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Hartley's Theorem on Representations of the General Linear Groups and Classical Groups

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To the memory of Brian Hartley

Abstract

We suggest a new proof of Hartley's theorem on representations of the general linear groups $\mathrm{GL}_n(K)$ where K is a field. Let H be a subgroup of $\mathrm{GL}_n(K)$ and E the natural $\mathrm{GL}_n(K)$-module. Suppose that the restriction $E|_H$ of E to H contains a regular KH-module. The theorem asserts that this is then true for an arbitrary $\mathrm{GL}_n(K)$-module M provided $\dim M > 1$ and H is not of exponent 2. Our proof is based on the general facts of representation theory of algebraic groups. In addition, we provide partial generalizations of Hartley's theorem to other classical groups.

Key Words: subgroups of classical groups, representation theory of algebraic groups

1. Introduction

In 1986 Brian Hartley [4] obtained the following interesting result:

Theorem 1.1 *Let K be a field, E the standard $\mathrm{GL}_n(K)$-module, and let M be an irreducible finite-dimensional $\mathrm{GL}_n(K)$-module over K with $\dim M > 1$. For a finite subgroup $H \subset \mathrm{GL}_n(K)$ suppose that the restriction of E to H contains a regular submodule, that is, $E \cong KH \oplus E_1$ where E_1 is a KH-module. Then M contains a free KH-submodule, unless H is an elementary abelian 2-groups.*

His proof is based on deep properties of the duality between irreducible representations of the general linear group $\mathrm{GL}_n(K)$ and the symmetric group S_n. We give here another

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proof which seems to be more conceptual and more transparent in the sense that it is based on general principles of representations theory of algebraic groups. We also expect that our method will have further applications. Observe that we do not consider here the exceptional case where H is of exponent 2, as it does not seem to be important to justify an additional work required with our approach.

Theorem 1.1 cannot be generalized to other classical groups without a stronger assumption imposed on H. Examples are available for H of order 3 and 5 (Proposition 6.2). However, we prove the following weaker version of Theorem 1.1:

Theorem 1.2 *Let $G = \mathrm{Sp}_{2n}(K)$ or $\mathrm{Spin}_n(K)$ and let $H \subset G$ be a subgroup of order coprime to the characteristic of K. Let E be the natural G-module and let M be an arbitrary irreducible G-module of dimension greater than 1. Suppose that $E|_H$ contains a free submodule of rank 2. Then $M|_H$ contains a regular KH-submodule.*

If $G = \mathrm{Sp}_{2n}(K)$ and $\mathrm{char} K \neq 2$ then Theorem 1.2 remains valid under assumption that $E|_H$ contains a regular KH-submodule and H is cyclic. This is not true for orthogonal groups. Our result in this case is the following:

Theorem 1.3 *Let $G = \mathrm{Spin}_m(K)$ where $m > 6$ and $\mathrm{char}\, K \neq 2$, and let E be the natural module for G. Let $g \in G$ be of odd order d and let E^g denote the subspace of vectors in E fixed by g. Suppose that g has d distinct eigenvalues on E. Then g has d distinct eigenvalues on every non-trivial irreducible G-module M except when $d = 3$ or 5 and $\dim E^g = m - d + 1$.*

We only consider below the case where the ground field K is algebraically closed, and representations of (and modules for) classical groups G in question are rational. The case where K is infinite but not algebraically closed follows from this one due to a theorem of Borel and Tits [1] as explained in [4, page 123]. If K is finite then every irreducible representation of G extends to a rational representation of the respective classical group over the algebraic closure of K. This is essentially a theorem of Steinberg [8, Theorem 43], see [4, page 114] for details.

2. Preliminaries

We first make easy reduction to a particular case of Theorem 1.1.

Lemma 2.1 *It suffices to prove the theorem for $n = |H|$.*

Proof. Indeed, express $E = E_1 \oplus E_2$ where $E_1 \cong KH$ is a regular KH-module (so $\dim E_1 = |H|$) and E_2 is a complement. Then $H \subset D := D_1 \times D_2$ where $D_i \cong GL(E_i)$ for

$i = 1, 2$. Therefore, the restriction $M|_D$ contains an irreducible submodule $L = L_1 \otimes L_2$ where L_i is a D_i-module and L_1 is not 1-dimensional. Then $L|_H = L_1|_H \otimes L_2|_H$. As the theorem is assumed to be true for E_1, the restriction $L_1|_H$ contains a regular KH-submodule. Then this is so for L_H and for M_H. $\square$

The following lemma by R. Bryant is stated in [4, page 116].

Lemma 2.2 *Let G be a finite group of order n and let $1 \leq m < n$ be an integer. Then there is a subset R of G of size m such that $xR = R$ implies $x = 1$ unless G has exponent 2 and $m = 2$ or $n - 2$.*

Let Ω be a finite set of size n and let Ω_m for $1 < m < n$ denote the set of all unordered m-tuples of elements from Ω (so $|\Omega_m| = \binom{n}{m}$). Let G be a subgroup of $S_n = \mathrm{Sym}\,(\Omega)$ such that G has a regular orbit on Ω. We need a result stated as Proposition 2.3 which easily follows from Bryant's lemma but it has not been stated in [4].

