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On Graded Secondary Modules

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Abstract

Let G be a group with identity e, and let R be a G-graded commutative ring. Here we study the graded primary submodules of a G-graded R-module and examine when graded submodules of a graded representable module are graded representable. A number of results concerning of these class of submodules are given.

Key Words: Graded secondary modules, Graded primary submodules.

1. Introduction

Secondary modules have been studied extensively by many authors (see [3], [6] and [2], for example). Here we study graded representable modules and the graded primary submodules of a graded module over a G-graded commutative ring. Various properties of such modules are considered. For example, we show that every graded primary submodule of a graded representable module over a G-graded ring is graded representable.

Before we state some results let us introduce some notation and terminology. Let G be an arbitrary group with identity e. A commutative ring R with non-zero identity is G-graded if it has a direct sum decomposition (as an additive group) $R = \oplus_{g \in G} R_g$ such that $1 \in R_e$; and for all $g, h \in G$, $R_g R_h \subseteq R_{gh}$. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \oplus_{g \in G} M_g$ such that for all $g, h \in G$, $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be homogeneous element. A submodule of $N \subseteq M$, where M is G-graded, is called G-graded if $N = \oplus_{g \in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes

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a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Clearly, 0 is a graded submodule of M. Also, we write $h(R) = \cup_{g\in G} R_g$ and $h(M) = \cup_{g\in G} M_g$. A graded ideal I of R is said to be a graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I, denoted by $\mathrm{Gr}(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. A graded ideal I of R is said to be a graded primary ideal if $I \neq R$; and whenever $a, b \in h(R)$ with $ab \in I$, then $a \in I$ or $b \in \mathrm{Gr}(I)$. In this case, $\mathrm{Gr}(I) = P$ is a graded prime ideal of R, and we say that I is a graded P-primary ideal of R (see [5, Lemma 1.8]).

Let S be a commutative ring and let M an R-module. Given an element a of S, we say that a divides M if $aM = M$, and we say that a is nilpotent on M if $a^n M = 0$ for some n. We say that M is secondary if it is non-zero and every $a \in S$ either divides M or is nilpotent on M; in this case the ideal $\mathrm{nirad}(M) = P$ is prime and we also say that M is P-secondary (see [3]).

2. Graded primary submodules

First, we give some basic facts concerning graded primary submodules of a graded module. Next, we study graded submodules of a graded representable module.

Definition 2.1 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M.*

(i) We say that M is a graded free R-module if it has an R-basis consisting of homogeneous elements.

(ii) N is a graded prime submodule of M if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a \in (N :_R M)$.

(iii) N is a graded primary submodule of M if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a^k \in (N :_R M)$ for some k.

(iv) N is a graded maximal submodule of M if $N \neq M$ and there is no graded submodule K of M such that $N \subsetneqq K \subsetneqq M$.

(v) We say that M is a graded simple module if it has only two graded submodules 0 and M.

The following lemma is known, but we write it here for the sake of reference.

Lemma 2.2 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M. Then the following hold:*

(i) N is a graded maximal submodule of M if and only if M/N is a graded simple R-module.

(ii) If $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then $(N :_R M)$ is a graded ideal of R, Rx, IN and rN are graded submodules of M.

The graded radical (resp. radical) of a graded submodule (resp. submodule) N of a graded module (resp. module) M, denoted by $\mathrm{Gr}(N)$, (resp. $\mathrm{rad}(N)$) is defined to be intersection of all graded prime (resp. prime) submodules of M containing N. Clearly, if N and K are graded submodules of M with $K \subseteq N$, then $\mathrm{Gr}(K) \subseteq \mathrm{Gr}(N)$. Let M be a graded module over a graded ring R. We say that an element $r \in h(R)$ is a graded zero-divisor on M if there exists $0 \neq m \in M$ such that $rm = 0$.

Lemma 2.3 *Let M be a graded simple module over a G-graded ring R. Then every graded zero-divisor on M is an annihilator of M.*

Proof. Let r be an arbitrary graded zero-divisor on M. Then there exists $0 \neq a \in h(M)$ such that $ra = 0$. Since M is a simple graded R-module, we get $Ra = M$. Hence, $rM = r(Ra) = (Rr)a = R(ra) = 0$. Thus, r is an annihilator of M. □

Proposition 2.4 *Let M be a graded module over a G-graded ring R. Then every graded maximal submodule of M is a graded prime.*

Proof. Let N be an arbitrary graded maximal submodule of M. Let $rm \in N$ where $r \in h(R)$ and $m \in h(M) - N$. Since $0 \neq (m + N) \in h(M/N)$ and $r(m + N) = 0$, we get r is a graded zero-divisor on graded module M/N; hence by Lemma 2.2 and Lemma 2.3, $r \in (N :_R M)$, as required. □

Proposition 2.5 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M. Then the following hold:*

(i) If N is a graded primary submodule of M, then $(N :_R M)$ is a graded primary ideal of R.

