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On the Unique Continuation Property for the Higher Order Nonlinear Schrödinger Equation With Constant Coefficients*

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Abstract

We solve the unique continuation property: If u is a solution of the higher order nonlinear Schrödinger equation with constant coefficients with $t_1 < t_2$ which is sufficiently smooth and such that $\operatorname{supp} u(\,.\,,\, t_j) \subset (a,\, b)$, $-\infty < a < b < \infty$, $j = 1,\, 2$, then $u \equiv 0$.

Key words and phrases: Higher order nonlinear Schrödinger equation, unique continuation property.

1. Introduction

We consider the initial value problem

$$(HSCHROD)\begin{cases} i\,u_t + \alpha\,u_{xx} + i\,\eta\,u_{xxx} + |u|^2\,u = 0, & x,\, t \in \mathbb{R} \\ u(x,\,0) = u_0(x), \end{cases}$$

where $\alpha,\, \eta \in \mathbb{R}$, $\eta \neq 0$ and u is a complex valued function. The above equation is a particular case of the equation

$$(Q)\begin{cases} i\,u_t + \alpha\,u_{xx} + i\,\eta\,u_{xxx} + \gamma\,|u|^2\,u + i\,\delta\,|u|^2\,u_x + i\,\epsilon\,u^2\,\overline{u}_x = 0, & x,\, t \in \mathbb{R} \\ u(x,\,0) = u_0(x), \end{cases}$$

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where α, η, γ, δ, with $\eta \neq 0$ and u is a complex valued function. This equation was first proposed by A. Hasegawa and Y. Kodama [10] as a model for the propagation of a signal in a optic fiber (see also [22]). The equation (Q) can be reduced to other well known equations. For instance, setting $\alpha = \gamma = 1$, $\eta = \epsilon = \delta = 0$ in (Q) we have the semi linear Schrödinger equation, i. e.,

$$i\,u_t + u_{xx} + |u|^2\,u = 0. \qquad (Q_1)$$

If we let $\eta = \gamma = 0$ in (Q) we obtain the derivative nonlinear Schrödinger equation

$$i\,u_t + \alpha\,u_{xx} + i\,\delta\,|u|^2\,u_x + i\,\epsilon\,u^2\,\overline{u}_x = 0. \qquad (Q_2)$$

Letting $\alpha = \gamma = \epsilon = 0$ in (Q), the equation that arises is the complex modified Korteweg-de Vries equation,

$$u_t + \eta\,u_{xxx} + \delta\,|u|^2\,u_x = 0. \qquad (Q_3)$$

The initial value problem for equations (Q_1), (Q_2) and (Q_3) has been extensively studied in the last few years. See for instance [1, 7, 14, 31] and the references therein. In 1992, Laurey C. [24] considered equation (Q) and proved local well-posedness of the initial value problem associated for data in $H^s(\mathbb{R})$ with $s > 3/4$, and global well-posedness in $H^s(\mathbb{R})$, where $s \geq 1$. In 1997, Staffilani G. [33] established local well-posedness for data in $H^s(\mathbb{R})$ with $s \geq 1/4$, improving Laurey's result. Similar results were given in [5, 6] for (Q) where $w(t)$, $\beta(t)$ are real functions. Recently, Vera O. [35] showed that C^∞ solutions $u(x,\,t)$ are obtained for all $t > 0$ if the initial data $u_0(x)$ decays faster than polynomially on $\mathbb{R} = \{x \in \mathbb{R} :\ x > 0\}$ and has certain initial Sobolev regularity.

For the case of the $(HSCHROD)$ we consider the Gauge transformation

$$\begin{aligned} u(x,\,t) &= e^{i\,\frac{\alpha}{3\eta}\,x - i\,2\,\frac{\alpha^3}{27\eta^2}\,t}\,v\left(x - \frac{\alpha^2}{3\eta}\,t,\,t\right) \\ &\equiv e^{\theta}\,v\,(\mu,\,\xi)\,, \end{aligned}$$

where $\theta = i\,\frac{\alpha}{3\eta}\,x - i\,2\,\frac{\alpha^3}{27\eta^2}\,t$, $\mu = x - \frac{\alpha^2}{3\eta}\,t$ and $\xi = t$, to have

$$\begin{aligned} u_t &= -\,i\,2\frac{\alpha^3}{27\eta^2}\,e^{\theta}\,v - \frac{\alpha^2}{3\eta}\,e^{\theta}\,v_\mu + e^{\theta}\,v_\xi \\ u_{xx} &= -\,\frac{\alpha^2}{9\eta^2}\,e^{\theta}\,v + 2\,i\,\frac{\alpha}{3\eta}\,e^{\theta}\,v_\mu + e^{\theta}\,v_{\mu\,\mu} \\ u_{xxx} &= -\,i\,\frac{\alpha^3}{27\eta^3}\,e^{\theta}\,v - 3\,\frac{\alpha^2}{9\eta^2}\,e^{\theta}\,v_\mu + 3\,i\,\frac{\alpha}{3\eta}\,e^{\theta}\,v_{\mu\mu} + e^{\theta}\,v_{\mu\mu\mu}. \end{aligned}$$

Replacing in $(HSCHROD)$, we obtain

$$(KdVm)\begin{cases} i\,v_\xi + i\,\eta\,v_{\mu\mu\mu} + |v|^2\,v = 0, & \mu,\,\xi\in\mathbb{R}\\ v(\mu,\,0) = v_0(\mu), \end{cases}$$

then the $(HSCHROD)$ equation is reduced to the complex modified Korteweg-de Vries type equation. This is our main motivation, to think that the $(HSCHROD)$ equation has the unique continuation property.

A partial differential equation $\mathcal{L}u = 0$ in some open, connected domain Ω of $\mathbb{R}^n$ is said to have the weak unique continuation property (UCP) if every solution u of $\mathcal{L}u = 0$ (in a suitable function space), which vanishes on some nonempty open subset of Ω, vanishes in Ω.

This paper is concerned with unique continuation results for the higher order nonlinear Schrödinger equation with constant coefficients. The equation $(HSCHROD)$ with initial data and periodic boundary conditions has the form

$$i\,u_t + \alpha\,u_{xx} + i\,\eta\,u_{xxx} + |u|^2\,u + i\,\delta\,u_x = 0 \tag{1.1}$$

$$u(x,\,0) = u_0(x) \tag{1.2}$$

$$\partial_x^k u(0,\,t) = \partial_x^k u(1,\,t), \quad k = 0,\,1,\,2, \tag{1.3}$$

where $\alpha,\,\eta,\,\delta\in\mathbb{R}$, $\eta\neq 0$ and u is a complex valued function on the domain $[0,\,1]$, $t\in\mathbb{R}$.

For the UCP the first results are due to J. C. Saut and B. Scheurer [30]. They considered some dispersive operators in one space dimension of the type $L = i\,D_t + \alpha\,i^{2k+1}\,D^{2k+1} + R(x,\,t,\,D)$, $(x,\,t)\in\mathbb{R}^n\times\mathbb{R}$, where $\alpha\neq 0$, $D = \frac{1}{i}\frac{\partial}{\partial x}$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ and $R(x,\,t,\,D) = \sum_{j=0}^{2k} r_j(x,\,t)\,D^j$, $r_j\in L^\infty_{loc}(\mathbb{R}:\,L^2_{loc}(\mathbb{R}))$. They proved that, if $u\in L^2_{loc}(\mathbb{R}:\,H^{2k+1}_{loc}(\mathbb{R}))$ is a solution of $Lu = 0$, which vanishes in some open set Ω_1 of $\mathbb{R}_x\times\mathbb{R}_t$, then u vanishes in the horizontal component of Ω_1. As a consequence of the uniqueness of the solutions of the KdV equation in $L^\infty_{loc}(\mathbb{R}:\,H^3(\mathbb{R}))$, their result immediately yields the following theorem

Theorem 1.1. *If $u\in L^\infty_{loc}(\mathbb{R}:\,H^3(\mathbb{R}))$ is a solution of the KdV equation*

$$u_t + u_{xxx} + u\,u_x = 0, \tag{1.4}$$

and vanishes on an open set of $\mathbb{R}_x\times\mathbb{R}_t$, then $u(x,\,t) = 0$ for $x\in\mathbb{R}$, $t\in\mathbb{R}$.

In 1992, Zhang B. [36, 37] proved using inverse scattering transform and some results from Hardy function theory that if $u \in L^{\infty}_{Loc}(\mathbb{R} : H^m(\mathbb{R}))$, $m > 3/2$ is a solution of the KdV equation (1.4), then it cannot have compact support at two different moments unless it vanishes identically. As a consequence of the Miura transformation, the above results for the KdV equation (1.4) are also true for the modified Korteweg-de Vries equation

$$u_t + u_{xxx} - u^2\, u_x = 0. \tag{1.5}$$

A variety of techniques such as spherical harmonics [32], singular integral operators [25], inverse scattering [36], and others have been used. However the Carleman's methods which consist of in establishing a priori estimates containing a weight has influenced a lot the development on the subject. In 2002, C. Kenig *et al.* [17] studied the generalized KdV equation

$$u_t + u_{xxx} + F(u_{xx},\, u_x,\, u,\, x,\, t) = 0, \quad x \in \mathbb{R},\ t \in [t_1,\, t_2] \tag{1.6}$$

under suitable assumptions on F and on the class of solutions considered in [18]. For this case, they used a unique continuation result due to [30]. C. Kenig *et al.* [19] studied uniqueness continuation properties of solution for the nonlinear Schrödinger equations of the form

$$i\, u_t + \triangle u + F(u,\, \overline{u}) = 0, \quad (x,\, t) \in \mathbb{R} \times \mathbb{R}. \tag{1.7}$$

More precisely, they showed the following theorem.

Theorem 1.2. *Let* $u_1,\, u_2 \in C([0,\, 1] :\ H^s(\mathbb{R}^n))$, $s \geq max\{n/2^+;\ 2\}$, *be two solutions of the equations* (1.7)*, where* $F \in C^{[\![s]\!]+1}(\mathbb{C} :\ \mathbb{C})$ *with*

$$|\nabla F(u,\, \overline{u})| \leq c\,(|u|^{p_1 - 1} + |u|^{p_2 - 1}), \quad p_1,\, p_2 > 1. \tag{1.8}$$

If there exists a convex cone Γ *strictly contained in a half-space such that*

$$u_1(x,\, 0) = u_2(x,\, 0) \quad : \quad u_1(x,\, 1) = u_2(x,\, 1), \qquad \forall\, x \notin \Gamma + y_0, \quad y_0 \in \mathbb{R}^n, \tag{1.9}$$

then $u_1 \equiv u_2$.

Subsequent work in these type of equation were given by [8, 18, 23] (and see references therein). According to the characteristic of equations (1.1)–(1.3) and considering the above argument, the following basic question arises: Let $u = u(x,\, t)$ be a solution for the higher order nonlinear Schrödinger equation with constant coefficients

$$i\, u_t + \alpha\, u_{xx} + i\, \eta\, u_{xxx} + |u|^2\, u + i\, \delta\, u_x = 0, \quad (x,\, t) \in \mathbb{R} \times (t_1,\, t_2)$$

with $t_1 < t_2$ which is sufficiently smooth and such that

$$\operatorname{supp}\,(\,.\,,\,t_j) \subseteq (a,\,b), \quad -\infty < a < b < \infty, \quad j = 1,\,2.$$

Is $u \equiv 0$?