Proposition 2.3 *Let G be a finite group acting on a set Ω of size n. Suppose that G has a regular orbit on Ω. Then G has a regular orbit on Ω_m for $1 < m < n$, except for the case where G is of exponent 2, acts transitively on Ω and $m = 2$ or $n - 2$.*

Proof. Suppose first that G is transitive on Ω, that is, $n = |G|$. Then the elements of Ω can be labeled by elements of G, more precisely, Ω and G are isomorphic G-sets where the action of G on itself is defined via the left multiplication. By Lemma 2.2, there is an m-tuple $R \subset \Omega_m$ such that $xR = R$ for $x \in G$ implies $x = 1$ unless G is of exponent 2 and $m = 2$ or $n - 2$. It follows that GR is a regular G-orbit on Ω_m unless we are not in the exceptional case.

Suppose next that G is intransitive. If $m \neq 2, n - 2$ then the result immediately follows from that for the transitive case. Let $m = 2$. Let $a \in \Omega$ be such that the orbit Ga is regular and $b \in \Omega$ with $b \notin Ga$. Then $\{xa, xb\} = \{a, b\}$ for $x \in G$ implies $x = 1$ and the result follows. As the G-sets Ω_m and Ω_{n-m} are isomorphic, this also implies the result for $m = n - 2$. $\square$

Remark. The stabilizer of a point in Ω_m is isomorphic to $S_{n-m} \times S_m$, and if $n - m \geq m$ then $S_{n-m} \times S_m$ is a Young subgroup of S_n corresponding to the Young diagram $\lambda = [n - m, m]$.

Proposition 2.4 *Let G be a finite group, not of exponent 2, acting on a set Ω of size n. Suppose that G has a regular orbit on Ω. Then G has a regular orbit on S_n/Y_λ where Y_λ is any Young subgroup of S_n whose Young diagram consists more than one row.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \geq \cdots \geq \lambda_k > 0$ is a partition of n. Consider the partition $\mu = (n-m, m)$ where $n - m = \lambda_1 + \cdots + \lambda_{k-1}$ and $m = \lambda_k$. Then we can assume that Y_λ is contained in Y_μ. By Lemma 2.4, G has a regular orbit on S_n/Y_μ, which is equivalent to saying that $G \cap g^{-1}Y_\mu g = 1$ for some $g \in S_n$. Then $G \cap g^{-1}Y_\lambda g = 1$. This is equivalent to saying that G has a regular orbit on S_n/Y_λ. □

3. Algebraic groups and their modules

We first make short comments to explain the notions of weight and weight space to readers unfamiliar with the representation theory of algebraic groups. Let K be algebraically closed field of characteristic $p \geq 0$.

The set T of diagonal matrices in $\mathrm{SL}_n(K)$ is an example of a maximal torus in an algebraic group. Denote by $K^\times$ the multiplicative group of K and set $T^* := \mathrm{Hom}(T, K^\times)$. For $\lambda \in T^*$ and for an $\mathrm{SL}_n(K)$-module M we set $M_\lambda = \{x \in M : tx = \lambda(t)x \text{ for all } t \in T\}$. If $M_\lambda \neq 0$ then T is called the *T-weight space of M of weight* λ or just the weight space of weight λ if T is fixed. We set $T^*(M) = \{\lambda \in T^* : M_\lambda \neq 0\}$ and call $T^*(M)$ *the set of weights of M*.

Observe that $\mathrm{GL}_n(K) = Z \cdot \mathrm{SL}_n(K)$ where Z denotes the group of non-zero scalar matrices. It follows that every irreducible $\mathrm{GL}_n(K)$-module remains irreducible under restriction to $\mathrm{SL}_n(K)$. Therefore, if M is an $\mathrm{GL}_n(K)$-module then we can and shall keep the same meaning for M_λ, the weight space of weight λ. Let $S \cong S_n$ be the group of permutational matrices in $\mathrm{GL}_n(K)$. Then S normalizes T, as it is the group of diagonal matrices of determinant 1, and moreover S acts faithfully on T. The conjugation action of S on T induces the dual action of S on T^* in an obvious way, namely, for $\lambda \in T^*$ and $s \in S$ one defines $\lambda^s(t) = \lambda(s^{-1}ts)$. Therefore, if $x \in M_\lambda$ and $s \in S$ then $sx \in M_{\lambda^s}$. This tells us that S permutes the T-weight spaces of M and this action is isomorphic to the action of S on $T^*(M)$. Therefore, it does not depend in a sense on the choice of M, as it is described in terms of the action of S on T^*.

Theorem 3.1 *Let ω be any non-zero weight of a $\mathcal{G}$-module M where $\mathcal{G}$ is a simple algebraic group. Let $C_W(\omega)$ denote the stabilizer of ω in W, the Weyl group of $\mathcal{G}$. Then $C_W(\omega)$ is generated by reflections. In particular, if $\mathcal{G} = \mathrm{SL}_n(K))$ where $W \cong S_n$ then $C_W(\omega)$ is a Young subgroup of S_n.*

Proof. This is essentially a particular case of [8, Addendum, Theorem 36]. In fact, the weights of rational representations of an algebraic group $\mathcal{G}$ are elements of a vector space V (over the rational numbers) on which W acts 'naturally', that is, as a group generated by reflections. If $\mathcal{G} = \mathrm{SL}_n(K))$ then $\dim V = n-1$ and reflections are exactly transpositions in S_n. It is well known and easy to justify that every subgroup of S_n generated by transpositions is a Young subgroup. $\square$

4. Proof of Hartley's Theorem

By Lemma 2.1, it suffices to prove the following particular case of Theorem 1:

Proposition 4.1 *Let E be the natural module for $\mathrm{GL}_n(K)$ and let H be a finite subgroup of $\mathrm{GL}_n(K)$ not of exponent 2. Suppose that E itself is a regular KH-module. Let M be an irreducible rational $\mathrm{GL}_n(K)$-module of dimension greater than 1. Then $M|_H$ contains a regular H-submodule.*

Proof. Observe first that a regular KH-module is isomorphic to the permutational KH-module associated with the action of H on itself via the left multiplications. It follows that there is a basis $e_1, \dots, e_n$ of E such that $\{e_1, \dots, e_n\}$ is a regular H-set. Denote by T the set of diagonal matrices in $\mathrm{SL}_n(K)$ with respect to this basis. Then H normalizes T so we can assume that $H \subset S$ where S is the group of permutational matrices. Moreover, H is a transitive subgroup of $S \cong S_n$ and $n = |H|$.

