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Some Remarks on the $L^p - L^q$ Boundedness of uC_φ

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Abstract

In this paper we will consider the weighted composition operators between two different L^p- spaces and then we characterize the functions u and transformations φ that induce weighted composition operator uC_φ between $L^p(X, \Sigma, \mu)$-spaces by using some properties of conditional expectation operator, pair (u, φ) and the measure space (X, Σ, μ).

Key Words: Weighted composition operator, conditional expectation, multiplication operator.

1. Preliminaries And Notations

Let (X, Σ_X, μ) be a sigma finite measure space. By $L(X)$, we denote the linear space of all Σ_X-measurable functions on X. When we consider any sub-sigma algebra $\mathcal{A}$ of Σ_X, we assume they are completed. For any sigma finite algebra $\mathcal{A} \subseteq \Sigma_X$ and $1 \leq p \leq \infty$ we abbreviate the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ to $L^p(\mathcal{A})$, and denote its norm by $\|.\|_p$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma_X)$ and as a Banach space. We define the support of a function $f \in L(X)$ as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set.

Next, let (Y, Σ_Y, ν) be another sigma finite measure space. Similarly, we use the symbols $L(Y)$ and $L^p(\Sigma_Y)$ to denote the linear space of all Σ_Y-measurable functions on Y and the L^p-space $L^p(Y, \Sigma_Y, \nu)$, respectively. Take a function $u \in L(Y)$ and let $\varphi : Y \to X$ be a non-singular measurable function; i.e. $\varphi^{-1}(\Sigma_X) \subseteq \Sigma_Y$ and

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$\nu\circ\varphi^{-1}(A)=\nu(\varphi^{-1}(A))=0$ for all $A\in\Sigma_X$ such that $\mu(A)=0$. Then the non-singularity of φ means that $\nu\circ\varphi^{-1}$ is absolutely continuous with respect to μ (we write $\nu\circ\varphi^{-1}\ll\mu$, as usual). Let $h_\varphi\in L(X)$ be the Radon-Nikodym derivative $h_\varphi=d\nu\circ\varphi^{-1}/d\mu$.

Associated with each sigma algebra $\mathcal{A}\subseteq\Sigma_Y$, there exists an operator $E_\nu^{\mathcal{A}}=E$, which is called *conditional expectation* operator, on the set of all non-negative measurable functions f or for each $f\in L^q(\Sigma_Y)$ for any q, $1\leq q\leq\infty$, and is uniquely determined by the conditions

(i) $E(f)$ is $\mathcal{A}$- measurable, and

(ii) if A is any $\mathcal{A}$- measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu=\int_A E(f)d\mu$.

This operator is at the central idea of our work, and we list here some of its useful properties:

E1. If g is $\mathcal{A}$-measurable then $E(fg)=E(f)g$.

E2. $E(1)=1$.

E3. If $f>0$ then $E(f)>0$.

E4. If $f\geq 0$ then $E(f)\geq 0$ and $\sigma(f)\subseteq\sigma(E(f))$.

Properties E1 and E2 imply that E is an idempotent; and as operator on $L^q(\Sigma_Y)$ we have $E(L^q(\Sigma_Y))=L^q(\mathcal{A})$. Hence E is the identity operator I on $L^q(\Sigma_Y)$, if and only if $\mathcal{A}=\Sigma_Y$. If we put $\mathcal{A}=\varphi^{-1}(\Sigma_X)$, it is easy to show that, for each non-negative Σ_Y-measurable function f or for each $f\in L^q(\Sigma_Y)$, there exists a Σ_X-measurable function g such that $E(f)=g\circ\varphi$. We can assume that $\sigma(g)\subseteq\sigma(h_\varphi)$, and there exists only one g with this property. We then write $g=E(f)\circ\varphi^{-1}$, though we make no assumptions regarding the invertibility of φ (see [1]). For a deeper study of the properties of E see [5]. For any non-singular measurable function φ from Y into X and $u\in L(Y)$, the pair (u,φ) induce a linear operator uC_φ from $L^p(\Sigma_X)$ into $L(Y)$ defined by

$$uC_\varphi(f)=u.f\circ\varphi\qquad(f\in L^p(\Sigma_X)).$$

Here, the non-singularity of φ guarantees that uC_φ is well defined as a mapping of equivalence classes of functions on $\sigma(u)$. If uC_φ takes $L^p(\Sigma_X)$ into $L^q(\Sigma_Y)$, then uC_φ is bounded, by the closed graph theorem. In this case we call uC_φ a weighted composition operator $L^p(\Sigma_X)$ into $L^q(\Sigma_Y)$. If $X=Y$ and φ is the identity, we write uC_φ as M_u and call it the multiplication operator induced by u. In case that $u\equiv 1$ we write $uC_\varphi=M_uC_\varphi$ as C_φ and call it the composition operator induced by φ.