In this work, motivated by the work of C. Kenig *et al.* [18], we pretend to give an answer to this question. This paper is organized as follows: Before describing the main results, section 2 outlines briefly the notation and terminology to be used subsequently and presents a statement of some Lemmas. We prove the local existence theorem. In section 3, we obtain the main theorem. Our main result reads as follows.

Theorem 1.3. *Suppose that $u(x,\,t)$ is a sufficiently smooth solution of the* (1.1)–(1.3). *If*

$$\operatorname{supp} u(\,.\,,\,t_j) \subseteq (-\infty,\,b), \qquad j = 1,\,2. \tag{1.10}$$

or

$$\operatorname{supp} u(\,.\,,\,t_j) \subseteq (a,\,\infty), \qquad j = 1,\,2, \tag{1.11}$$

then $u(x,\,t) \equiv 0$.

2. Preliminaries

The notation we use is standard: We write the time derivative by $u_t = \frac{\partial u}{\partial t} = \partial_t u$. Spatial derivatives are denoted by $u_x = \frac{\partial u}{\partial x} = \partial u$, $u_{xx} = \frac{\partial^2 u}{\partial x^2} = \partial^2 u, \dots, \frac{\partial^j u}{\partial x^j} = \partial^j u$ and we abbreviate $u_j = \partial^j u$.

If $\mathbb{E}$ is any Banach space, its norm is written as $||\cdot||_{\mathbb{E}}$. For $1 \le p \le +\infty$, the usual class of p^{th}-power Lebesgue-integrable (essentially bounded if $p = +\infty$) real-valued functions defined on the open set Ω in $\mathbb{R}^n$ is written by $L^p(\Omega)$ and its norm is abbreviated as $||\cdot||_p$. The Sobolev space of L^2-functions whose derivatives up to order m also lie in L^2 is denoted by H^m.

We denote $[H^{m_1}(\Omega),\,H^{m_2}(\Omega)] = H^{(1-\theta)\,m_1+\theta\,m_2}(\Omega)$ for all $m_i > 0\,(i = 1,\,2)$, $m_2 < m_1$, $0 < \theta < 1$ (with equivalent norms) the interpolation of $H^m(\Omega)$-spaces. If a function belongs, locally, to L^p or H^m we write $f \in L^p_{loc}$ or $f \in H^m_{loc}$. $C(0,\,T : \mathbb{E})$ to denote the class of all continuous maps $u : [0,\,T] \to \mathbb{E}$ equipped with the norm $||u||_{C(0,T;\mathbb{E})} = \sup_{0\le t\le T}||u||_{\mathbb{E}}$. $u(x,\,t) \in C^{3,\,1}(\mathbb{R}^2)$ if $\partial u, \partial^2 u, \partial^3 u, \partial_t u \in C(\mathbb{R}^2)$. $u(x,\,t) \in C_0^{3,\,1}(\mathbb{R}^2)$

if $u \in C^{3,1}(\mathbb{R}^2)$ and u with compact support. Throughout this paper c is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities. The next proposition is well known and it will be used frequently

Proposition 2.1. *Let K be a non empty compact set and F a close subset of $\mathbb{R}$ such that $K \cap F = \emptyset$. Then there is $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ in K, $\psi = 0$ in F and $0 \le \psi(x) \le 1$, $\forall\, x \in \mathbb{R}$.*

Definition 2.2. *Let $\mathcal{L}$ be an evolution operator acting on functions defined on some connected open set Ω of $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_t$. $\mathcal{L}$ is said to have the horizontal unique continuation property if every solution u of $\mathcal{L}u = 0$ that vanishes on some nonempty open set $\Omega_1 \subset \Omega$ vanishes in the horizontal component of Ω_1 in Ω, i.e., in*

$$\Omega_h = \{(x,\, t) \in \Omega / \; \exists\, x_1,\; (x_1,\, t) \in \Omega_1\}.$$

The following results, Theorem 2.3 and Corollary 2.4 will be important in the proof of the main theorem of this paper. The proof is given in the paper of J. C. Saut and B. Scheurer [30].

Theorem 2.3. *Assume that $u = u(x,\, t)$ satisfies the equation*

$$\partial_t u + \partial^3 u + \sum_{j=0}^{2} r_j\, \partial^j u = 0, \qquad (x,\, t) \in (a,\, b) \times (t_1,\, t_2) \subseteq \mathbb{R} \times \mathbb{R} \tag{2.1}$$

with

$$r_j \in L^\infty((t_1,\, t_2):\; L^2_{loc}(a,\, b)). \tag{2.2}$$

If u vanishes on an open set $\Omega_1 \subseteq (a,\, b) \times (t_1,\, t_2)$, then u vanishes in the horizontal components of Ω_1 in $(a,\, b) \times (t_1,\, t_2)$, i. e., the set

$$\{(x,\, t) \in (a,\, b) \times (t_1,\, t_2):\; \exists\, x_1 \text{ such that } (x_1,\, t) \in \Omega_1\}. \tag{2.3}$$

As a consequence they obtained the following result.

Corollary 2.4. *If u is a sufficiently smooth solution of the equation (2.1) with*

$$supp\, u(\,.\,,\, t) \subseteq (a,\, b), \qquad \forall\, t \in (t_1,\, t_2), \tag{2.4}$$

then $u \equiv 0$.

Remark. The key step is the following Carleman's estimate: Assume that $(0,\, 0) \in \Omega$ then

$$\iint_\Omega |\partial_t + \partial^3 u|^2\, e^{2\,\lambda\,\varphi}\, dx\, dt$$

$$\geq \lambda \iint_\Omega |\partial^2 u|^2\, e^{2\,\lambda\,\varphi}\, dx\, dt + \lambda^2 \iint_\Omega |\partial u|^2\, e^{2\,\lambda\,\varphi}\, dx\, dt + \lambda^4 \iint_\Omega |u|^2\, e^{2\,\lambda\,\varphi}\, dx\, dt \quad (2.5)$$

with $\lambda\,\delta \geq M$, $0 < \delta < \delta_0$ and $\varphi(x,\, t) = (x - \delta)^2 + \delta^2\, t^2$.

Lemma 2.5. *Equation* (1.1)–(1.3) *has the following conservation law*

$$\partial_t \int_0^1 u^2\, dx = 0. \quad (2.6)$$

Proof. Straightforward. □

Lemma 2.6. *For all* $u \in H^1(\Omega)$

$$||u||_{L^\infty(\Omega)} \leq c\, ||u||^{1/2}_{L^2(\Omega)} \left(||u||_{L^2(\Omega)} + ||\, \partial u\, ||_{L^2(\Omega)} \right)^{1/2}, \quad (2.7)$$

and for all $u \in H^3(\Omega)$,

$$||u||_{L^4(\Omega)} \leq c\, ||u||^{11/12}_{L^2(\Omega)} \left(||u||_{L^2(\Omega)} + ||\, \partial^3 u\, ||_{L^2(\Omega)} \right)^{1/12} \quad (2.8)$$

$$||\, \partial u\, ||_{L^4(\Omega)} \leq c\, ||u||^{7/12}_{L^2(\Omega)} \left(||u||_{L^2(\Omega)} + ||\, \partial^3 u\, ||_{L^2(\Omega)} \right)^{5/12} \quad (2.9)$$

Proof. See [34]. □

Lemma 2.7.

$$\left|\left| \int_{\mathbb{R}} e^{i\,(x,\,t)\cdot(\xi,\,\xi^3)}\, \widehat{\widehat{u}}(\xi,\, \xi^3)\, d\xi \right|\right|_{L^8(\mathbb{R}^2)} \leq c\, ||u||_{L^{8/7}(\mathbb{R}^2)}$$

where $\widehat{\ }$ denotes the Fourier transform in $\mathbb{R}$.

Proof. See [15] □

Lemma 2.8. *For any* $u \in C_0^{3,1}(\mathbb{R}^2)$,

$$||u||_{L^8(\mathbb{R}^2)} \leq c\,||\{\partial_t + \partial^3 + a\}u||_{L^{8/7}(\mathbb{R}^2)}$$

with c *independent of* $a \in \mathbb{R}$.

Proof. See [18]. □

Theorem 2.9 (Local Existence). *Let* $u_0 \in H^1(0,\,1)$ *with* $u_0(0) = u_0(1)$. *Then there exist* $T > 0$ *and* u *such that* u *is a solution of* (1.1)–(1.3). $u \in L^\infty(0,\,T;\,H^1(0,\,1))$ *and the initial data* $u(x,\,0) = u_0(x)$ *is satisfied.*

Proof. For $\epsilon > 0$ we approximate the equation (1.1)–(1.3) by the parabolic equation

$$i\,\partial_t u^\epsilon + \alpha\,\partial^2 u^\epsilon + i\,\eta\,\partial^3 u^\epsilon + |u^\epsilon|^2\,u^\epsilon + i\,\delta\,\partial u^\epsilon + i\,\epsilon\,\partial^4 u^\epsilon = 0 \tag{2.10}$$

$$u^\epsilon(x,\,0) = u_0(x) \tag{2.11}$$

$$\partial^k u^\epsilon(0,\,t) = \partial^k u^\epsilon(1,\,t), \quad k = 0,\,1,\,2. \tag{2.12}$$

Multiplying equation (2.10) by $\overline{u^\epsilon}$, we have

$$i\,\overline{u^\epsilon}\,\partial_t u^\epsilon + \alpha\,\overline{u^\epsilon}\,\partial^2 u^\epsilon + i\,\eta\,\overline{u^\epsilon}\,\partial^3 u^\epsilon + |u^\epsilon|^4 + i\,\delta\,\overline{u^\epsilon}\,\partial u^\epsilon + i\,\epsilon\,\overline{u^\epsilon}\,\partial^4 u^\epsilon = 0$$

$$-\,i\,u^\epsilon\,\overline{\partial_t u^\epsilon} + \alpha\,u^\epsilon\,\overline{\partial^2 u^\epsilon} - i\,\eta\,u^\epsilon\,\overline{\partial^3 u^\epsilon} + |u^\epsilon|^4 - i\,\delta\,u^\epsilon\,\overline{\partial u^\epsilon} - i\,\epsilon\,u^\epsilon\,\overline{\partial^4 u^\epsilon} = 0$$

(applying conjugate).

