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Linear Automorphism Groups of Relatively Free Groups

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Abstract

Let G be a free group in a variety of groups, but G is not absolutely free. We prove that the group of automorphisms $Aut(G)$ is linear if and only if G is a finitely generated virtually nilpotent group.

1. Introduction

Let $F = F(x_1, x_2, \dots)$ be an absolutely free group with basis $x_1, x_2, \dots$. Recall that a group H satisfies an identity $w(x_1, \dots, x_n) = 1$ for a word $w(x_1, \dots, x_n) = w \in F$ if w vanishes under every homomorphism $F \to H$. A *variety* of groups is a class of groups consisting of all groups that satisfy some set of identities.

A *relatively free group* G is a free group in a group variety $\mathcal{V}$, i.e., G belongs to $\mathcal{V}$, and G is generated by a set Y such that every mapping $Y \to H$, where $H \in \mathcal{V}$, extends to a homomorphism $G \to H$. The (free) rank of G is the cardinality of Y. (See details in [8].) We denote by F_n, by $A_n = F_n/F_n'$, and by $M_n = F_n/F_n''$ the absolutely free group, the free abelian and the free metabelian groups of rank n, respectively.

The group of inner automorphisms of a group G is normal in $Aut(G)$, and so the factor group $G/Z(G)$, where $Z(G)$ is the center of G, can be canonically identified with a normal subgroup of $Aut(G)$. Since the center of M_n is trivial for $n \geq 2$ (see [8], 25.63), one can identify M_n with the normal subgroup of $Aut(M_n)$ consisting of the inner automorphisms of M_n.

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Our paper is inspired by the following result of V.P.Platonov that answers a question raised by H.Mochizuki [7].

Theorem 1.1 *(V.P.Platonov [9]) (1) Let ρ be a finite-dimensional linear representation of the automorphism group $Aut(M_n)$ over a field k. Then the image $\rho(M_n)$ is a virtually nilpotent group. (2) It follows that the group $Aut(M_n)$ is not linear for $n > 1$.*

As usual, a group G is called *linear* if it is isomorphic to a subgroup of $GL_m(k)$ for some field k and some integer $m \geq 1$.

In Section 2, we give an alternative and shorter proof of Theorem 1.1. Then a similar approach and the utilization of some known properties of group varieties lead to the complete description of relatively free groups G for which $Aut(G)$ is a linear group.

Theorem 1.2 *Let G be a relatively free but not absolutely free group. The automorphism group $Aut(G)$ is linear if and only if G is a finitely generated virtually nilpotent group.*

Furthermore, if the group G is finitely generated but not virtually-nilpotent, then there is an automorphism ϕ of G such that the extension P of $G/Z(G)$ by ϕ is a non-linear subgroup of $Aut(G)$; and if G is finitely generated and virtually nilpotent, then the holomorph $Hol(G)$ is linear over $\mathbb{Z}$.

Recall that a group G is *virtually nilpotent* if it contains a (normal) nilpotent subgroup of finite index. (The "if" part of the statement does not need the hypothesis that G is relatively free.)

Remark 1.3 *The automorphism group $Aut(F_n)$ is not linear for $n \geq 3$ (Formanek, Procesi [2]) but the group $Aut(F_2)$ is linear (Krammer [5]).*

Remark 1.4 *Note that the formulation of Theorem 1.2 is similar to those contained in the papers of O.M.Mateiko and O.I.Tavgen' [6] and A.A.Korobov [4]. Nevertheless we prove Theorem 1.2 here for the following reasons. (1) There is a mistake in both [6] and [4]. Namely, the proofs essentially use the "known property" of Fitting subgroups (i.e. the product of all nilpotent normal subgroups) to be fully characteristic. But this does not hold even for relatively free groups. (For example, the Fitting subgroup is not fully characteristic in the free group of rank $n > 1$ of the variety generated by the alternating group $Alt(5)$; see [8], 44.47.) (2) The formulation of the main theorems is not quite correct in both [6] and [4] because it is not proved there that the virtually nilpotency of* *a free group of rank $n > 1$ in a variety implies virtually nilpotency of free groups having rank $> n$. (3) Our proof is simpler. (The authors of both papers [6] and [4] refer, in particular, to the statement that all locally finite groups of exponent dividing m form a variety. This claim is equivalent to the restricted Burnside problem for groups of exponent m, and the affirmative solution is bases on the Classification Hypothesis for finite simple groups.)*

2. Free metabelian case

The following Kolchin–Mal'cev Theorem is the most–known fact on linear solvable groups.