Let $W := N_{\mathrm{SL}_n(K)}(T)/T$ denote the Weyl group of $\mathrm{SL}_n(K)$ and let Z be the center of $\mathrm{GL}_n(K)$. Then $W = N_{\mathrm{GL}_n(K)}(T)/TZ$ and the image of S under the projection $\pi : N_{\mathrm{GL}_n(K)}(T) \to N_{\mathrm{GL}_n(K)}(T)/TZ$ coincides with W. It follows that the action of H on T by conjugation coincides with the action of $\pi(H)$ as a subgroup of the Weyl group of $\mathrm{SL}_n(K)$.

Let ω be a non-zero weight of $M|_{\mathrm{SL}(n,K)}$. By Theorem 3.1, the orbit $W\omega$ is isomorphic to S_n/Y_λ for some Young subgroup Y_λ of S_n. By Proposition 2.4, H has a regular orbit on S_n/Y_λ. Hence there is a weight $\mu \in W\omega$ such that the orbit $H\mu$ is regular. Let U_μ be the T-weight space of weight μ in M. Then $hU_\mu = U_\mu$ for $h \in H$ implies $h = 1$. Therefore for any non-zero vector $0 \neq u \in U_\mu$ the vectors $\{hu : h \in H\}$ belong to distinct weight spaces, and hence they are linear independent. It follows that the K-span of the vectors $\{hu : h \in H\}$ forms a regular KH-submodule of M. $\square$

Remark. In fact, we have shown that every orbit $W\omega$ for non-zero weight ω contains a regular H-orbit. As for distinct dominant weights ω, ω' the orbits $W\omega, W\omega'$ are disjoint, it follows that $M|_H$ contains a free KH-submodule of rank r which is not less than the number of non-zero dominant weights of M. Moreover, one can refine this to claim that r is not less than the sum of the dimensions of the weight spaces M_ω where ω runs over all non-zero dominant weights of M.

5. Other classical groups

In this section we assume that the reader is familiar with standard facts of the theory of algebraic groups and their representations.

Lemma 5.1 *Let V be an orthogonal or symplectic vector space over a algebraically closed field K and let $I(V)$ denote the group of isometries of V. Let H be a subgroup of $I(V)$ such that V is a homogeneous KH-module. (This means that V is the sum of irreducible KH-submodules isomorphic to each other.) Suppose that V has no totally isotropic (totally singular) KH-submodule. Then V is irreducible.*

Proof. Suppose first that V is not an orthogonal space in characteristic 2. Then there exists an anti-automorphism σ of the matrix algebra $M_n(K)$ such that $I(V) = \{x \in M_n(K) : \sigma(x)x = 1_n\}$. (Here 1_n stands for the identity matrix.) As V is a homogeneous KH-module, the centralizer C of H in $M_n(K)$ is isomorphic to $M_k(K)$ where k is the composition length of V as a KH-module. As $\sigma(H) = H$, it follows that $\sigma(C) = C$. Let $C^\sigma = \{x \in C : \sigma(x)x = 1_n\}$. Then $C^\sigma = C \cap I(V)$. Let $U = K^k$ be the natural $M_k(K)$-module. It is known (see for instance Wagner [12]) that there exists a non-degenerate bilinear form f on U such that $C^\sigma \cong I(U)$. Suppose (arguing by contradiction) that the lemma is false. Then $k > 1$ and hence U contains a non-zero isotropic 1-dimensional subspace U', say. (This is true in the orthogonal case with $k = 2$ as K is algebraically closed.) Moreover, there is a basis in U such that $I(U)$ contains the subgroup $T = \{t_a : 0 \neq a \in K\}$ such that $TU' = U'$ and $t_a := \mathrm{diag}(a, 1_{k-2}, a^{-1})$ for all $0 \neq a \in K$. Moreover, we can assume that $t_a u = au$ for $u \in U'$. Now view T as a subgroup of C due to an isomorphism $C^\sigma \cong I(U)$. Set $V' = \{v \in V : t_a v = av$ for $a \in K, a \neq 0\}$. Evidently, $\dim V' = n/k$ and $HV' = V'$. We show that V' is totally isotropic. Suppose the contrary. Then $V' \neq R$ where $R := V' \cap (V')^\perp$ is the radical of V'. Moreover, R is a KH-module whose KH-module complement R', v is non-degenerate. Therefore, $t_a|_{R'}$ belongs to $I(R')$. However, $t_a|_{R'} = a \cdot \mathrm{Id}$ is scalar but the only scalar matrices in $I(V')$ are $\pm\mathrm{Id}$, so we have a contradiction. We are left with the case where V is an orthogonal space in characteristic 2. Let Q be the quadratic form defining the orthogonal structure on V and let f be the associated bilinear form. The above argument shows that V contains an H-submodule V' such that $f(v,w)=0$ for any $v,w\in V'$ and $t_a|_{V'}=a\cdot\mathrm{Id}$. If V' is not totally singular then $Q(v)\neq 0$ for some $v\in V'$. Then $Q(v)=Q(t_a v)=Q(av)=a^2Q(v)$ which is false for $a\neq 1$. $\square$