Subtracting and integrating over $x \in \Omega = (0,\,1)$, we obtain

$$i\,\partial_t \int_0^1 |u^\epsilon|^2\,dx + i\,\eta \int_0^1 \overline{u^\epsilon}\,\partial^3 u^\epsilon\,dx + i\,\eta \int_0^1 u^\epsilon\,\overline{\partial^3 u^\epsilon}\,dx$$

$$+\,i\,\epsilon \int_0^1 \overline{u^\epsilon}\,\partial^4 u^\epsilon\,dx + i\,\epsilon \int_0^1 u^\epsilon\,\overline{\partial^4 u^\epsilon}\,dx = 0. \tag{2.13}$$

Each term in (2.13) is calculated separately. Integrating by parts, there is

$$\int_0^1 \overline{u^\epsilon}\,\partial^3 u^\epsilon dx = -\int_0^1 \overline{\partial u^\epsilon}\,\partial^2 u^\epsilon\,dx$$

$$\int_0^1 u^\epsilon\,\overline{\partial^3 u^\epsilon} dx = -\int_0^1 \partial^2 u^\epsilon \overline{\partial^2 u^\epsilon} dx$$

$$\int_0^1 \overline{u^\epsilon}\, \partial^4 u^\epsilon dx = \int_0^1 |\partial^2 u^\epsilon|^2 dx$$

$$\int_0^1 u^\epsilon\, \overline{\partial^4 u^\epsilon} dx = \int_0^1 |\partial^2 u^\epsilon|^2 dx$$

then in (2.13), we obtain

$$\begin{aligned} & i\,\partial_t \int_0^1 |u^\epsilon|^2 dx - i\,\eta \int_0^1 \overline{\partial u^\epsilon}\, \partial^2 u^\epsilon dx - i\,\eta \int_0^1 \partial u^\epsilon\, \overline{\partial^2 u^\epsilon} dx \\ & + i\,\epsilon \int_0^1 |\partial^2 u^\epsilon|^2 dx + i\,\epsilon \int_0^1 |\partial^2 u^\epsilon|^2 dx = 0. \end{aligned}$$

hence

$$i\,\partial_t\, ||u^\epsilon||^2_{L^2(0,\,1)} - i\,\eta \int_0^1 \partial(|\partial u^\epsilon|^2)\, dx + 2\,i\,\epsilon \int_0^1 |\partial^2 u^\epsilon|^2 dx = 0,$$

thus

$$\partial_t\, ||u^\epsilon||^2_{L^2(0,\,1)} + 2\,\epsilon\, ||\partial^2 u^\epsilon||^2_{L^2(0,\,1)} = 0.$$

Integrating over $t \in [0,\, T]$, we obtain

$$||u^\epsilon||^2_{L^2(0,\,1)} + 2\,\epsilon \int_0^T ||\partial^2 u^\epsilon||^2_{L^2(0,\,1)} dt = ||u_0^\epsilon||^2_{L^2(0,\,1)}.$$

In particular,

$$||u^\epsilon||_{L^\infty(0,\,T:\,L^2(0,\,1))} \le c \qquad : \qquad \sqrt{\epsilon}\, ||\partial^2 u^\epsilon||_{L^2(0,\,T:\,L^2(0,\,1))} \le c$$

if and only if

$$||u^\epsilon||_{L^\infty(0,\,T:\,L^2(0,\,1))} \le c \qquad : \qquad \sqrt{\epsilon}\, ||\partial^2 u^\epsilon||_{L^2(Q)} \le c,$$

or

$$u^\epsilon \in L^\infty(0,\, T:\, L^2(0,\,1)) \tag{2.14}$$

$$\sqrt{\epsilon}\, u^\epsilon \in L^2(0,\, T:\, H^2(0,\,1)). \tag{2.15}$$

On the other hand, multiplying (2.10) by $\overline{\partial^2 u^\epsilon}$ and integrating over $x \in \Omega = (0,\,1)$ we have

$$i\int_0^1 \overline{\partial^2 u^\epsilon}\,\partial_t u^\epsilon dx + \alpha\int_0^1 |\partial^2 u^\epsilon|^2 dx + i\,\eta\int_0^1 \overline{\partial^2 u^\epsilon}\,\partial^3 u^\epsilon dx$$
$$+\int_0^1 |u^\epsilon|^2\,u^\epsilon\,\overline{\partial^2 u^\epsilon} dx + i\,\delta\int_0^1 \partial u^\epsilon\,\overline{\partial^2 u^\epsilon}\,dx + i\,\epsilon\int_0^1 \overline{\partial^2 u^\epsilon}\,\partial^4 u^\epsilon dx = 0. \tag{2.16}$$

Each term is treated separately. Integrating by parts, we have

$$\int_0^1 \overline{\partial^2 u^\epsilon}\,\partial_t u^\epsilon dx = -\int_0^1 \overline{\partial u^\epsilon}\,\partial_t(\partial u^\epsilon)\,dx$$

$$\int_0^1 |u^\epsilon|^2\,u^\epsilon\,\overline{\partial^2 u^\epsilon} dx = -\int_0^1 \partial(|u^\epsilon|^2)\,u^\epsilon\,\overline{\partial u^\epsilon} dx - \int_0^1 |u^\epsilon|^2\,|\partial u^\epsilon|^2 dx$$

$$\int_0^1 \overline{\partial^2 u^\epsilon}\,\partial^4 u^\epsilon dx = -\int_0^1 |\partial^3 u^\epsilon|^2 dx.$$

Replacing in (2.16) we obtain

$$-\,i\int_0^1 \overline{\partial u^\epsilon}\,\partial_t(\partial u^\epsilon)\,dx + \alpha\int_0^1 |\partial^2 u^\epsilon|^2 dx + i\,\eta\int_0^1 \overline{\partial^2 u^\epsilon}\,\partial^3 u^\epsilon dx$$
$$-\int_0^1 \partial(|u^\epsilon|^2)\,u^\epsilon\,\overline{\partial u^\epsilon} dx - \int_0^1 |u^\epsilon|^2 |\partial u^\epsilon|^2 dx + i\,\delta\int_0^1 \partial u^\epsilon\,\overline{\partial^2 u^\epsilon} dx$$
$$-i\,\epsilon\int_0^1 |\partial^3 u^\epsilon|^2 dx = 0. \tag{2.17}$$

Applying conjugate in (2.10), we get

$$-\,i\,\overline{\partial_t u^\epsilon} + \alpha\,\overline{\partial^2 u^\epsilon} - i\,\eta\,\overline{\partial^3 u^\epsilon} + |u^\epsilon|^2\,\overline{u^\epsilon} - i\,\delta\,\overline{\partial u^\epsilon} - i\,\epsilon\,\overline{\partial^4 u^\epsilon} = 0. \tag{2.18}$$

We multiply (2.18) by $\partial^2 u^\epsilon$ and we integrate over $x \in \Omega = (0,\,1)$ to fird

$$-\,i\int_0^1 \partial^2 u^\epsilon\,\overline{\partial_t u^\epsilon} dx + \alpha\int_0^1 |\partial^2 u^\epsilon|^2 dx - i\,\eta\int_0^1 \partial^2 u^\epsilon\,\overline{\partial^3 u^\epsilon} dx$$
$$+\int_0^1 |u^\epsilon|^2\,\overline{u^\epsilon}\,\partial^2 u^\epsilon dx - i\,\delta\int_0^1 \overline{\partial u^\epsilon}\,\partial^2 u^\epsilon dx - i\,\epsilon\int_0^1 \partial^2 u^\epsilon\,\overline{\partial^4 u^\epsilon} dx = 0 \tag{2.19}$$

Each term is treated separately. Integrating by parts, we hav

$$\int_0^1 \partial^2 u^\epsilon\, \overline{\partial_t u^\epsilon} dx = -\int_0^1 \partial u^\epsilon\, \overline{\partial_t(\partial u^\epsilon)} dx$$

$$\int_0^1 |u^\epsilon|^2\, \overline{u^\epsilon}\, \partial^2 u^\epsilon dx = -\int_0^1 \partial(|u^\epsilon|^2)\, \overline{u^\epsilon}\, u_x^\epsilon dx - \int_0^1 |u^\epsilon|^2 |\partial u^\epsilon|^2 dx$$

$$\int_0^1 \partial^2 u^\epsilon\, \overline{\partial^4 u^\epsilon} dx = -\int_0^1 |\partial^3 u^\epsilon|^2 dx.$$

Replacing in (2.19) we obtain

$$i \int_0^1 \partial u^\epsilon\, \overline{\partial_t(\partial u^\epsilon)} dx + \alpha \int_0^1 |\partial^2 u^\epsilon|^2 dx - i\,\eta \int_0^1 \partial^2 u^\epsilon\, \overline{\partial^3 u^\epsilon} dx - \int_0^1 \partial(|u^\epsilon|^2)\, \overline{u^\epsilon}\, \partial u^\epsilon dx$$
$$- \int_0^1 |u^\epsilon|^2 |\partial u^\epsilon|^2 dx - i\,\delta \int_0^1 \overline{\partial u^\epsilon}\, \partial^2 u^\epsilon dx + i\,\epsilon \int_0^1 |\partial^3 u^\epsilon|^2 dx = 0. \tag{2.20}$$

Subtracting (2.17) and (2.20), using Lemma 2.7 and performing straightforward calculations we obtain

$$\begin{aligned}
&\partial_t\, ||\partial u^\epsilon||^2_{L^2(0,1)} + 2\,\epsilon\, ||\partial^3 u^\epsilon||^2_{L^2(0,1)} \\
&= -2\, Im \int_0^1 \partial(|u^\epsilon|^2)\, u^\epsilon\, \overline{\partial u^\epsilon} dx \\
&\leq 4 \int_0^1 |u^\epsilon|^2 |\partial u^\epsilon|^2\, dx \\
&\leq 4\, ||u^\epsilon||^2_{L^4(0,1)} ||\partial u^\epsilon||^2_{L^4(0,1)} \\
&\leq 4 \Big[c^2 ||u^\epsilon||^{11/12}_{L^2(0,1)} [||u^\epsilon||_{L^2(0,1)} + ||\partial^3 u^\epsilon||_{L^2(0,1)}]^{1/12} c^2 ||u^\epsilon||^{7/12}_{L^2(0,1)} [||u^\epsilon||_{L^2(0,1)} \\
&\quad + ||\partial^3 u^\epsilon||_{L^2(0,1)}]^{5/12} \Big]^2 \\
&= 4c^4 ||u^\epsilon||^{11/6}_{L^2(0,1)} \left[||u^\epsilon||_{L^2(0,1)} + ||\partial^3 u^\epsilon||_{L^2(0,1)}\right]^{1/6} ||u^\epsilon||^{7/6}_{L^2(0,1)} \left[||u^\epsilon||_{L^2(0,1)} \right. \\
&\quad \left. + ||\partial^3 u^\epsilon||_{L^2(0,1)}\right]^{5/6} \\
&= 4c^4 ||u^\epsilon||^3_{L^2(0,1)} \left[||u^\epsilon||_{L^2(0,1)} + ||\partial^3 u^\epsilon||_{L^2(0,1)}\right] \\
&= 4c^4 ||u^\epsilon||^4_{L^2(0,1)} + 4c^4 ||u^\epsilon||^3_{L^2(0,1)} ||\partial^3 u^\epsilon||_{L^2(0,1)}.
\end{aligned}$$

Using that $2\,a\,b \le a^2 + b^2$, and Lemma 2.6, we obtain

$$\partial_t\,||\partial u^\epsilon||^2_{L^2(0,\,1)} + \epsilon\,||\partial^3 u^\epsilon||^2_{L^2(0,\,1)} \le c.$$