Lemma 2.1 *Every linear solvable group has a subgroup H of finite index such that the derived subgroup H' is nilpotent.* □

Let ϕ be an automorphism of the free abelian group A_n and $B_\phi = \langle A_n, \phi\rangle$ the extension of A_n by the automorphism ϕ. Assume that no root of the characteristic polynomial of ϕ is an k-th root of 1 for any integer $k > 0$. The following property of the group B_ϕ is folklore.

Lemma 2.2 *Let C be a subgroup of finite index in B_ϕ. Then the derived subgroup C' has finite index in A_n.*

Proof. Since C is of finite index in B_ϕ, it must contain ϕ^k and mA_n for some positive integers m and k. (We use the additive notation for A_n in this proof.) Therefore C' contains the subgroup $[mA_n, \phi^k] = \{\phi^k(ma) - ma \mid a \in A_n\}$.

Proving by contradiction, assume that the index $(A_n : [mA_n, \phi^k])$ is infinite. Then the image of mA_n under the mapping $\phi^k - id$ is of rank $< n$. Hence 0 is an eigenvalue of $\phi^k - id$, and so $\lambda^k = 1$ for a characteristic root λ of ϕ; a contradiction. □

Proof of Theorem 1.1. There is nothing to prove if $n = 1$. For every $n \geq 2$, there exists an automorphism ϕ of A_n whose characteristic roots are not roots of 1. (One can easily find such automorphisms for $n = 2, 3$ and note for $n > 3$ that A_n is a direct sum of subgroups isomorphic to A_2 or A_3.) We keep the same notation ϕ for a lifting of ϕ to $Aut(M_n)$. (Recall that $A_n \simeq M_n/M_n'$, and the induced homomorphism $Aut(M_n) \to Aut(A_n)$ is surjective by [8], 41.21.)

Let $P = \langle M_n, \phi\rangle$ be the extension of M_n by the automorphism ϕ. This P is a subgroup of $Aut(M_n)$ since $\langle\phi\rangle \cap M_n = \{1\}$; and P is solvable because the factor-group P/M_n is cyclic. By Lemma 2.1, there is a normal subgroup T of finite index in P such that the subgroup $\rho(T')$ is nilpotent.

The canonical image $C = TM_n'/M_n'$ of T in $B_\phi = P/M_n'$ has finite index, and therefore C' is of finite index in $A_n = M_n/M_n'$ by Lemma 2.2. Hence the inverse image $D = T'M_n'$ is of finite index in M_n.

The normal subgroup $\rho(M_n')$ of $\rho(M_n)$ is abelian since M_n is metabelian. Thus, $\rho(D)$ is nilpotent being a product of two nilpotent normal in $\rho(P)$ subgroups $\rho(T')$ and $\rho(M_n')$. Since $[M_n : D] < \infty$, the theorem is proved. □

3. Few lemmas on varieties of groups

The *product* $\mathcal{UV}$ of two group varieties contains all the groups G having a normal subgroup N such that $N \in \mathcal{U}$ and $G/N \in \mathcal{V}$; $\mathcal{UV}$ is also a group variety ([8], 21.12).

Lemma 3.1 *([11]). Let L be a free group of rank $n \geq 2$ in a product of varieties $\mathcal{UV}$. Assume that the free group of rank n in the variety $\mathcal{V}$ is infinite and $\mathcal{U}$ contains a non-trivial group. Then the center of L is trivial.* □

We denote by $\mathcal{A}$ (by $\mathcal{A}_k$) the variety of all abelian groups (of all abelian groups of exponent dividing k), and denote by $M_{k,n}$ the free group of rank n in the variety $\mathcal{M}_k = \mathcal{A}_k\mathcal{A}$.