Boundedness of uC_φ

Boundedness of composition operators in L^p-spaces ($1 \leq p < \infty$) for finite measures appeared already in the Dunford-Schwarz book [2, Lemma 7, pp.664–665] and for σ-finite measures in [6] and [7]. In this section we turn attention to the follow-up problem.

Which function $u \in L(Y)$ and measurable function $\varphi : Y \to X$ induce a weighted composition operator from $L^p(\Sigma_X)$ into $L^q(\Sigma_Y)$ in the case $1 \leq q \leq p < \infty$?

The next lemma will be crucial in what follows. In fact, it is a slight generalization of proposition 2.1 in [3].

Lemma 1 *Suppose* $1 \leq p,\ q < \infty$, $u \in L(Y)$ *and let the pair* (u, φ) *induce a weighted composition operator from* $L^p(\Sigma_X)$ *into* $L^q(\Sigma_Y)$*. Then for any* $f \in L^p(\Sigma_X)$ *we have*

$$\|uC_\varphi f\|_{L^q(\Sigma_Y)} = \|M_J f\|_{L^q(\Sigma_X)},$$

where $J = (h_\varphi E(|u|^q) \circ \varphi^{-1})^{\frac{1}{q}}$.

Proof. Let $f \in L^p(\Sigma_X)$. As an application of the properties of the conditional expectation and using the change of variable formula we have

$$\|uC_\varphi f\|^q_{L^q(\Sigma_Y)} = \int_Y |u.f \circ \varphi|^q d\nu = \int_Y E(|u|^q)|f|^q \circ \varphi d\nu$$

$$= \int_X E(|u|^q) \circ \varphi^{-1} |f|^q d\nu \circ \varphi^{-1} = \int_X (h_\varphi E(|u|^q) \circ \varphi^{-1})|f|^q d\mu$$

$$= \int_X |Jf|^q d\mu = \|M_J f\|^q_{L^q(\Sigma_X)} .$$

So we proved that the pair (u, φ) induce a weighted composition operator $uC_\varphi : L^p(\Sigma_X) \to L^q(\Sigma_Y)$ if and only if J induces a multiplication operator $M_J : L^p(\Sigma_X) \to L^q(\Sigma_X)$ and $\|uC_\varphi\| = \|M_J\|$. □

The proof of the following proposition can be obtained by adapting the proof of theorem 2.3 in [4].

Proposition 2 *Suppose* $1 \leq q < p < \infty$ *and* $\frac{1}{p}+\frac{1}{r}=\frac{1}{q}$. *Let* $u \in L(Y)$ *and* $\varphi : Y \to X$ *be a non-singular measurable function. Then the pair* (u, φ) *induce a weighted composition operator* uC_φ *from* $L^p(\Sigma_X)$ *into* $L^q(\Sigma_Y)$ *if and only if* $J \in L^r(\Sigma_X)$ *and its norm given by* $\|uC_\varphi\| = \|J\|_{L^r(\Sigma_X)}$.

Corollary 3 *Under the same assumptions as in proposition 2.2,* φ *induces a composition operator* $C_\varphi : L^p(\Sigma_X) \to L^q(\Sigma_Y)$ *if and only if* $h_\varphi \in L^{\frac{r}{q}}(\Sigma_X)$. *Also, if* $X = Y$, u *induces a multiplication operator* $M_u : L^p(\Sigma_X) \to L^q(\Sigma_X)$ *if and only if* $u \in L^r(\Sigma_X)$. *In these cases we have* $\|uC_\varphi\| = \|h_\varphi{}^{\frac{1}{q}}\|_{L^r(\Sigma_X)}$ *and* $\|M_u\| = \|u\|_{L^r(\Sigma_X)}$.

If $p = q$, then r must be ∞. So $uC_\varphi(L^p(\Sigma_X)) \subseteq L^p(\Sigma_Y)$ if and only if $J \in L^\infty(\Sigma_X)$. In this case $\|uC_\varphi\| = \|J\|_{L^\infty(\Sigma_X)}$. This fact is well known. For direct proof see [6].