Integrating over $t \in [0,\,T]$ we have

$$||\partial u^\epsilon||^2_{L^2(0,\,1)} + \epsilon \int_0^T ||\partial^3 u^\epsilon||^2_{L^2(0,\,1)}\,dt \le c;$$

then

$$||\partial u^\epsilon||^2_{L^2(0,\,1)} + \epsilon\,||\partial^3 u^\epsilon||^2_{L^2(0,\,T:\,L^2(0,1))} \le c. \tag{2.21}$$

In particular

$$||\partial u^\epsilon||_{L^\infty(0,\,T:\,L^2(0,1))} \le c \qquad : \qquad \sqrt{\epsilon}\,||\partial^3 u^\epsilon||_{L^2(0,\,T:\,L^2(0,1))} \le c$$

if and only if

$$||\partial u^\epsilon||_{L^\infty(0,\,T:\,L^2(0,1))} \le c \qquad : \qquad \sqrt{\epsilon}\,||\partial^3 u^\epsilon||_{L^2(Q)} \le c$$

or

$$\begin{aligned} \partial u^\epsilon &\in L^\infty(0,\,T:\;L^2(0,\,1)) \\ \partial^3 u^\epsilon &\in L^2(0,\,T:\;L^2(0,\,1)), \end{aligned}$$

where

$$u^\epsilon \in L^\infty(0,\,T:\;H^1(0,\,1)) \cap L^2(0,\,T:\;H^3(0,\,1)). \tag{2.22}$$

Hence from (2.14)–(2.15) and (2.22) we have the existence of subsequence $u^{\epsilon_j} \overset{def}{=} u^\epsilon$ such that

$$\begin{aligned} u^\epsilon &\overset{*}{\rightharpoonup} u \quad \text{weakly in} \quad L^\infty(0,\,T:\;L^2(0,\,1)) \hookrightarrow L^2(0,\,T:\;L^2(0,\,1)) = L^2(Q) \\ \partial u^\epsilon &\overset{*}{\rightharpoonup} \partial u \quad \text{weakly in} \quad L^\infty(0,\,T:\;L^2(0,\,1)) \hookrightarrow L^2(0,\,T:\;L^2(0,\,1)) = L^2(Q). \end{aligned}$$

Thus from the equation (2.10) we deduce that

$$\partial_t u^\epsilon \overset{*}{\rightharpoonup} \partial_t u \quad \text{weakly in} \quad L^2(0,\,T:\;H^{-2}(0,\,1)).$$

On the other hand, we have $H^1(0,\,1) \overset{c}{\hookrightarrow} L^2(0,\,1) \hookrightarrow H^{-2}(0,\,1)$. Using Lions-Aubin's compactness Theorem

$$u^\epsilon \to u \quad \text{strongly in} \quad L^2(Q).$$

Then

$$|u^\epsilon|^2\, u^\epsilon = u^\epsilon\,\overline{u^\epsilon}\,u^\epsilon \longrightarrow u\,\overline{u}\,u = |u|^2\,u \quad \text{in} \quad \mathcal{D}'(0,\,1).$$

The other terms are calculated in a similar way and therefore we can pass to the limit in the equation (2.10)–(2.12). Finally, u is solution of the equation (1.1)–(1.3) and the theorem follows. □

3. The Main Theorem

The first result is concerned with the decay properties for the higher order nonlinear Schrödinger equation with constant coefficients. The idea goes back to Kato T. [14]. □

Lemma 3.1. *Let* $|\alpha| \le 3\,\eta$. *Let* u *be a solution to* (1.1)–(1.3) *and* $e^{\beta\,x}\,u_0 \in L^2(\mathbb{R})$, *then*

$$e^{\beta\,x}\,u \in C([0,\,1] : \, L^2(\mathbb{R})) \tag{3.1}$$

Proof. Let $\varphi_n \in C^\infty(\mathbb{R})$ be defined by

$$\varphi_n(x) = \begin{cases} e^{\beta\,x} & , \quad \text{for } \; x \le n \\ e^{2\,\beta\,x} & , \quad \text{for } \; x > 10\,n \end{cases}$$

with

$$\varphi_n(x) \le e^{2\,\beta\,x} \;\; : \quad 0 \le \varphi_n'(x) \le \beta\,\varphi_n(x) \;\; : \quad |\varphi_n^{(j)}(x)| \le \beta^j\,\varphi_n(x) \qquad j = 2,\,3. \tag{3.2}$$

□

Example.
Multiplying the equation (1.1) by $\overline{u}\,\varphi_n$, we have

$$i\,\overline{u}\,\partial_t u\,\varphi_n + \alpha\,\overline{u}\,\partial^2 u\,\varphi_n + i\,\eta\,\overline{u}\,\partial^3 u\,\varphi_n + |u|^4\,\varphi_n + i\,\delta\,\overline{u}\,\partial u\,\varphi_n = 0$$

$$-\,i\,u\,\partial_t\overline{u}\,\varphi_n + \alpha\,u\,\partial^2\overline{u}\,\varphi_n - i\,\eta\,u\,\partial^3\overline{u}\,\varphi_n + |u|^4\,\varphi_n - i\,\delta\,u\,\partial\overline{u}\,\varphi_n = 0$$

(applying conjugate).

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$i\,\partial_t \int_{\mathbb{R}} |u|^2\,\varphi_n dx + \alpha \int_{\mathbb{R}} \overline{u}\,\partial^2 u\,\varphi_n dx - \alpha \int_{\mathbb{R}} u\,\partial^2\overline{u}\,\varphi_n dx + i\,\eta \int_{\mathbb{R}} \overline{u}\,\partial^3 u\,\varphi_n dx$$

$$+\, i\,\eta \int_{\mathbb{R}} u\,\partial^3\overline{u}\,\varphi_n dx + i\,\delta \int_{\mathbb{R}} \partial(|u|^2)\,\varphi_n dx = 0. \tag{3.3}$$

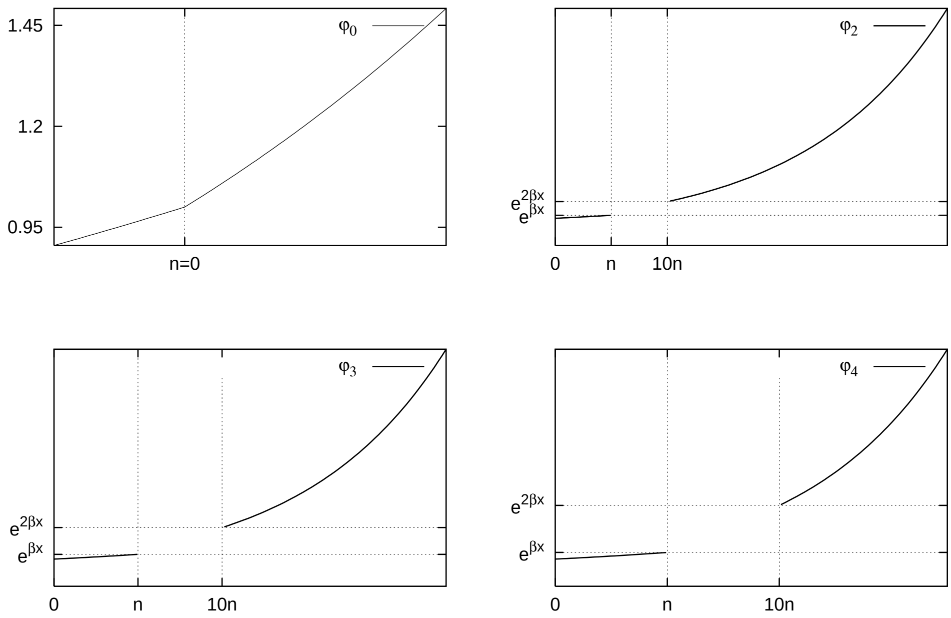


Figure 1. These are sample figures for different values of n.

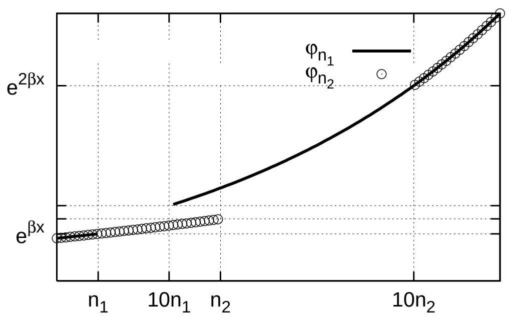


Figure 2. This is a figure comparing two functions with different values of n.

Each term is calculated separately using integrating by parts:

$$\begin{aligned}
\int_{\mathbb{R}} \overline{u}\,\partial^2 u\,\varphi_n dx &= -\int_{\mathbb{R}} \overline{u}\,\partial u\,\varphi_n' dx - \int_{\mathbb{R}} |\partial u|^2\,\varphi_n dx \\
\int_{\mathbb{R}} u\,\partial^2 \overline{u}\,\varphi_n dx &= -\int_{\mathbb{R}} u\,\partial \overline{u}\,\varphi_n' dx - \int_{\mathbb{R}} |\partial u|^2\,\varphi_n dx \\
\int_{\mathbb{R}} \overline{u}\,\partial^3 u\,\varphi_n\,dx &= \int_{\mathbb{R}} \overline{u}\,\partial u_x\,\varphi_n'' dx + \int_{\mathbb{R}} |\partial u|^2\,\varphi_n'\,dx - \int_{\mathbb{R}} \partial\overline{u}\,\partial^2 u\,\varphi_n dx \\
\int_{\mathbb{R}} u\,\partial^3 \overline{u}\,\varphi_n\,dx &= \int_{\mathbb{R}} u\,\partial \overline{u}\,\varphi_n'' dx + \int_{\mathbb{R}} |\partial u|^2\,\varphi_n' dx - \int_{\mathbb{R}} \partial u\,\partial^2\overline{u}\,\varphi_n dx.
\end{aligned}$$

