,e_j)). \tag{1.5}$$

For any $p\in M$ and for any $X\in T_pM$, we put $JX=PX+FX$, where $PX\in T_pM, FX\in T_p^{\perp}M$.

We put

$$\|P\|^2=\sum_{i,j=1}^{n}g^2(Pe_i,e_j). \tag{1.6}$$

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1,...,e_k\}$ of L such that $e_1=X$.

Define the *Ricci curvature* Ric_L of L at X by

$$Ric_L(X)=K_{12}+K_{13}+...+K_{1k}, \tag{1.7}$$

where K_{ij} denotes the *sectional curvature* of the 2-plane section spanned by e_i, e_j. We simply called such a curvature a *k-Ricci curvature*.

The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}. \tag{1.8}$$

For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad p \in M, \tag{1.9}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Recall that for a submanifold M in a Riemannian manifold, the *relative null space* of M at a point $p \in M$ is defined by

$$N(p) = \{X \in T_pM \,|\, h(X,Y) = 0, \forall Y \in T_pM\}. \tag{1.10}$$

2. Sectional curvature and shape operator

The notion of a slant submanifold of an almost Hermitian manifold was introduced by B. Y. Chen [1].

Definition. A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be a *slant submanifold* if for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between JX and the tangent space T_pM is constant ($= \theta$).

We prove an inequality for an n-dimensional slant submanifold M into a $2m$-dimensional generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature c and of constant type α.

Theorem 2.1. *Let $x : M \to \widetilde{M}(c, \alpha)$ be an isometric immersion of an n-dimensional θ-slant submanifold into a $2m$-dimensional generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature $c > \alpha > 0$. If there exists a point $p \in M$ and a number $b > \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{2n}\cos^2\theta$ such that $K \geq b$ at p, then the shape operator at the* *mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}[b - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}\cos^2\theta]I_n, \ \ at\ p, \tag{2.1}$$

where I_n is the identity map.

Proof. Let $p \in M$ and a number $b > \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{2u}\cos^2\theta$ such that $K \geq b$ at p. We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector H and $e_1, ..., e_n$ diagonalize the shape operator A_{n+1}.

Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & ... & 0 \\ 0 & a_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & a_n \end{pmatrix}, \tag{2.2}$$

$$A_r = (h_{ij}^r), i, j = 1, ..., n, r = n+2, ..., 2m, \text{trace } A_r = \sum_{i=1}^{n} h_{ii}^r = 0. \tag{2.3}$$

For $i \neq j$, we denote by

$$u_{ij} = a_i a_j. \tag{2.4}$$

From Gauss equation for $X = Z = e_i, Y = W = e_j$, we get

$$u_{ij} \geq b - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4}g^2(e_i, Je_j) - \sum_{r=n+2}^{2m}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{2.5}$$

We prove that u_{ij} have the following properties:

1. For any fixed $i \in \{1, ..., n\}$, we have

$$\sum_{i \neq j} u_{ij} \geq (n-1)(b - \frac{c+3\alpha}{4}) - 3\frac{c-\alpha}{4}\cos^2\theta > 0.$$

2. $u_{ij} \neq 0$, for $i \neq j$.

3. For distinct $i, j, k \in \{1, ..., n\}$, $a_i^2 = \dfrac{u_{ij}u_{ik}}{u_{jk}}$.

4. We denote by $S_k = \{B \subset \{1, ..., n\}; |B| = k\}$ and for any $B \in S_k$ we denote by $\overline{B} = \{1, ..., n\} \setminus B$. Then, for a fixed $k, 1 \leq k \leq \left[\frac{n}{2}\right]$ and each $B \in S_k$, we have

$$\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} > 0.$$

5. For distinct $i, j \in \{1, ..., n\}, u_{ij} > 0$.

1. From (2.3), (2.4) and (2.5), we have:

$$\sum_{j \neq i} u_{ij} \geq (n-1)(b - \frac{c+3\alpha}{4}) - 3\frac{c-\alpha}{4} \|Pe_i\|^2 - \sum_{r=n+2}^{2m} [h^r_{ii}(\sum_{j \neq i} h^r_{jj}) - \sum_{j \neq i} (h^r_{ij})^2] =$$