Lemma 5.2 *Let $G=\mathrm{Sp}_{2n}(K)$ or $\mathrm{Spin}_n(K)$ and let $H\subset G$ be a subgroup of order coprime to the characteristic of K. Let E be the natural G-module. Suppose that $E|_H$ contains a free submodule of rank 2. Then $E|_H$ contains a regular totally isotropic (totally singular) H-submodule.*

Proof. Let τ be an irreducible KH-module, and let τ' be the dual of τ. Denote by E_τ the homogeneous τ-component of E, that is, the sum of all irreducible KH-submodules of $E|_H$ isomorphic to τ. Let d_τ denote the composition length of H on E_τ. Then $d_\tau\geq 2\dim\tau$. By [7, Lemma 3.3], E_τ is either (i) non-degenerate or (ii) totally isotropic (totally singular), and in the latter case $E_{\tau'}$ is totally isotropic (totally singular), $\tau\not\cong\tau'$, $\dim E_\tau=\dim E_{\tau'}$ and $E_\tau+E_{\tau'}$ is non-degenerate. It follows that $d_\tau=d_{\tau'}$.

Suppose first that (i) hold. Let Y be a maximal totally isotropic (totally singular) KH-submodule of E_τ. Then $L:=(E_\tau\cap Y^\perp)/Y$ contains no totally isotropic (totally singular) KH-submodule. Set $t=\dim L$. We show that either $t=0$ or $t=\dim\tau$. Let $t\neq 0$. Then L inherits the bilinear (quadratic) form which defines G so H_L is a subgroup of $D:=Sp_t(K)$ or $SO_t(K)$. By Lemma 5.1, L is irreducible KH-module and $\dim L=\dim\tau$. As Y and $E_\tau/(E_\tau\cap Y^\perp)$ are isomorphic KH-modules, the composition length of H on Y is $d_\tau/2\geq\dim\tau$ if $t=0$ and $(d_\tau-1)/2$ otherwise. In the latter case d_τ is odd, so $d_\tau>2\dim\tau$. Hence the composition length of H on Y is at least $\dim\tau$ in both the cases. Let (ii) hold. Then E_τ has a submodule X, say, of composition length $\dim\tau$. In addition, the composition length of H on $E_{\tau'}/(E_{\tau'}\cap X^\perp)$ is equal to that of X which is $\dim\tau$. Therefore, the composition length of $E_{\tau'}\cap X^\perp$ is at least $d_\tau-\dim\tau\geq\dim\tau$. It follows that $E_{\tau'}\cap X^\perp$ contains a submodule X', say, of composition length $\dim\tau'$. The module $X+X'$ is totally isotropic (totally singular) as $X'\subseteq E_{\tau'}\cap X^\perp$.

Obviously, $E=\oplus E_\tau$ where τ runs over the set $\mathrm{Irr}H$ of irreducible KH-modules. Moreover, E_τ is orthogonal to E_σ unless $\sigma=\tau$ or τ'. If E_τ is non-degenerate, let Y_τ be KH-submodule Y of E_τ constructed above and X_τ its submodule of composition length d_τ. If E_τ is totally isotropic (totally singular) then let $X_\tau\subseteq E_\tau$ and $X_{\tau'}\subseteq E_{\tau'}$ be submodules orthogonal to each other as described above. Then $R:=\oplus_{\tau\in\mathrm{Irr}H}X_\tau$ is a totally isotropic (totally singular) submodule of E isomorphic to the regular KH-module. $\square$

Proof of Theorem 1.2. There is an orthogonal decomposition $E = E_1 \oplus E_2$ where $E_1 = R + R'$ and R, R' are totally isotropic (totally singular) KH-modules, each isomorphic to KH, and E_2 is non-degenerate. Indeed, let R be a regular totally isotropic (totally singular) submodule of $E|_H$ constructed in Lemma 5.2. Define E_2 to be a KH-module complement of R in $R^\perp$ and $E_1 = E_2^\perp$. Then E_2, and hence E_1, is non-degenerate. Take for R' any KH-module complement of R in E_1. Suppose that E is not an orthogonal space in characteristic 2. Then R' is totally isotropic (totally singular) as otherwise R' would contain a non-degenerate submodule D, say, and if we let $D_1 = D^\perp \cap E_1$ then D_1 is non-degenerate and contains R. Then $2 \dim R \leq \dim D_1$ which is false. Suppose next that E is an orthogonal space in characteristic 2. Then we refine this argument as follows. Let R_0 be the sum of all non-trivial KH-submodules of R. Then the above argument applied to R_0 in place of R yields a similar submodule R_0'. As R_0' is dual to R_0, it has no non-zero vectors fixed by H. We show that R_0' is totally singular. Indeed, if not, set X be the set of all singular elements in R_0'. Then X is a subspace of codimention 1 in R_0', hence H acts trivially on R_0'/X contrary to the fact that H acts fixed point freely on R_0'. Thus, R_0' is totally singular. Next we can pick from $(R_0 + R_0')^\perp$ two singular vectors v, v' fixed by H such that $\langle v, v'\rangle$ is non-degenerate. (These v, v' are available as the subspace of H-fixed points in E is non-degenerate.) Then set $R = \langle R_0, v\rangle$ and $R' = \langle R_0', v'\rangle$ to obtain $E_1 = R + R'$ with properties required.