Replacing in (3.3), we have

$$\begin{aligned}
& i\,\partial_t \int_{\mathbb{R}} |u|^2\,\varphi_n dx - \alpha \int_{\mathbb{R}} \overline{u}\,\partial u\,\varphi_n' dx - \alpha \int_{\mathbb{R}} |\partial u|^2\,\varphi_n dx + \alpha \int_{\mathbb{R}} u\,\partial\overline{u}\,\varphi_n' dx \\
& +\alpha \int_{\mathbb{R}} |\partial u|^2\,\varphi_n dx i\,\eta \int_{\mathbb{R}} \overline{u}\,\partial u\,\varphi_n'' dx + i\,\eta \int_{\mathbb{R}} |\partial u|^2\,\varphi_n' dx - i\,\eta \int_{\mathbb{R}} \partial\overline{u}\,\partial^2 u\,\varphi_n dx \\
& i\,\eta \int_{\mathbb{R}} u\,\partial\overline{u}\,\varphi_n''\,dx + i\,\eta \int_{\mathbb{R}} |\partial u|^2\,\varphi_n' dx - i\,\eta \int_{\mathbb{R}} \partial u\,\partial^2\overline{u}\,\varphi_n dx - i\,\delta \int_{\mathbb{R}} |u|^2\,\varphi_n' dx = 0,
\end{aligned}$$

hence

$$\begin{aligned}
& i\,\partial_t \int_{\mathbb{R}} |u|^2\,\varphi_n\,dx - 2\,i\,\alpha\, Im \int_{\mathbb{R}} \overline{u}\,\partial u\,\varphi_n' dx + i\,\eta \int_{\mathbb{R}} \partial(|u|^2)\,\varphi_n''\,dx \\
& + 2\,i\,\eta \int_{\mathbb{R}} |\partial u|^2\,\varphi_n' dx - i\,\eta \int_{\mathbb{R}} \partial(|\partial u|^2)\,\varphi_n dx - i\,\delta \int_{\mathbb{R}} |u|^2\,\varphi_n' dx = 0;
\end{aligned}$$

then

$$\begin{aligned}
& \partial_t \int_{\mathbb{R}} |u|^2\,\varphi_n\,dx - 2\,\alpha\, Im \int_{\mathbb{R}} \overline{u}\,\partial u\,\varphi_n' dx - \eta \int_{\mathbb{R}} |u|^2\,\varphi_n''' dx \\
& + 2\,\eta \int_{\mathbb{R}} |\partial u|^2\,\varphi_n' dx + \eta \int_{\mathbb{R}} |\partial u|^2\,\varphi_n' dx - \delta \int_{\mathbb{R}} |u|^2\,\varphi_n' dx = 0,
\end{aligned}$$

thus

$$\begin{aligned}
& \partial_t \int_{\mathbb{R}} |u|^2\,\varphi_n dx + 3\,\eta \int_{\mathbb{R}} |\partial u|^2\,\varphi_n' dx - \eta \int_{\mathbb{R}} |u|^2 \varphi_n''' dx - \delta \int_{\mathbb{R}} |u|^2 \varphi_n' dx \\
= \;& 2\,\alpha\, Im \int_{\mathbb{R}} \overline{u}\,\partial u\,\varphi_n' dx \le |\alpha| \int_{\mathbb{R}} |u|^2\,\varphi_n' dx + |\alpha| \int_{\mathbb{R}} |\partial u|^2 \varphi_n'\,dx
\end{aligned}$$

hence we have

$$\partial_t \int_{\mathbb{R}} |u|^2 \varphi_n dx + (3\,\eta - |\alpha|) \int_{\mathbb{R}} |\partial u|^2 \varphi_n' dx - \eta \int_{\mathbb{R}} |u|^2 \varphi_n''' dx - \delta \int_{\mathbb{R}} |u|^2 \varphi_n' dx$$
$$\leq |\alpha| \int_{\mathbb{R}} |u|^2 \varphi_n' dx$$

then

$$\begin{aligned} & \partial_t \int_{\mathbb{R}} |u|^2 \varphi_n dx + (3\eta - |\alpha|) \int_{\mathbb{R}} |\partial u|^2 \varphi_n' dx \\ \leq \quad & \eta \int_{\mathbb{R}} |u|^2 \varphi_n''' dx + \delta \int_{\mathbb{R}} |u|^2 \varphi_n' dx + |\alpha| \int_{\mathbb{R}} |u|^2 \varphi_n' dx. \end{aligned}$$

Using that $|\alpha| \leq 3\,\eta$ and (3.2), we obtain

$$\partial_t \int_{\mathbb{R}} |u|^2 \varphi_n dx + \beta\,(3\,\eta - |\alpha|) \int_{\mathbb{R}} |\partial u|^2 \varphi_n dx \leq (\eta\,\beta^3 + \delta\,\beta + |\alpha|\,\beta) \int_{\mathbb{R}} |u|^2 \varphi_n dx.$$

Integrating over $t \in [0,\,1]$ we have

$$\begin{aligned} & \int_0^1 \partial_t \int_{\mathbb{R}} |u|^2 \varphi_n \, dx \, dt + \beta\,(3\,\eta - |\alpha|) \int_0^1 \int_{\mathbb{R}} |\partial u|^2 \varphi_n \, dx \, dt \\ \leq \quad & (\eta\,\beta^3 + \delta\,\beta + |\alpha|\,\beta) \int_0^1 \int_{\mathbb{R}} |u|^2 \varphi_n dx \, dt \end{aligned}$$

hence

$$\begin{aligned} & \int_{\mathbb{R}} |u|^2 \varphi_n \, dx \, dt + \beta\,(3\,\eta - |\alpha|) \int_0^1 \int_{\mathbb{R}} |\partial u|^2 \varphi_n \, dx \, dt \\ \leq \quad & \int_{\mathbb{R}} |u_0|^2 \varphi_n \, dx + (\,\eta\,\beta^3 + \delta\,\beta + |\alpha|\,\beta\,) \int_0^1 \int_{\mathbb{R}} |u|^2 \varphi_n \, dx \, dt \end{aligned} \tag{3.4}$$

then

$$\int_{\mathbb{R}} |u|^2 \varphi_n \, dx \, dt \leq \int_{\mathbb{R}} |u_0|^2 \varphi_n \, dx + (\,\eta\,\beta^3 + \delta\,\beta + |\alpha|\,\beta\,) \int_0^1 \int_{\mathbb{R}} |u|^2 \varphi_n \, dx \, dt.$$

Using straightforward calculations, we have

$$\int_{\mathbb{R}} |u|^2 \varphi_n \, dx \leq \left[\int_{\mathbb{R}} |u_0|^2 \varphi_n \, dx \right] \, e^{c_1\,t},$$

where $c = \eta\,\beta^3 + \delta\,\beta + |\alpha|\,\beta$. Thus

$$\sup_{t\in[0,\,1]} \int_{\mathbb{R}} |u|^2\,\varphi_n\,dx \le \left[\int_{\mathbb{R}} |u_0|^2\,\varphi_n\,dx\right]\; e^{c_1} \le \left[\int_{\mathbb{R}} |u_0|^2\,e^{2\,\beta\,x}\,dx\right]\; e^{c_1}.$$

Now, taking $n \to \infty$ we obtain

$$\sup_{t\in[0,\,1]} \int_{\mathbb{R}} |u|^2\,e^{2\,\beta\,x}\,dx \le \left[\int_{\mathbb{R}} |u_0|^2\,e^{2\,\beta\,x}\,dx\right]\; e^{c_1}.$$

where

$$\sup_{t\in[0,\,1]} ||u||^2_{L^2(\,e^{2\,\beta\,x}\,dx)} \le ||u_0||^2_{L^2(e^{2\,\beta\,x}\,dx)}\; e^{c_1}.$$

This way

$$\sup_{t\in[0,\,1]} ||u||_{L^2(\,e^{2\,\beta\,x}\,dx)} \le ||u_0||_{L^2(e^{2\,\beta\,x}\,dx)}\; e^{c}.$$

with $c = 2\;(\,\eta\,\beta^3 + \delta\,\beta + |\alpha|\,\beta\,)$. □

Remark. Since (3.4) we see that if $e^{\beta\,x}\,u_0 \in L^2(\mathbb{R})$, then $e^{\beta\,x}\,u \in H^1(\mathbb{R})$. Hence there is a gain in regularity.

We have the following extension to higher derivatives.

Lemma 3.2. *Let* $|\alpha| \le 3\,\eta$ *and* $m \in \mathbb{N}$. *Let* u *be a solution of the* (1.1)–(1.3) *equation such that*

$$\sup_{t\in[0,\,1]} \;||u(\,.\,,\,t)||_{H^m(\mathbb{R})} < +\infty$$

and

$$e^{\beta\,x}\,u_0,\;\ldots\;,\;e^{\beta\,x}\,\partial^m u_0 \in L^2(\mathbb{R}), \qquad \forall\;\beta > 0$$

then

$$\sup_{t\in[0,\,1]} \;||u(t)||_{C^{m-1}(\mathbb{R})} \le c_m = c_m(u_0,\,c_1)$$

with $c_1 = \beta^3 + \frac{3}{2}\,\beta\,||u||_{L^\infty(\mathbb{R}\times[0,\,1])}\;||v||_{L^\infty(\mathbb{R}\times[0,\,1])}$.

Proof. Differentiating m times in variable x we have

$$i\,\partial_t(\partial^m u) + \alpha\,\partial^{m+2}u + i\,\eta\,\partial^{m+3}u + \partial^m(|u|^2 u) + i\,\delta\,\partial^{m+1}u = 0.$$

Multiplying by $\partial^m\overline{u}\,\varphi_n$, we obtain

$$\begin{aligned}
& i\,\partial^m\overline{u}\,\partial_t(\partial^m u)\,\varphi_n + \alpha\,\partial^m\overline{u}\,\partial^{m+2}u\,\varphi_n + i\,\eta\,\partial^m\overline{u}\,\partial^{m+3}u\,\varphi_n \\
& + \partial^m\overline{u}\,\partial^m(|u|^2u)\,\varphi_n + i\,\delta\,\partial^m\overline{u}\,\partial^{m+1}u\,\varphi_n = 0
\end{aligned}$$

$$\begin{aligned}
& -i\,\partial^m u\,\partial_t(\partial^m\overline{u})\,\varphi_n + \alpha\,u_m\,\overline{u}_{m+2}\,\varphi_n - i\,\eta\,\partial^m u\,\partial^{m+3}\overline{u}\,\varphi_n \\
& + \overline{\partial^m\overline{u}\,\partial^m(|u|^2u)}\,\varphi_n - i\,\delta\,\partial^m u\,\partial^{m+1}\overline{u}\,\varphi_n = 0 \quad \text{(applying conjugate).}
\end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$, we have

$$\begin{aligned}
& i\,\partial_t\int_{\mathbb{R}}|\partial^m u|^2\varphi_n\,dx + \alpha\int_{\mathbb{R}}\partial^m\overline{u}\,\partial^{m+2}u\,\varphi_n\,dx - \alpha\int_{\mathbb{R}}\partial^m u\,\partial^m\overline{u}\,\varphi_n\,dx \\
& + i\,\eta\int_{\mathbb{R}}\partial^m\overline{u}\,\partial^{m+3}u\,\varphi_n\,dx + i\,\eta\int_{\mathbb{R}}u\,\partial^{m+3}\overline{u}\,\varphi_n dx + 2\,i\,Im\int_{\mathbb{R}}\partial^m\overline{u}\,\partial^m(|u|^2u)\,\varphi_n dx \\
& + i\,\delta\int_{\mathbb{R}}\partial(|\partial^m u|^2)\varphi_n\,dx = 0. \qquad (3.5)
\end{aligned}$$