Examples. (i) Suppose $X = [0, a^4]$ and $Y = [-a^2, a^2]$ for some $a > 0$. Let $\varphi : (Y, \Sigma_Y, \nu) \to (X, \Sigma_X, \mu)$ be defined on Lebesgue measure spaces by $\varphi(x) = a^4 - x^2$. If we consider $uC_\varphi : L^2(\Sigma_X) \to L^2(\Sigma_Y)$ as $uC_\varphi f(x) = xf(a^4 - x^2)$, then a simple computation gives $h_\varphi = 1/(2\sqrt{a^4 - x}) \notin L^\infty(\Sigma_X)$. Then C_φ does not define a bounded composition operator. However it is easy to see that

$$J(x) = \left(\frac{1}{2\sqrt{a^4 - x}} \left[2(a^4 - x) \right] \right)^{\frac{1}{2}} = \sqrt[4]{a^4 - x} \in L^\infty(\Sigma_X).$$

So uC_φ is bounded and $\|uC_\varphi\| = a$.

(ii) Let (X, Σ_X, μ) be the unit circle in complex plane and Lebesgue measurable sets equipped with normalized Lebesgue measure, and $\varphi(z) = z^2$. If we consider uC_φ from $L^2(X, \Sigma_X, \mu)$ into $L^2(X, \Sigma_X,\ \mu \circ \varphi^{-1})$, then we have

$$\|uC_\varphi\|^2_{L^2(X, \Sigma_X,\ \mu\circ\varphi^{-1})} = \int_X h_\varphi |u|^2 |f \circ \varphi|^2 d\mu \circ \varphi^{-1}$$

$$= \int_X h_\varphi E(h_\varphi |u|^2) \circ \varphi^{-1} |f|^2 d\mu = \int_X G|f|^2 d\mu,$$

where $G = h_\varphi E(h_\varphi |u|^2) \circ \varphi^{-1}$. Hence uC_φ is bounded if and only if $G \in L^\infty(X, \Sigma_X, \mu)$. We note that by a simple computation we have

$$G(z) = \frac{1}{2} \sum_{\zeta^2 = z} |u(\zeta)|^2 h(\zeta), \qquad (z \in X).$$

Now, we try to give another characterization of boundedness for uC_φ from $L^p(\Sigma_X)$ into $L^q(\Sigma_Y)$. Let $u \in L(Y)$ and $\varphi : Y \to X$ be a non-singular measurable function. Define the measure $\mu_{u,\varphi}$ by

$$\mu_{u,\varphi}(A) = \int_{\varphi^{-1}(A)} |u|^q d\nu, \qquad (A \in \Sigma_X) .$$

Since $\nu \circ \varphi^{-1} \ll \mu$, then for each $A \in \Sigma_X$ with $\mu(A) = 0$, we have $\nu(\varphi^{-1}(A)) = 0$; so $\mu_{u,\varphi}(A) = 0$. Then $\mu_{u,\varphi} \ll \mu$. Put $\theta = (\frac{d\mu_{u,\varphi}}{d\mu})^{1/q}$ which, of course, is a non-negative Σ_X-measurable function.

Lemma 4 *Fixing $1 \leq q < \infty$ and given $u \in L(Y)$. Then, for any non-negative Σ_X-measurable function f,*

$$\int_X f d\mu_{u,\varphi} = \int_Y |u|^q f \circ \varphi \; d\nu$$

in the sense that, if one of the Integrals exists, then so does the other, and they are equal.

Proof. Let $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $A_i \in \Sigma_X$ and $0 < \mu(A_i) < \infty$. We have that

$$\int_X f d\mu_{u,\varphi} = \sum_{i=1}^n \alpha_i \mu_{u,\varphi}(A_i)$$

$$= \sum_{i=1}^n \alpha_i \int_{\varphi^{-1}(A_i)} |u|^q d\nu = \int_Y |u|^q \left(\sum_{i=1}^n \alpha_i \chi_{\varphi^{-1}(A_i)} \right) d\nu$$

$$= \int_Y |u|^q \left(\sum_{i=1}^n \alpha_i \chi_{A_i} \right) \circ \varphi \; d\nu = \int_Y |u|^q f \circ \varphi \; d\nu.$$