$$= (n-1)(b - \frac{c+3\alpha}{4}) - 3\frac{c-\alpha}{4} \cos^2\theta - \sum_{r=n+2}^{2m} [h^r_{ii}(-h^r_{ii}) - \sum_{j \neq i} (h^r_{ij})^2] =$$

$$= (n-1)(b - \frac{c+3\alpha}{4}) - 3\frac{c-\alpha}{4} \cos^2\theta + \sum_{r=n+2}^{2m} \sum_{j=1}^{n} (h^r_{ij})^2 \geq$$

$$\geq (n-1)(b - \frac{c+3\alpha}{4}) - 3\frac{c-\alpha}{4} \cos^2\theta > 0.$$

2. If $u_{ij} = 0$, for $i \neq j$, then $a_i = 0$ or $a_j = 0$. $a_i = 0$ implies that $u_{it} = a_i a_t = 0, \forall t \in \{1, ..., n\}, t \neq i$.

It follows that

$$\sum_{j \neq i} u_{ij} = 0,$$

in contradiction with 1.

3. $\dfrac{u_{ij} u_{ik}}{u_{jk}} = \dfrac{a_i a_j a_i a_k}{a_j a_k} = a_i^2.$

4. Since we can change the order of $e_1, ..., e_n$, we may assume $B = \{1, ..., k\}$ and $\overline{B} = \{k+1, ..., n\}$. Then

$$\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} = k(n-k)(b - \frac{c+3\alpha}{4}) - 3\frac{c-\alpha}{4} \sum_{j=1}^{k} \sum_{t=k+1}^{n} g^2(Je_j, e_t) -$$

$$- \sum_{r=n+2}^{2m} \{\sum_{j=1}^{k} \sum_{t=k+1}^{n} [h^r_{jj} h^r_{tt} - (h^r_{jt})^2]\} \geq$$

$$\geq k(n-k)(b-\frac{c+3\alpha}{4})-3k\frac{c-\alpha}{4}\cos^2\theta+$$

$$+\sum_{r=n+2}^{2m}[\sum_{j=1}^{k}\sum_{t=k+1}^{n}(h_{jt}^r)^2+\sum_{j=1}^{k}(h_{jj}^r)^2]\geq$$

$$\geq k(n-k)(b-\frac{c+3\alpha}{4})-3k\frac{c-\alpha}{4}\cos^2\theta>0.$$

5. Assume $u_{1n}<0$. From 3, we get $u_{1i}u_{in}<0$, for $1<i<n$.

Without loss of generality, we may assume

$$\begin{cases} u_{12},...,u_{1l},u_{(l+1)n},...,u_{(n-1)n}>0, \\ u_{1(l+1)},...,u_{1n},u_{2n},...,u_{\mathrm{ln}}<0, \end{cases} \tag{2.6}$$

for some $\left[\frac{n+1}{2}\right]\leq l\leq n-1$.

If $l=n-1$, then $u_{1n}+u_{2n}+...+u_{(n-1)n}<0$, which contradicts to 1. Thus, $l<n-1$.

From 3, we get

$$a_n^2=\frac{u_{in}u_{tn}}{u_{it}}>0, \tag{2.7}$$

where $2\leq i\leq l, l+1\leq t\leq n-1$. By (2.6) and (2.7), we obtain $u_{it}<0$, which implies

$$\sum_{i=1}^{l}\sum_{t=l+1}^{n}u_{it}=\sum_{i=2}^{l}\sum_{t=l+1}^{n-1}u_{it}+\sum_{i=1}^{l}u_{in}+\sum_{t=l+1}^{n}u_{1t}<0.$$

This contradicts to 4.

Now, we return to the proof of Theorem 2.1.

From 5, it follows that $a_1,...,a_n$ have the same sign. Assume $a_j>0,\forall j\in\{1,...,n\}$. Then

$$\sum_{j\neq i}u_{ij}=a_i(a_1+...+a_n)-a_i^2\geq(n-1)(b-\frac{c+3\alpha}{4})-3\frac{c-\alpha}{4}\cos^2\theta.$$

From the above relation and from (2.2), we have

$$a_i n\left\|H\right\|\geq(n-1)(b-\frac{c+3\alpha}{4})-3\frac{c-\alpha}{4}\cos^2\theta+a_i^2>$$

$$> (n-1)(b - \frac{c+3\alpha}{4}) - 3\frac{c-\alpha}{4}\cos^2\theta.$$

This equation implies

$$a_i \|H\| > \frac{n-1}{n}[b - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}\cos^2\theta],$$

and consequently (2.1). ■

In particular, for $\alpha = 0$, we refind Theorem 3.1 from [6].