Each term is calculated separately integrating by parts

$$\int_{\mathbb{R}}\partial^m\overline{u}\,\partial^{m+2}u\,\varphi_n\,dx \;=\; -\int_{\mathbb{R}}\partial^m\overline{u}\,\partial^{m+1}u\,\varphi_n' dx - \int_{\mathbb{R}}|\partial^{m+1}u|^2\varphi_n dx$$

$$\int_{\mathbb{R}}\partial^m u\,\partial^{m+2}\overline{u}\,\varphi_n dx \;=\; -\int_{\mathbb{R}}\partial^m u\,\partial^{m+1}\overline{u}\,\varphi_n' dx - \int_{\mathbb{R}}|\partial^{m+1}u|^2\varphi_n dx$$

$$\int_{\mathbb{R}}\partial^m\overline{u}\,\partial^{m+3}u\,\varphi_n dx \;=\; \int_{\mathbb{R}}\partial^m\overline{u}\,\partial^{m+1}u\,\varphi_n'' dx + \int_{\mathbb{R}}|\partial^{m+1}u|^2\varphi_n' dx - \int_{\mathbb{R}}\partial^{m+1}\overline{u}\,\partial^{m+2}u\,\varphi_n dx$$

$$\int_{\mathbb{R}}\partial^m u\,\partial^{m+3}\overline{u}\,\varphi_n dx \;=\; \int_{\mathbb{R}}\partial^m u\,\partial^{m+1}\overline{u}\,\varphi_n'' dx + \int_{\mathbb{R}}|\partial^{m+1}u|^2\varphi_n' dx - \int_{\mathbb{R}}\partial^{m+1}u\,\partial^{m+2}\overline{u}\,\varphi_n dx.$$

Replacing in (3.5), we have

$$\begin{aligned}
& i\,\partial_t \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n dx - \alpha \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^{m+1} u\, \varphi_n' dx - \alpha \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n dx + \alpha \int_{\mathbb{R}} u\, \partial^{m+1} \overline{u}\, \varphi_n' dx \\
& + \alpha \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n dx + i\,\eta \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^{m+1} u\, \varphi_n'' dx + i\,\eta \int_{\mathbb{R}} |\partial^{m+1} u|^2\, \varphi_n' dx \\
& - i\,\eta \int_{\mathbb{R}} \partial^{m+1} \overline{u}\, \partial^{m+2} u\, \varphi_n dx + i\,\eta \int_{\mathbb{R}} \partial^m u\, \partial^{m+1} \overline{u}\, \varphi_n'' dx + i\,\eta \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n' dx \\
& - i\,\eta \int_{\mathbb{R}} \partial^{m+1} u\, \partial^{m+2} \overline{u}\, \varphi_n dx + 2\, i\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2 u)\, \varphi_n dx + i\,\delta \int_{\mathbb{R}} \partial(|\partial^m u|^2)\, \varphi_n dx = 0,
\end{aligned}$$

hence

$$\begin{aligned}
& i\,\partial_t \int_{\mathbb{R}} |\partial^m u|^2\, \varphi_n dx - 2\, i\, \alpha\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^{m+1} u\, \varphi_n' dx + i\,\eta \int_{\mathbb{R}} \partial(|\partial^m u|^2)\, \varphi_n'' dx \\
& + 2\, i\, \eta \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n' dx - i\,\eta \int_{\mathbb{R}} \partial(|\partial^{m+1} u|^2)\, \varphi_n dx \\
& + 2\, i\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2 u)\, \varphi_n dx - i\,\delta \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n' dx = 0,
\end{aligned}$$

then

$$\begin{aligned}
& \partial_t \int_{\mathbb{R}} |\partial^m u|^2\, \varphi_n dx - 2\, \alpha\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^{m+1} u\, \varphi_n' dx - \eta \int_{\mathbb{R}} |\partial^m u|^2\, \varphi_n''' dx \\
& + 2\, \eta \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n' dx + \eta \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n' dx + 2\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2 u)\, \varphi_n dx \\
& - \delta \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n' dx = 0,
\end{aligned}$$

thus

$$\begin{aligned}
& \partial_t \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n dx + 3\, \eta \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n' dx - \eta \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n''' dx \\
& + 2\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2 u)\, \varphi_n dx - \delta \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n' dx \\
= \;\; & 2\, \alpha\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^{m+1} u\, \varphi_n' dx \le |\alpha| \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n' dx + |\alpha| \int_{\mathbb{R}} |\partial^{m+1} u|^2 \varphi_n' dx,
\end{aligned}$$

hence we have

$$\partial_t \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n dx + (3\,\eta - |\alpha|) \int_{\mathbb{R}} |\partial^{m+1} u|^2\, \varphi_n' dx - \eta \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n''' dx$$

$$+ 2\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2\, u)\, \varphi_n dx - \delta \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n' dx \leq |\alpha| \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n' dx,$$

then

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n dx \\ \leq\ & -(3\,\eta - |\alpha|) \int_{\mathbb{R}} |\partial^{m+1} u|^2\, \varphi_n' dx + \eta \int_{\mathbb{R}} |\partial^m u|^2\, \varphi_n''' dx + \delta \int_{\mathbb{R}} |\partial^m u|^2\, \varphi_n' dx \\ & + |\alpha| \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n' dx - 2\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2 u)\, \varphi_n dx. \end{aligned}$$

Using that $|\alpha| \leq 3\,\eta$ and (3.2) we obtain

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n dx \\ &\leq (\eta\, \beta^3 + \delta\, \beta + |\alpha|\, \beta) \int_{\mathbb{R}} |\partial^m u|^2 \varphi_n dx - 2\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2 u)\, \varphi_n dx. \end{aligned} \tag{3.6}$$

But

$$\begin{aligned} \partial^m (|u|^2\, u) &= \sum_{n=1}^{m-1} \frac{n!}{n!(m-n)!}\, \partial^{m-n} (|u|^2)\, \partial^n u \\ &= \sum_{n=1}^{m} \frac{n!}{n!(m-n)!}\, \partial^{m-n} (|u|^2)\, \partial^n u + |u|^2 \partial^m u, \end{aligned}$$

hence

$$\begin{aligned} & 2\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^m (|u|^2 u)\, \varphi_n\, dx \\ =\ & 2 \sum_{n=1}^{m} \frac{n!}{n!(m-n)!}\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^{m-n} (|u|^2)\, \partial^n u\, \varphi_n dx + 2\, Im \int_{\mathbb{R}} |u|^2 |\partial^m u|^2 \varphi_n dx \\ =\ & 2 \sum_{n=1}^{m} \frac{n!}{n!(m-n)!}\, Im \int_{\mathbb{R}} \partial^m \overline{u}\, \partial^{m-n} (|u|^2)\, \partial^n u\, \varphi_n dx. \end{aligned} \tag{3.7}$$

Replacing (3.7) in (3.6), integrating over $t \in [0, 1]$ and doing straightforward calculations as in above Lemma, the result follows. □

Lemma 3.3 (Carleman's Estimate). *Let* $u \in C_0^{3,1}(\mathbb{R}^2)$ *and* $\eta \in \mathbb{R}$, *then*

$$||e^{\lambda x} u||_{L^8(\mathbb{R}^2)} \leq ||e^{\lambda x}\{\partial_t + \eta\, \partial^3\}\, u||_{L^{8/7}(\mathbb{R}^2)} \equiv ||e^{\lambda x}\, \mathcal{L}u||_{L^{8/7}(\mathbb{R}^2)} \tag{3.8}$$

for all $\lambda \in \mathbb{R}$, *with* c *independent of* λ.

Proof. We will prove that if $u \in C_0^{3,1}(\mathbb{R}^2)$, then

$$||e^{\lambda x} u||_{L^8(\mathbb{R}^2)} \leq ||e^{\lambda x}\, \{\partial_t + \eta\, \partial^3\}u||_{L^{8/7}(\mathbb{R}^2)}, \quad \forall\, \lambda \in \mathbb{R} \tag{3.9}$$

with c independent of λ. □

Claim 1. It suffices to consider the cases $\lambda = \pm 1$ in (3.9).

In fact, we observe that the case $\lambda = 0$ follows from the case $\lambda \neq 0$ by taking the limit as $\lambda \to 0$. So we can restrict ourselves to the case $\lambda \neq 0$.

We consider the case $\lambda > 0$ (the proof $\lambda < 0$ is similar). Assume that

$$||e^{x}\, u||_{L^8(\mathbb{R}^2)} \leq ||e^{x}\, \{\partial_t + \eta\, \partial^3\}u||_{L^{8/7}(\mathbb{R}^2)}, \qquad \forall\, u \in C_0^{3,1}(\mathbb{R}^2) \tag{3.10}$$

with c independent of λ.

Let $u_\lambda(x, t) = u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right)$, then

$$\{\partial_t + \eta\, \partial^3\}u_\lambda(x, t) = \frac{1}{\lambda^3}\left[\partial_t u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right) + \eta\, \partial^3 u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right)\right].$$

We consider the change of variable

$$y = \frac{x}{\lambda} \;\; : \qquad s = \frac{t}{\lambda^3};$$

then

$$\frac{\partial(y, s)}{\partial(x, t)} = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda^3} \end{vmatrix} = \frac{1}{\lambda^4},$$

hence $dy\,ds = \frac{1}{\lambda^4}\,dx\,dt$, then $\lambda^4\,dy\,ds = dx\,dt$. This way

$$
\begin{aligned}
||e^x\,u_\lambda||_{L^8(\mathbb{R}^2)} &= \left[\int_{\mathbb{R}^2} |e^x\,u_\lambda|^8\,dx\,dt\right]^{1/8} \\
&= \lambda^{4/8}\left[\int_{\mathbb{R}^2} |e^{\lambda\,y}\,u_\lambda|^8\,dy\,ds\right]^{1/8} \\
&= \lambda^{1/2}\,||e^{\lambda\,y}\,||_{L^8(\mathbb{R}^2)} \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
||e^x\,\{\partial_t + \eta\,\partial^3\}\,u_\lambda||_{L^{8/7}(\mathbb{R}^2)} &= \frac{\lambda^{4\cdot\frac{7}{8}}}{\lambda^3}\,||e^{\lambda\,y}\,\{\partial_s + \eta\,\partial^3\}\,u||_{L^{8/7}(\mathbb{R}^2)} \\
&= \lambda^{1/2}\,||e^{\lambda\,y}\,\{\partial_s + \eta\,\partial^3\}\,u||_{L^{8/7}(\mathbb{R}^2)}, \qquad (3.12)
\end{aligned}
$$

hence (3.11) and (3.12) in (3.10) we obtain (3.9). □

Claim 2. To prove (3.10) it suffices to establish the inequality

$$||u||_{L^8(\mathbb{R}^2)} \le c\,||\,\{\partial_t + \eta\,\partial^3 - 3\,\eta\,\partial^2 + 3\,\eta\,\partial - \eta\}u\,||_{L^{8/7}(\mathbb{R}^2)}, \quad \forall\,u \in C_0^{3,1}(\mathbb{R}^2). \qquad (3.13)$$

In fact, let $v(x,\,t) = e^x\,u(x,\,t)$, hence

$$\{\partial_t + \eta\,\partial^3 - 3\,\eta\,\partial^2 + 3\,\eta\,\partial - \eta\}v = e^x\,\{\partial_t + \eta\,\partial^3\}u$$

then (3.10) follows. □

Claim 3. It suffices to prove the inequality (3.13) without the term on the right hand side involving the derivatives of order 1 in the x-variable. Indeed, to prove (3.13) is suffices to show

$$||w||_{L^8(\mathbb{R}^2)} \le c\,||\,\{\partial_t + \eta\,\partial^3 - 3\,\eta\,\partial^2 - \eta\}w\,||_{L^{8/7}(\mathbb{R}^2)}, \qquad \forall\,w \in C_0^{3,1}(\mathbb{R}^2). \qquad (3.14)$$