(i) If N is a graded prime submodule of M, then $(N :_R M)$ is a graded prime ideal of R.

Proof. (i) Clearly, $(N :_R M) \neq R$. Let $ab \in (N :_R M)$ with $b \notin (N :_R M)$ where $a, b \in h(R)$, so there exists $m \in h(M) - N$ such that $bm \notin N$. As $abm \in N$, N graded primary gives $a^k M \subseteq N$ for some k, as needed.

(ii) The proof is similar to that of (i). □

Proposition 2.6 *Let R be a G-graded ring, M a graded free R-module and I an ideal of R. Then the following hold:*

(i) If I is a graded primary ideal of R, then IM is a graded primary submodule of M.

(ii) If I is a graded prime ideal of R, then IM is a graded prime submodule of M.

Proof. (i) As M is a cancellation module and $I \neq R$, we get $IM \neq M$. Assume that M is the graded free R-module with a homogeneous basis $\{x_g : g \in G\}$ and let $rm \in IM$ with $m \notin IM$ where $r \in h(R)$ and $m \in h(M)$. We can write $m = \sum_{i=1}^{n} r_i x_{g_i}$ with $r_i \in R$. Since $m \notin IM$, there exists an integer j such that $r_j \notin I$. There are elements $b_1, ..., b_n \in I$ such that $\sum_{i=1}^{n} (rr_i) x_{g_i} = \sum_{i=1}^{n} b_i x_{g_i}$, so $rr_i = b_i$ for every $i = 1, ..., n$; hence $rr_j \in I$. Since $r_j = \sum_{i=1}^{s} r_{g_i} \notin I$ with $r_{g_i} \neq 0$, we obtain that $r_{g_t} \notin I$ for some t. It follows that $rr_{g_t} \in I$ since I is graded ideal, so $r^m \in I$ for some m; hence $r^m M \subseteq IM$, as required.

(ii) The proof is similar to that of (i). □

One approach to the graded case is simply to redefine all of the terminology to involve only homogeneous elements and graded submodules. In this vein, a non-zero graded module M is graded secondary if every homogeneous element of R either divides M or is nilpotent on M, in which case $\mathrm{Gr}(\mathrm{ann}M) = P$ is a graded prime ideal of R, and M is said to be graded P-secondary (see [6, Proposition 2.2]). A graded module M is said to be graded secondary representable if it can be written as a sum $M = M_1 + ... + M_k$ with each M_i graded secondary, and if such a representation exists (and is irredundant) then the graded attached primes of M are $\mathrm{Att}(M) = \{\mathrm{Gr}(\mathrm{ann}M), ..., \mathrm{Gr}(\mathrm{ann}M)\}$. Note that a graded secondary module, in general, is not secondary. For example, as discussed in Sharp ([6, p. 215]), if $R = k[x]$ is a polynomial ring in one variable with the natural Z-graded ring and $M = k[x, 1/x]$, then M is graded secondary but is not secondary. So the graded secondary and secondary modules are different concepts.

A graded submodule N of M is said to be graded pure submodule if $aN = N \cap aM$ for every $a \in h(R)$. We have the following proposition.

Proposition 2.7 *Let R be a G-graded ring, M a graded R-module and N a non-zero graded pure R-submodule of M. Then M is a graded P-secondary if and only if both N and M/N are graded P-secondary.*

Proof. Assume that M is P-secondary and let $a \in h(R)$. If $a \in P$, then $a^s N \subseteq a^s M = 0$ and $a^s(M/N) = 0$ for some s, so a is nilpotent on N and M/N. If $a \notin P$, then $aN = N \cap aM = N$ and $a(M/N) = M/N$, so a divides N and M/N; hence N and M/N are P-secondary. Conversely, assume that N and M/N are P-secondary and let $b \in h(R)$. If $b \in P$, then $b^t M \subseteq N$ and $0 = b^t N = N \cap b^t M = b^t M$ for some t, so b is nilpotent on M. If $b \notin P$, then $N = bN = N \cap bM$ and $b(M/N) = M/N$, so $bM = M$, as required. □