For totally real submanifolds, we have the following

Corollary 2.2. *Let $x : M \to \widetilde{M}(c,\alpha)$ be an isometric immersion of an n-dimensional totally real submanifold into an $2m$-dimensional generalized complex space form $\widetilde{M}(c,\alpha)$. If there exists a point $p \in M$ and a number $b > \frac{c+3\alpha}{4}$ such that $K \geq b$ at p, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}(b - \frac{c+3\alpha}{4})I_n, \text{ at } p,$$

where I_n is the identity map.

3. k-Ricci curvature and shape operator

We prove an inequality for a slant submanifold M of a $2m$-dimensional generalized complex space form $\widetilde{M}(c,\alpha)$ of constant holomorphic sectional curvature c and of constant type α.

Theorem 3.1. *Let $x : M \to \widetilde{M}(c,\alpha)$ be an isometric immersion of an n-dimensional θ-slant submanifold M into a $2m$-dimensional generalized complex space form $\widetilde{M}(c,\alpha)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have:*

i) If $\Theta_k(p) \neq \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)}\cos^2\theta$, then the shape operator at the mean curvature satisfies

$$A_H > \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}\cos^2\theta]I_n, \text{ at } p, \tag{3.1}$$

where I_n denotes the identity map of T_pM.

ii) If $\Theta_k(p) = \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)}\cos^2\theta$*, then* $A_H \geq 0$ *at* p*.*

iii) A unit vector $X \in T_pM$ *satisfies*

$$A_H X = \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}\cos^2\theta]X \tag{3.2}$$

if and only if $\Theta_k(p) = \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)}\cos^2\theta$ *and* $X \in N(p)$*.*

iv) $A_H = \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}\cos^2\theta]I_n$ *at* p *if and only if* p *is a totally geodesic point.*

Proof. i) Let $\{e_1, ...e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1...i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. It is easily seen by the definitions

$$\tau(L_{i_1...i_k}) = \frac{1}{2}\sum_{i\in\{i_1,...,i_k\}} Ric_{L_{i_1...i_k}}(e_i), \tag{3.3}$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\leq i_1<...<i_k\leq n} \tau(L_{i_1...i_k}). \tag{3.4}$$

Combining (3.3) and (3.4), we find

$$\tau(p) \geq \frac{n(n-1)}{2}\Theta_k(p). \tag{3.5}$$

From the equation of Gauss for $X = Z = e_i, Y = W = e_j$, by summing, we obtain

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - \frac{c+3\alpha}{4}n(n-1) - 3\frac{c-\alpha}{4}\|P\|^2. \tag{3.6}$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, ..., e_n$ diagonalize the shape operator A_{n+1}. Then we have the relations (2.2) and (2.3).

From (3.6), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - \tag{3.7}$$

$$-\frac{c+3\alpha}{4}n(n-1) - 3\frac{c-\alpha}{4}\|P\|^2.$$

On the other hand, since

$$0 \leq \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = (\sum_{i=1}^n a_i)^2 = \sum_{i=1}^n a_i^2 + 2\sum_{i<j} a_i a_j \leq n \sum_{i=1}^n a_i^2, \tag{3.8}$$

which implies

$$\sum_{i=1}^n a_i^2 \geq n \|H\|^2 .$$

We have from (3.7)

$$n^2 \|H\|^2 \geq 2\tau + n\|H\|^2 - \frac{c+3\alpha}{4} n(n-1) - 3\frac{c-\alpha}{4}\|P\|^2, \tag{3.9}$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4n(n-1)}\|P\|^2 . \tag{3.10}$$

Since M is a slant submanifold, from (3.5) and (3.10), we obtain

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4n(n-1)}\|P\|^2 = \tag{3.11}$$

$$= \Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}\cos^2\theta.$$

This shows that $H(p) = 0$ may occurs only when $\Theta_k(p) \leq \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)}\cos^2\theta$. Consequently, if $H(p) = 0$, statements i) and ii) hold automatically. Therefore, without loss of generality, we may assume $H(p) \neq 0$.