Now, if f is a non-negative function in $L(X)$, we take an increasing sequence $\{f_n\}_{n=1}^\infty$ of non-negative simple functions such that $f_n \to f$ a.e. Then we have $\int_X f_n d\mu_{u,\varphi} \to \int_X f d\mu_{u,\varphi}$. On the other hand $\{|u|^q f_n \circ \varphi\}_{n=1}^\infty$ is an increasing sequence such that $|u|^q f_n \circ \varphi \to |u|^q f \circ \varphi$ a.e., so $\int_X f_n d\mu_{u,\varphi} = \int_Y |u|^q f_n \circ \varphi \; d\nu \to \int_Y |u|^q f \circ \varphi \; d\nu$. $\square$

Now, we present the main result of this paper.

Theorem 5 *Suppose* $1 \leq q < p < \infty$ *and* $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. *Let* $u \in L(Y)$ *and* $\varphi : Y \to X$ *be a non-singular measurable function. Then the following assertions are equivalent:*
(i) The pair (u, φ) *induce a weighted composition operator* uC_φ *from* $L^p(\Sigma_X)$ *into* $L^q(\Sigma_Y)$.
(ii) θ *belongs to* $L^r(\Sigma_X)$.
(iii) There is a partition $\{F_n\}_{n=1}^{\infty}$ *of* X *such that* $\sum_{n=1}^{\infty} \|\theta_{|_{F_n}}\|_\infty^r \mu(F_n) < \infty$, *where* $\|\theta_{|_{F_n}}\|_\infty = ess \sup_{x \in F_n} \theta(x)$.

Proof. Suppose that (i) holds, and $f \in L^p(\Sigma_X)$. By using lemma 2.4 we have

$$\|uC_\varphi\|_{L^q(\Sigma_Y)}^q = \int_Y |u|^q |f|^q \circ \varphi \, d\nu = \int_X |f|^q d\mu_{u,\varphi}$$

$$= \int_X |\theta f|^q d\mu = \|M_\theta f\|_{L^q(\Sigma_X)}^q .$$

Hence by corollary 2.3, uC_φ is bounded if and only if $\theta \in L^r(\Sigma_X)$. Thus we obtain the equivalence of (i) and (ii).

Assume that (iii) dose not hold. Choose a number $a > 1$ arbitrarily, and set $F_0 = \{x \in X : \theta(x) = 0\}$, $F_{2n} = \{x \in X : a^{n-1} < \theta(x)^r \leq a^n\}$ and $F_{2n-1} = \{x \in X : a^{-n} \leq \theta(x)^r < a^{-n+1}\}$. Then $\{F_n\}_{n=0}^{\infty}$ clearly becomes a partition of X. So we have

$$\int_X \theta^r d\mu = \sum_{i=1}^{\infty} \int_{F_{2n}} \theta^r d\mu + \sum_{i=1}^{\infty} \int_{F_{2n-1}} \theta^r d\mu$$

$$\geq \sum_{i=1}^{\infty} a^{n-1} \mu(F_{2n}) + \sum_{i=1}^{\infty} a^{-n} \mu(F_{2n-1})$$

$$\geq \frac{1}{a} \left[\sum_{n=1}^{\infty} \|\theta_{|_{F_{2n}}}\|_\infty^r \mu(F_{2n}) + \sum_{n=1}^{\infty} \|\theta_{|_{F_{2n-1}}}\|_\infty^r \mu(F_{2n-1}) \right]$$

$$\geq \frac{1}{a} \sum_{n=1}^{\infty} \|\theta_{|_{F_n}}\|_\infty^r \mu(F_n) = +\infty.$$

This means that $\theta \notin L^r(\Sigma_X)$. Hence we proved the implication (ii) $\Rightarrow$ (iii).

Finally, let $\{F_n\}_{n=0}^{\infty}$ be a partition of X such that $\sum_{n=1}^{\infty} \|\theta_{|_{F_n}}\|_\infty^r \mu(F_n) < \infty$, we have

$$\int_X \theta^r d\mu = \sum_{i=1}^{\infty} \int_{F_n} \theta^r d\mu \leq \sum_{n=1}^{\infty} \|\theta_{|_{F_n}}\|_\infty^r \mu(F_n) < \infty.$$

Thus we proved the implication (iii)$\Rightarrow$ (i) ($\Leftrightarrow$ (ii)). □

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