Lemma 3.2 *If $n, k \geq 2$, then the group $M_{k,n}$ is not virtually nilpotent.*

Proof. The wreath product $W = \mathbb{Z}_k wr \mathbb{Z}$ of a cyclic group of order k and an infinite cyclic group is 2-generated, and it belongs to the variety $\mathcal{M}_k$. Therefore W is a homomorphic image of the group $M_{k,n}$. Since W is not virtually nilpotent, $M_{k,n}$ is not virtually nilpotent too. □

A variety $\mathcal{V}$ is called *solvable* if all the groups of $\mathcal{V}$ are solvable.

Lemma 3.3 *([3]). Let $\mathcal{S}$ be a solvable variety of groups. Then either $\mathcal{S}$ contains as a subvariety the product $\mathcal{M}_p = \mathcal{A}_p\mathcal{A}$ for some prime p or every finitely generated group in $\mathcal{S}$ is virtually nilpotent.* □

Further we call a variety $\mathcal{V}$ *proper* if it does not contain all groups, or equivently, the absolutely free group F_2 does not belong to $\mathcal{V}$. It is easy to see that a product of two proper varieties is proper. (See also [8].) The minimal variety containing a group Q is denoted by $varQ$. Given a group G and a variety $\mathcal{V}$, the *verbal subgroup* $V(G)$ corresponding to $\mathcal{V}$ is the smallest normal subgroup N of G such that $G/N \in \mathcal{V}$.

4. Proof of Theorem 1.2

By Auslander – Baumslag's theorem [1], the holomorph of any finitely generated (virtually) nilpotent group G is linear over $\mathbb{Z}$. Thus, it remains to consider a non-virtually-nilpotent free group G of rank $n \geq 2$ in a proper variety $\mathcal{V}$ and construct a automorphism ϕ required for the second statement of Theorem 1.2. (A non-trivial relatively free group of infinite rank admits the automorphisms from an infinite symmetric group that is not linear.) Now the quotient $H = G/Z(G)$ is non-virtually-nilpotent normal subgroup of $Aut(G)$. We may assume that H is a linear group since otherwise there is nothing to prove.

Since both G and H satisfy a non-trivial identity and H is linear, the group H is virtually solvable by Platonov's theorem [10], i.e., the solvable radical R of H is of finite index in H. Therefore R is a finitely generated but not virtually nilpotent solvable group.

By Lemma 3.3, there is a prime p such that $\mathcal{M}_p \subseteq varR \subseteq varH \subseteq varG$.

Therefore there are canonical epimorphisms $G \to M_{p,n}$ and $G \to A_n$. The kernels are the verbal subgroups of G corresponding to the varieties $\mathcal{M}_p$ and $\mathcal{A}$, respectively. The latest kernel is just the derived subgroup G', and we denote by $M_p(G)$ the former one.

The center of $M_{p,n}$ is trivial by Lemma 3.1, and so $Z(G)$ is contained in $M_p(G) \subseteq G'$. Consequently, we have isomorphisms $G/M_p(G) \simeq H/M_p(H) \simeq M_{p,n}$ and $G/G' \simeq H/H' \simeq A_n$.

Now, as in the proof of Theorem 1.1, we introduce an automorphism ϕ of A_n whose action has no characteristic roots equal to any root of 1. As there (by [8], 41.21) one can lift ϕ to $Aut(G)$ and also to $Aut(H)$ since the center $Z(G)$ is a characteristic subgroup of G. Denote by $P = \langle H, \phi \rangle$ the extension of H by the automorphism ϕ. It is a subgroup of $Aut(G)$ as in the proof of Theorem 1.1.

Proving by contradiction, assume that P is a linear group. Since $P \in \mathcal{V}\mathcal{A}$, the group P satisfies a non-trivial identity, and by [10], P must have a solvable normal subgroup of finite index. By Lemma 2.1, P contains a normal subgroup T of finite index with nilpotent derived subgroup T'. Applying Lemma 2.2 to the image of T in $B_\phi = P/H'$, we have $(H : (T'H')) < \infty$.

The quotient $H'/M_p(H)$ is an abelian normal subgroup of $H/M_p(H) \simeq M_{p,n}$. Since T' is nilpotent and normal in P, the image of the subgroup $T'H'$ under the canonical epimorphism $H \to M_{p,n}$ is nilpotent too. But this image is of finite index in $M_{p,n}$ because $(H : (T'H')) < \infty$. This contradicts the statement of Lemma 3.2, and so the theorem is proved. □

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