In fact, let $y = -\frac{x}{3\,\eta} + t$ and $s = t$, then $x = 3\,\eta\,(s - y)$ and $t = s$. Hence

$$\frac{\partial(y,\,s)}{\partial(x,\,t)} = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial t} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3\,\eta} & 1 \\ 0 & 1 \end{vmatrix} = -\frac{1}{3\,\eta}$$

then $dy\,ds = -\frac{1}{3\eta}\,dx\,dt$. Thus for $w(y,\,s) \sim u(x,\,t)$ we have

$$\partial_x u = -\frac{1}{3\eta}\,\partial_y w_y \;:\quad \partial_x^2 u = \frac{1}{9\eta^2}\,\partial_y^2 w \;:\quad \partial_x^3 u = -\frac{1}{27\eta^3}\,\partial_y^3 w \quad:\quad \partial_t u = \partial_s w - 3\,\eta\,\partial_x u.$$

Hence (3.13) can be written in the equivalent form

$$||w||_{L^8(\mathbb{R}^2)} \le c\,\left|\left|\left\{\partial_s - \frac{1}{27\,\eta^3}\,\partial_y^2 - \frac{1}{3\,\eta}\,\partial_y^2 - \eta\right\} w\right|\right|_{L^{8/7}(\mathbb{R}^2)}, \quad \forall\, w \in C_0^{3,\,1}(\mathbb{R}^2). \tag{3.15}$$

Now, making the change of variables $z = -\,3\,\eta\,y$ and $t = s$, we obtain in (3.15)

$$||w||_{L^8(\mathbb{R}^2)} \le c\,||\,\{\partial_s + \eta\,\partial_y^3 - 3\,\eta\,\partial_y^2 - \eta\}w\,||_{L^{8/7}(\mathbb{R}^2)}, \qquad \forall\, w \in C_0^{3,\,1}(\mathbb{R}^2) \tag{3.16}$$

and the claim follows. □

To complete the proof of Lemma we just need to prove (3.14), i. e.,

$$||w||_{L^8(\mathbb{R}^2)} \le c\,||\,\{\partial_t + \eta\,\partial_y^3 - 3\,\eta\,\partial_y^2 + 3\,\eta\,\partial_y - \eta\}w\,||_{L^{8/7}(\mathbb{R}^2)}, \quad \forall\, w \in C_0^{3,\,1}(\mathbb{R}^2). \tag{3.17}$$

Taking the Fourier transform, in space and time variables, on the right hand side of (3.17) we have

$$(i\,\tau - i\,\eta\,\xi^3 + 3\,\eta\,\xi^2 - \eta\,)\widehat{\widehat{w}}. \tag{3.18}$$

We consider the pair of points

$$P_\pm = (\xi_0^\pm,\,\tau_0^\pm) = \pm\left(\frac{1}{\sqrt{3}},\,\frac{\eta}{(\sqrt{3})^3}\right) \tag{3.19}$$

where the symbol in (3.18) vanishes. It suffices to prove (3.17) for any $w \in \mathcal{S}(\mathbb{R}^2)$, with $\widehat{\widehat{w}} = 0$ in a neighborhood of $P_\pm$.

So we are then reduced to showing the multiplier inequality

$$||Tw||_{L^8(\mathbb{R}^2)} = \left|\left|\left[\frac{1}{i\,(\,\tau - \eta\,\xi^3\,) + 3\,\eta\,\xi^2 - \eta}\,\widehat{\widehat{w}}\right]^{\vee\vee}\right|\right|_{L^8(\mathbb{R}^2)} \le c\,||w||_{L^{8/7}(\mathbb{R}^2)} \tag{3.20}$$

for such w's.

It suffices to prove (3.20) assuming that

$$\operatorname{supp}\widehat{\widehat{w}} \subseteq \{(\xi,\,\zeta): \quad \xi \ge 0\}, \tag{3.21}$$

since the proof for the case

$$\text{supp}\,\widehat{\widehat{w}} \subseteq \{(\xi,\,\zeta): \quad \xi < 0\} \tag{3.22}$$

is similar.

Remark. Using a variant of Littlewood-Paley theory we have the following. Let

$$\widehat{\widehat{L_k u}}(\xi,\,\zeta) = \chi_{[1/2,\,1]}\,(|\xi - \xi_0^+|/2^{-k})\,\widehat{\widehat{w}}(\xi,\,\zeta), \tag{3.23}$$

where $\chi_A(\cdot)$ is the characteristic function of the set A. Then for each $1 < p < \infty$ we have

$$||u||_{L^p(\mathbb{R}^2)} \simeq \left|\left|\left(\sum_{k\in\mathbb{Z}^+} |L_k(u)|^2\right)^{1/2}\right|\right|_{L^p(\mathbb{R}^2)}. \tag{3.24}$$

Thus it suffices to establish (3.20) for each $L_k w$ with a constant independent of k, since using Minskowski's integral inequality ($8/7 < 2 < 8$) one has

$$\begin{aligned}
&||Tw||_{L^8(\mathbb{R}^2)} \\
&\quad\simeq \left|\left|\left(\sum_{k\in\mathbb{Z}^+} |L_k(Tw)|^2\right)^{1/2}\right|\right|_{L^8(\mathbb{R}^2)} \\
&\quad= \left|\left|\left(\sum_{k\in\mathbb{Z}^+} |T(L_k w)|^2\right)^{1/2}\right|\right|_{L^8(\mathbb{R}^2)} \\
&\quad\leq \left(\sum_{k\in\mathbb{Z}^+} ||T(L_k w)||^2_{L^8(\mathbb{R}^2)}\right)^{1/2} \\
&\quad\leq c\left(\sum_{k\in\mathbb{Z}^+} ||L_k w||^2_{L^{8/7}(\mathbb{R}^2)}\right)^{1/2} \\
&\quad\leq c\left|\left|\left(\sum_{k\in\mathbb{Z}^+} |L_k w|^2\right)^{1/2}\right|\right|_{L^{8/7}(\mathbb{R}^2)} \\
&\quad\leq c\,||w||_{L^{8/7}(\mathbb{R}^2)}.
\end{aligned} \tag{3.25}$$

Therefore, we shall prove the multiplier estimate (3.20) when

$$\operatorname{supp}\widehat{\widehat{w}} \subseteq \{(\xi,\,\zeta):\quad \xi \geq 0,\quad 2^{-k-1} \leq |\xi - \xi_0^+| \leq 2^{-k}\}. \tag{3.26}$$

We split the proof of (3.26) into two cases

Case 1. $k \leq 0$. In this case, if $\xi \in \operatorname{supp}\widehat{\widehat{w}}$ then

$$|3\,\xi^2 - 1| \simeq |\xi - \xi_0^+|\,|\xi + \xi_0^+| \simeq 2^{-k}\,2^{-k}. \tag{3.27}$$

Using Lemma 2.8 we just need to bound the multiplier

$$\begin{aligned}
&\frac{1}{i\,(\tau - \eta\,\xi^3) + 3\,\eta\,\xi^2 - \eta} - \frac{1}{i\,(\tau - \eta\,\xi^3) + 2^{-2\,k}\eta}\\
&= \quad \frac{2^{-2\,k}\eta - (3\,\eta\,\xi^2 - \eta)}{[i\,(\tau - \eta\,\xi^3) + 3\,\eta\,\xi^2 - \eta][i\,(\tau - \eta\,\xi^3) + 2^{-2\,k}\eta]}\\
&= \quad \eta\,\frac{2^{-2\,k} - (3\,\xi^2 - 1)}{[i\,(\tau - \eta\,\xi^3) + 3\,\eta\,\xi^2 - \eta][i\,(\tau - \eta\,\xi^3) + 2^{-2\,k}\eta]}.
\end{aligned} \tag{3.28}$$

Using the change of variable $\tau = \lambda + \eta\,\xi^3$ we have

$$\begin{aligned}
&\eta\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\,(x,\,t)\cdot(\xi,\,\tau)}\,\frac{[\,2^{-2\,k} - (3\,\xi^2 - 1)]}{[i\,(\tau - \eta\,\xi^3) + 3\,\eta\,\xi^2 - \eta][i\,(\tau - \eta\,\xi^3) + 2^{-2\,k}\eta]}\,\widehat{\widehat{w}}(\xi,\,\tau)\,d\xi\,d\tau\\
&= \quad \eta\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\,(x,\,t)\cdot(\xi,\,\lambda+\eta\,\xi^3)}\frac{[2^{-2\,k} - (3\,\xi^2 - 1)]}{[i\,\lambda + 3\,\eta\,\xi^2 - \eta][i\,\lambda + 2^{-2\,k}\,\eta]}\,\widehat{\widehat{w}}(\xi,\,\lambda + \eta\,\xi^3)\,d\xi\,d\lambda\\
&= \quad \eta\int_{\mathbb{R}} e^{i\,\lambda\,t}\left[\int_{\mathbb{R}} e^{i\,(x,\,t)\cdot(\xi,\,\eta\,\xi^3)}\,\frac{[2^{-2\,k} - (3\,\xi^2 - 1)]}{[i\,\lambda + 3\,\eta\,\xi^2 - \eta][i\,\lambda + 2^{-2\,k}\,\eta\,]}\,\widehat{\widehat{w}}(\xi,\,\lambda + \eta\,\xi^3)\,d\xi\right]d\lambda\\
&\equiv \quad \Psi(x,\,t).
\end{aligned} \tag{3.29}$$

Let

$$\widehat{\widehat{w}}_\lambda(\xi,\,\xi^3) = \frac{[2^{-2\,k} - (3\,\xi^2 - 1)]}{[i\,\lambda + 3\,\eta\,\xi^2 - \eta][i\,\lambda + 2^{-2\,k}\eta]}\widehat{\widehat{w}}(\xi,\,\lambda + \eta\,\xi^3); \tag{3.30}$$

then in (3.29) we obtain

$$\Psi(x,\,t) \equiv \eta\int_{\mathbb{R}} e^{i\,\lambda\,t}\left[\int_{\mathbb{R}} e^{i\,(x,\,t)\cdot(\xi,\,\eta\,\xi^3)}\,\widehat{\widehat{w}}_\lambda(\xi,\,\xi^3)\,d\xi\right]d\lambda. \tag{3.31}$$

Because Minskowski's integral inequality we have

$$\begin{aligned} ||\Psi||_{L^8(\mathbb{R}^2)} &= \eta \left|\left| \int_{\mathbb{R}} e^{i\,\lambda\,t} \left[\int_{\mathbb{R}} e^{i\,(x,\,t)\cdot(\xi,\,\eta\,\xi^3)}\, \widehat{w}_\lambda(\xi,\, \xi^3)\, d\xi \right] d\lambda \right|\right|_{L^8(\mathbb{R}^2)} \\ &\le \eta \int_{\mathbb{R}} \left|\left| \int_{\mathbb{R}} e^{i\,(x,\,t)\cdot(\xi,\,\eta\,\xi^3)}\, \widehat{w}_\lambda(\xi,\, \xi^3)\, d\xi \right|\right|_{L^8(\mathbb{R}^2)} d\lambda. \end{aligned}$$