Set $d = |H|$. There is a vector $v \in R$ such that $\{hv : h \in H\}$ is a basis of R. Choose a dual basis in R' which always exists by [3, §11(1) and §16]. Let B be a basis in E such that $B \cap E_2$ is a basis of E_2 and $B \cap R$, $B \cap R'$ are dual bases in R, R' described above. Let $G_1 = \{g \in G : gR = R, gR' = R' \text{ and } g|_{E_2} = \mathrm{Id}\}$. Then if $g \in G_1$ then the matrix of $g|_E$ is of shape $\mathrm{diag}(A, {}^TA^{-1}, \mathrm{Id})$ where A is a square matrix of size $d = \dim R$ and ${}^TA$ is the transpose of A. Furthermore, if g runs over G_1 then A runs over a set containing all matrices of determinant 1. It follows that $G_1|_R$ is a subgroup of $\mathrm{GL}(R) \cong \mathrm{GL}_d(K)$ containing $\mathrm{SL}(R)$. Clearly, $H|_R \subset \mathrm{GL}(R)$ and the matrix of $h \in H$ under the above basis is of shape $\mathrm{diag}(h_1, {}^Th_1^{-1}, h_2)$ where h_1 is a $(d \times d)$-matrix and h_2 is a $(2n - d \times 2n - d)$-matrix.

As R, R', E_2 are KH-modules, it follow that H normalizes G_1. Observe next that G_1 has a subgroup L, say, isomorphic to $\mathrm{SL}_d(K)$. Indeed, $\mathrm{SL}_d(K)$ is simply connected, hence it does not have any non-split central extension as an algebraic group. Therefore, as G_1 is an algebraic group and $G_1/Z \cong \mathrm{SL}_d(K)$ for a suitable central subgroup Z of G_1, we conclude that G_1 is the direct product of Z and a subgroup isomorphic to $\mathrm{SL}_d(K)$. (If $G = Sp_{2n}(K)$ then $Z = 1$ so we do not need this argument.) Moreover, H normalizes L as L is the commutator subgroup of G_1.

Let $\alpha : L \to \mathrm{SL}(R)$ be the isomorphism defined by $g \to g|_R$ for $g \in L$. Let T be the group of diagonal matrices in $\mathrm{SL}(R) \cong \mathrm{SL}_d(K)$, and $T_L = \alpha^{-1}(T)$. As it is mentioned in the proof of Proposition 6.2, we can assume that $H|_R$ normalizes T. Then H normalizes T_L, and the actions of H on T_L and on T agree in the sense that $h\alpha^{-1}(t)h^{-1} = \alpha^{-1}(h_1 t h_1^{-1})$ for $t \in T$. Furthermore, α yields an isomorphism $\beta : \mathrm{Hom}(T_L, K^\times) \to \mathrm{Hom}(T, K^\times)$ defined for $\lambda \in \mathrm{Hom}(T_L, K^\times)$ as $\beta(\lambda)(t) = \lambda(\alpha^{-1}(t))$ ($t \in T$). Then the H-action on T_L by conjugation yields an H-action on $\mathrm{Hom}(T_L, K^\times)$. Moreover, if $\lambda \in \mathrm{Hom}(T_L, K^\times)$ then $h(\lambda)$ coincides with $\beta^{-1}(h_1(\beta(\lambda)))$ as the H-actions on T_L and on T agree.

We show that H has a regular orbit on T_L-weight spaces of M. Indeed, let M_L be a non-trivial irreducible constituent of the restriction $M|_L$ which can be viewed as an $\mathrm{SL}(R)$-module via α^{-1}. By Theorem 4.1 and the remark following it, $H|_R$ has a regular orbit on the T-weights of M_L, which can be regarded as T_L-weights via α. It follows from the comments in the previous paragraph that H has a regular orbit on the T_L-weights of M_L. Let, say, $H\lambda$ be a regular orbit for some T_L-weight λ of M_L. Pick a non-zero vector $m \in M_L \subseteq M$ from a T_L-weight space of weight λ. As vectors from distinct weight spaces are linear independent, $\langle hm : h \in H\rangle$ is a regular KH-submodule of M. (Observe that hm may not belong to M_L but we do not need it to belong.)

6. Cyclic subgroups

The following particular case of Theorem 1.1 is of interest for the study of the eigenvalues of semisimple elements of simple algebraic groups in their irreducible representations.

Theorem 6.1 *Let $g \in \mathrm{SL}_n(K)$ be a semisimple element of finite order d, and let E be the natural $\mathrm{SL}_n(K)$-module. Suppose that g has exactly d distinct eigenvalues on V. Then g has d distinct eigenvalues on every non-trivial $\mathrm{SL}_n(K)$-module.*

Proof. Indeed, let $H = \langle g\rangle$ and let M be an arbitrary non-trivial $\mathrm{SL}_n(K)$-module. Then V_H has a regular KH-submodule and, by Hartley's theorem, M_H has a regular KH-submodule. (The case $|H| = 2$ is not exceptional here.) As d is coprime to the characteristic of K, g is diagonalizable on M and has d distinct eigenvalues. □

A similar result is not true for orthogonal groups. Indeed, let $G = \mathrm{Spin}_{2n+1}(K)$ where $n = 1$ or 2. Pick $g \in G$ of order $d = 2n + 1$ such that g has d distinct eigenvalues on the natural KG-module E. Let $H = \langle g\rangle$. Assume that the characteristic of K is not 2. Then E is a regular KH-module. There are isomorphisms $\mathrm{Spin}_3(K) \cong \mathrm{Sp}_2(K)$ and

$\mathrm{Spin}_5(K) \cong \mathrm{Sp}_4(K)$. However, g does not have eigenvalue 1 on the natural module for $\mathrm{Sp}_2(K)$ and $\mathrm{Sp}_4(K)$.