Theorem 2.8 *Let R be a G-graded ring, M a graded secondary R-module and N a non-zero graded P-prime R-submodule of M. Then N is graded P-secondary.*

Proof. Assume that M is a graded Q-secondary R-module and let $r \in h(R)$. If $r \in Q$, then $r^s N \subseteq r^s M = 0$ for some s, so r is nilpotent on N. Suppose that $r \notin Q$; we show that r divides N. So assume that $a \in N$. Then there exists $b = \sum_{i=1}^{t} b_{g_i} \in M$ (with $b_{g_i} \neq 0$) such that $a = rb$. As N is graded, $rb_{g_i} \in N$ for every $i = 1, ..., t$, so for each i, N graded prime gives $b_{g_i} \in N$; hence $b \in N$. It follows that r divides N, so N is a graded Q-secondary R-module.

Now we need to show that $P = Q$. Since the inclusion $P \subseteq Q$ is trivial, we will prove the reverse inclusion. Suppose that $c = \sum_{i=1}^{n} c_{h_i} \in Q$ with $c_{h_i} \neq 0$. Then there are integers m_i such that $c_{h_i}^{m_i} M = 0$ for $i = 1, ..., n$ since Q is graded and M is graded Q-secondary. As $M \neq N$, there is an element $x = x_{g_1} + ... + x_{g_u} \in M$ (with $x_{g_i} \neq 0$) such that $x_{g_w} \notin N$ for some w. Therefore, for each $i = 1, ..., n$, $c_{h_i}^{m_i} x_{g_w} = 0 \in N$, so N graded prime gives $c_{h_i} \in P$; hence $c \in P$, as required. □

Lemma 2.9 *Let R be a G-graded ring, M a graded R-module and N a graded P-secondary R-submodule of M. Then the following hold:*

(i) If K is a graded primary submodule of M, then $N \cap K$ is graded P-secondary.

(ii) If K is a graded prime submodule of M, then $N \cap K$ is graded P-secondary.

Proof. (i) Assume that $a \in h(R)$ and let $a \in P$. Then $a^m(N \cap K) \subseteq a^m N = 0$ for some m, so a is nilpotent on $N \cap K$. Suppose that $a \notin P$; we show that a divides $N \cap K$. It suffices to show that $N \cap K \subseteq a(N \cap K)$. If $b \in N \cap K$, then $b = am$ for some $m = \sum_{i=1}^{s} m_{g_i} \in N$ with $m_{g_i} \neq 0$. Then for each $i = 1, ..., s$, $am_{g_i} \in K$ since K is a graded submodule of M. It follows that $m_{g_i} \in K$ for every i (otherwise, if $m_{g_j} \notin K$ for some j and $a^s \in (K :_R M)$ for some s, then $m_{g_j} \in N = a^s N \subseteq a^s M \subseteq K$ which is a contradiction), so $m \in K$; hence $b \in a(N \cap K)$ and the proof is complete.

(ii) This follows from (i). □

Theorem 2.10 *(i) Every graded primary submodule of a graded representable module over a G-graded ring is graded representable.*

(ii) Every graded prime submodule of a graded representable module over a G-graded ring is graded representable.

Proof. (i) Assume that $M = \sum_{i=1}^{k} S_i$ is a minimal graded secondary representation of M with $\mathrm{Att}(M) = \{P_1, ..., P_k\}$ and let N be a graded P-primary submodule of M. There exists a submodule S_i, say S_1, such that $S_1 \nsubseteq N$ since $N \neq M$. First we show that $P = P_1$. Let $a = a_{g_1} + ... + a_{g_t} \in P_1$ with $a_{g_i} \neq 0$. There are integers $n_1, ..., n_t$ and a homogeneous element $y_h \in S_1 - N$ such that $a_{g_i}^{n_i} y_h = 0$ for every i, so N graded primary gives $a_{g_i} \in P$ for every i; hence $a \in P$. Therefore, $P_1 \subseteq P$. For the other containment, suppose that there exists a homogeneous element $c_h \in P$ with $c_h \notin P_1$. Then $S_1 = c_h^s S_1 \subseteq c_h^s M \subseteq N$ for some s which is a contradiction. Thus, $P = P_1$. Likewise, if $S_j \nsubseteq N$ for $j \neq 1$, then $P = P_1 = P_j$ which is a contradiction. We will show that $S_i \subseteq N$ for $i = 2, .., k$. As $P \neq P_i$ we divide the proof into two cases:

Case 1 $P \nsubseteq P_i$.