From the equation of Gauss we get

$$a_i a_j = K_{ij} - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4} g^2(e_i, Je_j) - \sum_{r=n+2}^{2m}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{3.12}$$

By (3.12), we obtain

$$a_1(a_{i_2}+...+a_{i_k}) = Ric_{L_{1i_2\dots i_k}}(e_1)-(k-1)\frac{c+3\alpha}{4}- \tag{3.13}$$

$$-3\frac{c-\alpha}{4}\sum_{j=2}^{k}g^2(e_1,Je_{i_j})-\sum_{r=n+2}^{2m}\sum_{j=2}^{k}[h^r_{11}h^r_{i_ji_j}-(h^r_{1i_j})^2],$$

which yields

$$a_1(a_2+...+a_n)=\frac{1}{C^{k-2}_{n-2}}\sum_{2\le i_2<...<i_k\le n}Ric_{L_{1i_2\dots i_k}}(e_1)- \tag{3.14}$$

$$-(n-1)\frac{c+3\alpha}{4}-3\frac{c-\alpha}{4}\sum_{j=2}^{n}g^2(e_1,Je_j)+\sum_{r=n+2}^{2m}\sum_{j=1}^{n}(h^r_{1j})^2.$$

We find

$$a_1(a_2+...+a_n)\ge(n-1)[\Theta_k(p)-\frac{c+3\alpha}{4}-3\frac{c-\alpha}{4(n-1)}\cos^2\theta]. \tag{3.15}$$

Then

$$a_1(a_1+a_2+...+a_n)=a_1^2+a_1(a_2+...+a_n)\ge \tag{3.16}$$

$$\ge a_1^2+(n-1)[\Theta_k(p)-\frac{c+3\alpha}{4}-3\frac{c-\alpha}{4(n-1)}\cos^2\theta]\ge$$

$$\ge(n-1)[\Theta_k(p)-\frac{c+3\alpha}{4}-3\frac{c-\alpha}{4(n-1)}\cos^2\theta].$$

Since $n\left\|H\right\|=a_1+...+a_n$, the above equation implies

$$A_H\ge\frac{n-1}{n}[\Theta_k(p)-\frac{c+3\alpha}{4}-3\frac{c-\alpha}{4(n-1)}\cos^2\theta]I_n.$$

The equality does not hold, because in our case $H(p)\neq0$.

The assertion ii) is obvious.

iii) Let $X\in T_pM$ a unit vector satisfying (3.2). By (3.16) and (3.14) one has $a_1=0$ and $h^r_{1j}=0,\forall j\in\{1,...,n\},r\in\{n+2,...,2m\}$, respectively. The above conditions imply $\Theta_k(p)=\frac{c+3\alpha}{4}+3\frac{c-\alpha}{4(n-1)}\cos^2\theta$ and $X\in N(p)$.

The converse is clear.

iv) The equality (3.2) holds for any $X \in T_pM$ if and only if $N(p) = T_pM$, i.e. p is a totally geodesic point. ■

Remark. If we denote by λ_i the eigenvalues of A_H, i.e. $\lambda_i = a_i \|H\|$, $i \in \{1, ..., n\}$, we obtain the following inequality for arbitrary submanifolds of generalized complex space forms:

$$\lambda_i \geq \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}\|Pe_i\|^2].$$

In particular, for $\alpha = 0$, we obtain Theorem 4.1 from [6].

Corollary 3.2. *Let $x : M \to \widetilde{M}(c, \alpha)$ be an isometric immersion of an n-dimensional totally real submanifold M into a generalized complex space form $\widetilde{M}(c, \alpha)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have:*

i) If $\Theta_k(p) \neq \frac{c+3\alpha}{4}$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4}]I_n, \text{ at } p,$$

where I_n denotes the identity map of T_pM.

ii) If $\Theta_k(p) = \frac{c+3\alpha}{4}$, then $A_H \geq 0$ at p.

iii) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4}]X$$

if and only if $\Theta_k(p) = \frac{c+3\alpha}{4}$ and $X \in N(p)$.

iv) $A_H = \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4}]I_n$ at p if and only if p is a totally geodesic point.

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