One could think that these examples are exceptional due to the above isomorphisms. However, the following results makes evidence that this is not the case. In the proposition below the expression 'g acts regularly on V_1' means that V_1 is a regular $K\langle g\rangle$-module, equivalently, g has $|g|$ distinct eigenvalues on V_1.

Proposition 6.2 *Let E be the natural module for the orthogonal group $O_m(K)$ and let $E = E_1 \oplus E_2$ be the orthogonal sum of non-generate subspaces of E where $\dim E_1 = d$ and $d = 3$ or 5. Let M be the module of the spinor representation of $G = \mathrm{Spin}_m(K)$. Let $g \in G$ be an element of order d such that $gE_i = E_i$ for $i = 1, 2$, g acts trivially on E_2 and regularly on E_1. Then $g|_M$ has $d-1$ distinct eigenvalues and 1 is not an eigenvalue of $g|_M$.*

Proof. Set $G_i = \{x \in G : xE_1 = E_1,\ xE_2 = E_2,\ x|_{E_i} = Id\}$. Then $G_2 \cong \mathrm{Spin}_d(K)$. Suppose first $m = 2n+1$ is odd, so G is of type B_n. It is well known that $M|_{G_2}$ is a direct sum of modules isomorphic to each other and realizing the spinor representation of G_2. (For instance, this follows by induction from the comments at the end of §129 in [14] in the case where K is of characteristic 0. However, all weights of the spinor representation are of dimension 1 and are in a single orbit of the Weyl group. Therefore, the respective Weyl module in characteristic $p > 0$ is irreducible and its restriction to $\mathrm{Spin}_d(K)$ is again a direct sum of modules isomorphic to the the spinor module for $\mathrm{Spin}_d(K)$.) Thus, if m is odd, the result follows from that for dimension d as discussed above. Suppose that $m = 2n$ is even. Then $M|_X$ is irreducible for $X = \mathrm{Spin}(2n-1, K)$ and is isomorphic to the spinor X-module. (This follows from [14, §129] for characteristic 0 and remains true for prime characteristic by the same reason.) So again the result follows from the above. □

Remark. If m is even then the similar result is true for the semispinor irreducible representations of G, that is, with highest weight ω_{n-1}. (Here and below ω_i stands for a fundamental weight of a simple algebraic group; the fundamental weights are ordered as in Bourbaki [2].)

Proposition 6.2 can give impression that the above theorem remains true for any d. However, in fact the opposite is true.

Theorem 6.3 *Let $g \in \mathrm{Spin}_m(K)$ be of odd order $d > 5$ where $\mathrm{char}\, K \neq 2$. Suppose that g has d distinct eigenvalues on V. Then g has d distinct eigenvalues on every irreducible $\mathrm{Spin}_m(K)$-module M of dimension greater than 1.*

Proof. Clearly, g is semisimple. Suppose first that $d = \dim E$ and that $M = X$ is the spinor module, that is, the highest weight of E is ω_n where $n = (d-1)/2$. Following Bourbaki [2], denote the weights of E by $\varepsilon_1, \dots, \varepsilon_n, 0, -\varepsilon_n, \dots, -\varepsilon_1$. Then the weights of X are $(\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_n)/2$. We can reorder the eigenvalues of g so that the spectrum of g on E is $\{\beta, \beta^2, \dots, \beta^n, 1, \beta^{-n}, \dots, \beta^{-2}, \beta^{-1}\}$. Then the spectrum of g on M is $\{\beta^{(\pm 1 \pm 2 \pm \cdots \pm n)/2}\}$. Let U_n denote the set of all integers $\pm 1 \pm 2 \pm \cdots \pm n$. We show that $\{\beta^i : i \in U_n\}$ contains all d-roots if 1. It is easy to check that U_n consists of all integers l such that $-\frac{n(n+1)}{2} \leq l \leq \frac{n(n+1)}{2}$ and $\frac{n(n+1)}{2} - l$ is even. Suppose first that $\frac{n(n+1)}{2}$ is even. Then $0 \in U_n$ and β^i with $i > 0$ together with $\beta^i = \beta^{d+i}$ with $i < 0$ yield all d-roots of 1. Let $\frac{n(n+1)}{2}$ be odd. Then $d \in U_n$ for $d > 5$, and, similarly, $\beta^i = \beta^{d+i}$ with $i < 0$ together with β^i with $i > 0$ yield all d-roots of 1.

Keep the assumption that $\dim E = d$, and suppose that M is p-restricted. By [11, Proposition 2.3], the set of weights of M contains either the set of weights of E or the set of weights of X. Therefore, the set of eigenvalues of g on M contains either the spectrum of g on E or the spectrum of g on X. As we have shown, the spectrum of g on X contains all d-roots of 1 so the lemma follows for the case $\dim E = d$.

Next, suppose that M is not p-restricted. Then M is the tensor product $M_1 \otimes \cdots \otimes M_k$ where $M_1, \dots, M_k$ are Frobenius twists of non-trivial modules with restricted highest weights. Then $g|_M = g|_{M_1} \otimes \cdots \otimes g|_{M_k}$. Observe that if L is a G-module and $g|_L$ has d distinct eigenvalues then g has d distinct eigenvalues on every Frobenius twist of L. So the result follows from that for p-restricted modules.