There exists a homogeneous element $p_h \in P$ with $p_h \notin P_i$. Then $S_i = p_h^t S_i \subseteq p_h^t M \subseteq N$ for some t.

Case 2 $P_i \nsubseteq P$.

There is a homogeneous element $a_g \in P_i$ with $a_g \notin P$. Let $b = \sum_{i=1}^{m} b_{h_i} \in S_i$ with $b_{g_i} \neq 0$. Then there is an integer n such that $a_g^n b_{h_i} = 0 \in N$, so N graded primary gives $b_{h_i} \in N$ for $i = 1, ..., m$; hence $b \in N$. Thus, $S_i \subseteq N$. It follows that $N = N \cap M = N \cap S_1 + \sum_{i=2}^{k} S_i$. Now the assertion follows from Lemma 2.9. □

Corollary 2.11 *Let R be a G-graded ring, M a graded representable R-module and N a graded primary (resp. graded prime) R-submodule of M. Then* $\mathrm{Att}(N) \subseteq \mathrm{Att}(M)$.

Proof. This follows from Theorem 2.10. □

Let R be a G-graded ring. The graded dimension of R is defined as the supremom of all numbers n for which there exists a chain of graded prime ideals $P_0 \subseteq P_1 \subseteq ... \subseteq P_n$ in R and it is denoted by Gdim$R$. We say that R is a G-graded integral domain whenever $a, b \in h(R)$ with $ab = 0$ implies that either $a = 0$ or $b = 0$.

Lemma 2.12 *Let P be a graded prime ideal of a G-graded ring R, M a graded R-module and $\{N_i\}_{i \in I}$ a family of graded prime R-submodules of M such that $(N_i :_R M) = P$ for every $i \in I$. Then $\cap_{i \in I} N_i$ is a graded prime submodule of M.*

Proof. The proof is straightforward. □

Theorem 2.13 *Let R be a G-graded integral domain with* Gdim$R = 1$*, M a graded representable R-module and N a graded primary R-submodule of M. Then* Gr(N) *is graded representable.*

Proof. Consider the graded ideal $(K :_R M)$ for any graded prime submodule K containing N. These ideals are graded prime by Proposition 2.5 and $N \subseteq K$ implies $(N :_R M) \subseteq (K :_R M)$; hence by [5, Proposition 1.2], $\mathrm{Gr}(N :_R M) \subseteq \mathrm{Gr}(K :_R M)$ for all such K. For any one of these prime submodules K, we generate the chain of graded prime ideals $0 \subset \mathrm{Gr}(N :_R M) \subseteq (K :_R M)$ since by [5, Lemma 1.8], $\mathrm{Gr}(N :_R M)$ is a graded prime ideal of R. As Gdim$R = 1$, we must have $\mathrm{Gr}(N :_R M) = (K :_R M)$ for every graded prime submodule K containing N. By Lemma 2.12, $\mathrm{Gr}(N) = \bigcap_{N \subseteq K} K$ is a graded prime submodule of M. Now the assertion follows from Theorem 2.10. □

Lemma 2.14 *Let R be a G-graded ring, M a graded R-module and N a graded representable R-submodule of M. Then if K is a graded primary (resp. graded prime) submodule of M, then $N \cap K$ is graded representable.*

Proof. By Theorem 2.10, it suffices to show that $N \cap K$ is a graded primary submodule of N. Let $an \in N \cap K$ with $n \notin N \cap K$ where $a \in h(R)$ and $n \in h(N)$, so K graded primary gives $a^s M \subseteq K$ for some s; hence $a^s(N \cap K) \subseteq N$, as required. □

Theorem 2.15 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M such that N possess a graded primary decomposition. If K is a graded representable submodule of M, then $N \cap K$ can be expressed as an intersection of finitely many graded representable submodules.*

Proof. Let $N = \bigcap_{i=1}^{n} N_i$, where N_i is graded primary, be a normal decomposition. Then $N \cap K = (N_1 \cap K) \cap ... \cap (K \cap N_n)$. Now the assertion follows from Lemma 2.14. □

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