Let $\dim E > d$. We first show that there are non-degenerate g-stable subspaces E_1, E_2 in E such that $E = E_1 \oplus E_2$, g acts on E_1 regularly. Let μ be an eigenvalue of g on E and let E_μ be the respective μ-eigenspace. If $\mu \neq 1$ then $E_\mu \neq E_{\mu^{-1}}$ and E_μ is totally singular (otherwise E_μ has a non-degenerate subspace U, say, and $\mu \cdot \mathrm{Id} \in O(U)$ which is false). It is observed in the first paragraph of the proof of Lemma 5.2 that $E_\mu + E_{\mu^{-1}}$ is non-degenerate so we can find vectors $v_\mu \in E_\mu$ and $v_{\mu^{-1}} \in E_{\mu^{-1}}$ such that $\langle v_\mu, v_{\mu^{-1}} \rangle$ is non-degenerate. Furthermore, $E_\mu + E_{\mu^{-1}}$ is orthogonal to every E_ν for $\nu \neq \mu, \mu^{-1}$. These can be added by an anisotropic vector $v_1 \in E_\nu$ for $\nu = 1$. Then set $E_1 = \langle v_1, v_\mu \rangle$ where μ runs over all eigenvalues $\mu \neq 1$. It is easy to observe that E_1 is non-degenerate. Hence $E_2 = E_1^\perp$ is non-degenerate.

Define $G_1 = \{x \in G : xE_1 = E_1, xE_2 = E_2 \text{ and } x|_{E_2} = \mathrm{Id}\}$. Then $G_1 \cong \mathrm{Spin}_d(K)$. Let $G_1^+ = \langle G_1, g \rangle$ and let M_1 be an irreducible constituent of the restriction $M|_{G_1^+}$ of dimension greater than 1. Set $g' = g|_{E_1}$. Then $g' \in SO(E_1)$. Therefore, $g'' = \mathrm{diag}(g', \mathrm{Id})$ is of order d and belongs to $SO(E)$. Pick an element $g_1 \in G_1$ such that $g_1|_E = g''$. Then

$g_1^{-1}g \in Z(G_1^+)$ where $Z(G_1^+)$ is the center of G_1^+. Then $g|_{M_1}$ and $g_1|_{M_1}$ have the same number of distinct eigenvalues. This number is equal to d for $g_1|_{M_1}$. So the result follows. □

Proof of Theorem 1.3. If $d > 5$, the result is contained in Theorem 6.4. So we assume that $d \leq 5$. Then $m > d$ as $m > 6$. As in the proof of Theorem 6.4 we first observe that there is an orthogonal decomposition $E = E_1 \oplus E_2$ where E_1, E_2 are non-generate g-stable subspaces, $\dim E_1 = d$ and g has d distinct eigenvalues on E_1. As $\dim E^g \geq m-d$ and $m > 6$, it follows that g has at least two distinct eigenvalues on E_2.

Define $G_1 = \{x \in G : xE_1 = E_1, xE_2 = E_2 \text{ and } x|_{E_2} = \mathrm{Id}\}$ and $G_2 = \{x \in G : xE_1 = E_1, xE_2 = E_2 \text{ and } x|_{E_1} = \mathrm{Id}\}$. Then $G_1 \cong \mathrm{Spin}_d(K)$, $G_2 \cong \mathrm{Spin}_{m-d}(K)$. Then $g \in G_1G_2$. As d is odd, and the center of G is a 2-group, we can express $g = g_1g_2$ where $g_1 \in G_1$ and $g_2 \in G_2$. Suppose first that M is the spinor module. It is known $M|_{G_1G_2}$ contains an irreducible constituent L, say, which is the tensor product of the spinor modules for G_1 and G_2. Express this as $L = L_1 \otimes L_2$. Then $g|_L = g_1|_{L_1} \otimes g_2|_{L_2}$. By Proposition 6.2, $g_1|_{L_1}$ has $d-1$ distinct eigenvalues. If $\dim E_2 \neq m-d+1$ then E_2 is not contained in E^g. Hence $g|_{E_2}$ has an eigenvalue $\lambda \neq \pm 1$. In addition, $g|_{E_2} \neq \lambda \cdot \mathrm{Id}$ as E_2 is non-degenerate. Therefore, $g_2|_{L_2}$ is not scalar. It follows that $g|_L$ has d distinct eigenvalues.

Let M be an arbitrary p-restricted G-module. By [11, Proposition 3.2], $g|_M$ has d distinct eigenvalues. Let M be an arbitrary (not p-restricted). Then the argument used in the proof of Theorem 6.3 completes the proof.

Theorem 6.4 *Let $g \in \mathrm{Sp}_{2n}(K) = \mathrm{Sp}(E)$ where $\mathrm{char}\, K \neq 2$ and $d := |g|$. Suppose that g has d distinct eigenvalues on E. Then g has d distinct eigenvalues on every non-trivial irreducible $\mathrm{Sp}_m(K)$-module M.*

Proof. Obviously, g is semisimple. Observe first that eigenvalue 1 has even multiplicity. This is because of the well known fact that the 1-eigenspace of g on E is non-degenerate.

Suppose first that M is the adjoint module (of highest weight ω_2). Let $e_1, \ldots, e_{2n}$ be a hyperbolic basis of V consisting of eigenvectors of g, say, let $ge_i = \alpha_i e_i$. We can assume that $\alpha_1 = 1$ and $\alpha_{n+i} = \alpha_i^{-1}$. It is well known that the basis b_{ij} with $1 \leq i < j \leq 2n$ of the exterior square E_2 can be chosen so that $gb_{ij} = \alpha_i\alpha_j b_{ij}$. It follows that $gb_{1j} = \alpha_j b_{1j}$. Furthermore, E_2 is a reducible $\mathrm{Sp}_{2n}(K)$-module, and, in fact, $\mathrm{Sp}_{2n}(K)$ fixes a non-zero vector v, say, on E_2. If $\mathrm{char}\, K$ is coprime to n then $E_2 = \langle v \rangle \oplus E_2'$ where E_2' is irreducible and isomorphic to the adjoint module for $\mathrm{Sp}_{2n}(K)$. Otherwise, E_2 contains 2 trivial irreducible constituents and the adjoint module for $\mathrm{Sp}_{2n}(K)$ is isomorphic to the only non-trivial constituent E_2' (cf. [5]). Let m be the multiplicity of eigenvalue 1 on E_2.

As the multiplicity of eigenvalue 1 on E is at least 2, $m \geq (d+1)/2$ if d is odd, and $m \geq (d+2)/2$ if d is even. So if $d \geq 5$ then g has $|g|$ distinct eigenvalues on E_2' as required. Let $d = 4$. Then $n \geq 6$ as the multiplicity of eigenvalue 1 and -1 on E is at least 2. So again $m \geq 1$. Let $d = 3$. If $n > 2$ then $m > 2$. If $n = 2$ then $m = 2$ but in this case n is coprime to $\operatorname{char} K$ so g has an eigenvalue 1 on E_2'.

Next suppose that M is an arbitrary irreducible G-module whose highest weight is restricted. Recall that ω_1 is the highest weight of E and it is the only microweight for G. By [11, Proposition 2.3], the set of weights of M either contains all weights of E or all weights of the adjoint module of G. Therefore, g has $|g|$ distinct eigenvalues on M (as this is so for E and E_2').

Finally suppose that M is an arbitrary irreducible G-module and the highest weight of M is not restricted. Then M is the tensor product $M_1 \otimes \cdots \otimes M_k$ where $M_1, \ldots, M_k$ are Frobenius twists of non-trivial modules with restricted highest weights. It follows from the fact justified in the previous paragraph that g has $|g|$ distinct eigenvalues on M (the case $k = 1$ is almost trivial). □

For unipotent cyclic subgroups of classical groups I.D. Suprunenko [10, Theorem 1.3] obtains the following result which is very similar to the results in our Theorems 1.3 and 6.4 for seimisimple elements:

Theorem 6.5 *Let $G = Sp_{2n}(K)$ or $\mathrm{Spin}_n(K)$ where $\operatorname{char} K \neq 2$, and let E be the natural module for G. Let $g \in G$ be a unipotent element of order d. Suppose that the Jordan form of $g|_E$ contains a block of size d. Then for every non-trivial irreducible G-module M the Jordan form of $g|_M$ contains a block of size d, except in the cases where $G = \mathrm{Spin}_n(K)$, $d = 3$ or 5 and all but one Jordan blocks of $g|_E$ are of size 1.*

Moreover, in the exceptional cases of Theorem 6.5 the exceptional modules M are determined in [10, Theorem 1.3] as well. This can be done also for the semisimple element in our Theorem 1.3 with the same list as in [10]. Observe that our approach does not work for unipotent elements, so the method used in [10] is very different.

7. Some questions

1) Let E be the natural $\mathrm{GL}_n(K)$-module and $H \subset \mathrm{GL}_n(K)$ be a subgroup not of exponent 2. Suppose that $E|_H$ contains every irreducible representation of H as a constituent. Is this true for every $\mathrm{GL}_n(K)$-module of dimension greater than 1?

The method described above does not work anymore.

2) Let H be a subgroup of $S = S_n$ for $n > 6$, not of exponent 2. Suppose that H has a regular orbit on the natural S_n-set. Is it true that H has a regular orbit on every faithful S-set?

Proposition 2.3 is an essential evidence in support of the hope that this is true. However, no precise result of greater generality is known. There is an asymptotic result in this direction [6, Theorem 1.13] implying that 2) is true if $|H|^2 < n$ and n is large enough. (In our case option (b) in the conclusion of [6, Theorem 1.13] cannot hold and one does not need to require that $|H| > 8$ here.) One can observe that 2) fails for $n = 6$ and H cyclic of order 6.

3) A natural analog of Hartley's theorem for representations of S_n can be stated as follows. In assumption of 2), let M be an irreducible KS-module of dimension at least n. Is it true that $M|_H$ contains a regular submodule?

There are some exceptions to be kept in mind. For instance, if K is of characteristic 2 then 3) is not true as it is stated. Indeed, let $H = \langle h \rangle$ be of order $d = 3$ or 5 such that h fixes $n - d$ points on the natural S_n-set. Denote by $\tilde{A}_n$ the universal covering of the alternating group A_n and let $\tilde{H} = \langle \tilde{h} \rangle$ be the subgroup of $\tilde{A}_n$ of order d which projects to H under the homomorphism $\tilde{A}_n \to A_n \subset S_n$. It is shown in [13] that if $\tilde{M}$ is a $\mathbb{C}\tilde{A}_n$-module (over the complex number field $\mathbb{C}$) afforded to a so called basic representation of $\tilde{A}_n$ then $\tilde{h}$ does not have eigenvalue 1 on $\tilde{M}$. This remains true if one considers the Brauer reduction of $\tilde{M}$ modulo 2. Let M be any non-trivial irreducible constituent of $\tilde{M}$(mod 2). Then M can be viewed as KA_n-module as $Z(\tilde{A}_n)$ acts on M trivially. Clearly, h does does not have eigenvalue 1 on M, and hence M_{A_n} has no regular submodule. Obviously, this extends to KS_n-modules.

In case where K is of characteristic 0 there is an asymptotic result in support of 3), see [6, Theorem 1.8].

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