From Lemma 2.7 we obtain

$$||\Psi||_{L^8(\mathbb{R}^2)} \le c\,\eta \int_{\mathbb{R}} ||\, w_\lambda\, ||_{L^{8/7}(\mathbb{R}^2)}\, d\lambda. \tag{3.32}$$

Now for λ and k fixed we consider the multiplier in (3.30) in the variable ξ, with

$$\xi > 0, \quad |\xi - \xi_0^+| \simeq 2^{-\,k}, \quad |3\,\xi^2 - 1| \simeq 2^{-\,2\,k}$$

which has the norm bounded by

$$c\, \frac{2^{-\,2\,k}}{|\lambda|^2 + \eta\, 2^{-\,4\,k}}.$$

$$\begin{aligned} c\,\eta \int_{\mathbb{R}} ||\, w_\lambda\, ||_{L^{8/7}(\mathbb{R}^2)}\, d\lambda &\le c\,\eta \int_{\mathbb{R}} \frac{2^{-\,2\,k}}{|\lambda|^2 + \eta\, 2^{-\,4\,k}}\; ||\, \widetilde{w}_\lambda\, ||_{L^{8/7}(\mathbb{R}^2)}\, d\lambda \\ &\le c\, ||\, w\, ||_{L^{8/7}(\mathbb{R}^2)}\,, \end{aligned} \tag{3.33}$$

where $\widetilde{w}_\lambda(x,\, t) = e^{-\,i\,\lambda\,t}\, w(x,\, t)$. Thus (3.33) with (3.32) yields the proof of case 1.

Case 2. $k > 0$. In this case, if $\xi \in \operatorname{supp} \widehat{\widehat{w}}$ then

$$|3\,\xi^2 - 1| \simeq |\xi - \xi_0^+|\, |\xi + \xi_0^+| \simeq 2^{-\,k}\; |\xi - \xi_0^+|. \tag{3.34}$$

Using Lemma 2.8 to subtract

$$\frac{1}{i\,(\tau - \eta\,\xi^3) + 2^{-\,k}} \tag{3.35}$$

and argue in a similar way as before. The corresponding multiplier to (3.30) is

$$\frac{[\, 2^{-\,2\,k} + 1 - 3\,\xi^2\,]}{[\, i\,\lambda + 3\,\eta\,\xi^2 - \eta\,]\, [\, i\,\lambda + 2^{-\,2\,k}\,\eta\,]} \tag{3.36}$$

which has norm bounded by

$$c\,\frac{2^{-2k}}{|\lambda|^2+\eta\,2^{-2k}}.$$

The Lemma follows. □

Lemma 3.4. *Let* $u \in C^{3,1}(\mathbb{R}^2)$ *and* $\eta \in \mathbb{R}$ *such that*

$$\text{supp}\ u \subseteq [-M,\,M]\times[0,\,1] \tag{3.37}$$

and

$$u(x,\,0)=u(x,\,1)=0, \qquad \forall\, x\in\mathbb{R} \tag{3.38}$$

then

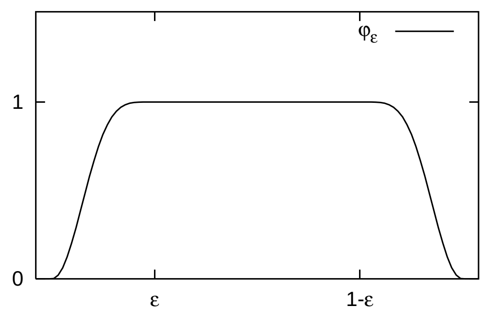
$$||e^{\lambda x}\,u||_{L^8(\mathbb{R}\times[0,\,1])} \le c\,||e^{\lambda x}\{\partial_t+\eta\,\partial^3\}\,u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \tag{3.39}$$

for all $\lambda\in\mathbb{R}$, *with* c *independent of* λ.

Proof. Let $\varphi_\epsilon \in C_0^\infty(\mathbb{R})$ with

$$\begin{aligned}&\text{supp}\ \varphi_\epsilon \subseteq [0,\,1]\\ &\varphi_\epsilon(t)=1 \quad \text{for} \quad t\in(\epsilon,\,1-\epsilon),\\ &0\le\varphi_\epsilon(t)\le 1 \quad \text{and} \quad |\varphi_\epsilon'(t)|\le c/\epsilon, \quad \text{c positive constant.}\end{aligned}$$

Example.



Let $u_\epsilon(x,\,t) = \varphi_\epsilon(t)\,u(x,\,t)$, hence

$$\text{supp}\ u_\epsilon(x,\,t) \quad\subseteq\quad \text{supp}\ \varphi_\epsilon(t)\cap\text{supp}\ u(x,\,t)\subseteq\text{supp}\ u(x,\,t)\subseteq[-M,\,M]\times[0,\,1].$$

Then, on one hand we have that

$$||e^{\lambda\,x}\,u_\epsilon||_{L^8(\mathbb{R}^2)}\longrightarrow||e^{\lambda\,x}\,u||_{L^8(\mathbb{R}\times[0,\,1])}\quad\text{as}\quad\epsilon\downarrow 0 \tag{3.40}$$

and on the other hand,

$$\{\partial_t+\eta\,\partial^3\}u_\epsilon(x,\,t)=\{\partial_t+\eta\,\partial^3\}[\varphi_\epsilon(t)\,u(x,\,t)]=\varphi_\epsilon(t)\,\{\partial_t+\eta\,\partial^3\}\,u+\varphi_\epsilon'(t)\,u \tag{3.41}$$

Hence,

$$||\varphi_\epsilon(t)\,\{\partial_t+\eta\,\partial^3\}\,u||_{L^{8/7}(\mathbb{R}^2)}\longrightarrow||\{\partial_t+\eta\,\partial^3\}\,u||_{L^{8/7}(\mathbb{R}^2)}$$

and using (3.37)

$$\begin{aligned}
&||\varphi_\epsilon'(t)\,u||_{L^{8/7}(\mathbb{R}^2)}\\
=\;&\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|\varphi_\epsilon'(t)\,u(x,\,t)|^{8/7}dx\,dt\right]^{7/8}\\
=\;&\left[\int_0^1\int_{-M}^{M}|\varphi_\epsilon'(t)|^{8/7}|u(x,\,t)|^{8/7}dx\,dt\right]^{7/8}\\
=\;&\left[\int_0^\epsilon\int_{-M}^{M}|\varphi_\epsilon'(t)|^{8/7}|u(x,\,t)|^{8/7}\,dx\,dt+\int_{1-\epsilon}^1\int_{-M}^{M}|\varphi_\epsilon'(t)|^{8/7}|u(x,\,t)|^{8/7}dx\,dt\right]^{7/8}\\
\le\;&\frac{c}{\epsilon}\left[\int_0^\epsilon\int_{-M}^{M}|u(x,\,t)|^{8/7}dx\,dt+\int_{1-\epsilon}^1\int_{-M}^{M}|u(x,\,t)|^{8/7}dx\,dt\right]^{7/8}.
\end{aligned} \tag{3.42}$$

Defining

$$G(t)=\int_{-M}^{M}|u(x,\,t)|^{8/7}dx$$

then, by (3.38) we have that $G(0)=G(1)=0$. G is continuous and differentiable with

$$G'(t)=\frac{8}{7}\int_{-M}^{M}|u(x,\,t)|^{1/7}\partial_t u(x,\,t)\,\text{sgn}(u(x,\,t))\,dx,$$

hence $G'(t)$ is continuous,

$$|G'(t)| \leq c\,|t| \quad , \quad |G'(t)| \leq c\,|1-t|$$

and

$$\int_0^{\epsilon} G(t)\,dt + \int_{1-\epsilon}^{1} G(t)\,dt \leq c\,\epsilon^2. \tag{3.43}$$

Inserting (3.43) in (3.42) we have

$$||\varphi_\epsilon'(t)\,u||_{L^{8/7}(\mathbb{R}^2)} \leq c\,\frac{1}{\epsilon}\,\epsilon^{7/4} \to 0 \quad \text{as} \quad \epsilon \downarrow 0$$

and (3.39) follows. □

Lemma 3.5. *Let* $u \in C^{3,1}(\mathbb{R} \times [0,\,1])$ *and* $\eta \in \mathbb{R}$. *Suppose that*

$$\sum_{j \leq 2} |\partial^j u(x,\,t)| \leq C_\beta\, e^{-\beta\,|x|}, \quad t \in [0,\,1],\ \forall\,\beta > 0, \tag{3.44}$$

and

$$u(x,\,0) = u(x,\,1) = 0, \quad \forall\,x \in \mathbb{R}.$$

Then

$$||e^{\lambda\,x}\,u||_{L^8(\mathbb{R}\times[0,\,1])} \quad \leq \quad c_0\,||e^{\lambda\,x}\{\partial_t + \eta\,\partial^3\}\,u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \tag{3.45}$$

for all $\lambda \in \mathbb{R}$, *with* c_0 *independent of* λ.

Proof. Let $\phi \in C_0^\infty(\mathbb{R})$ be an even, non increasing function for $x > 0$ with

$$\phi(x) = 1, \quad |x| \leq 1,$$
$$\text{supp}\ \phi \subseteq [-2,\,2].$$

For each M we consider the sequence $\{\phi_M\}$ in $C_0^\infty(\mathbb{R})$ defined by $\phi_M(x) = \phi(\frac{x}{M})$, then $\phi_M \equiv 1$ in a neighborhood of 0 and supp $\phi_M \subseteq [-M,\,M]$. Let $u_M(x,\,t) = \phi_M(x)\,u(x,\,t)$, then supp $u_M \subseteq [-M,\,M] \times [0,\,1]$. Hence,

$$\begin{aligned}
\{\partial_t + \eta\,\partial^3\}u_M(x,\,t) &= \{\partial_t + \eta\,\partial^3\}[\phi_M(x)\,u(x,\,t)] \\
&= \phi_M\{\partial_t + \eta\,\partial^3\}u(x,\,t) + 3\,\partial\phi_M\,\partial^2 u + 3\,\partial^2\phi_M\,\partial u + \partial^3\phi_M\,u \\
&= \phi_M\,\{\partial_t + \eta\,\partial^3\}\,u(x,\,t) + E_1 + E_2 + E_3,
\end{aligned} \tag{3.46}$$

and using Lemma 3.4 to $u_M(x,\, t)$ we get

$$\begin{aligned} & ||e^{\lambda\, x}\, u_M||_{L^8(\mathbb{R}\times[0,\,1])} \\ & \quad \le \; c\,||e^{\lambda\, x}\{\partial_t + \eta\,\partial^3\}u_M||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \\ & \quad \le \; c\,||e^{\lambda\, x}\,\phi_M\{\partial_t + \eta\,\partial^3\}u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} + c\sum_{j=1}^{3}||e^{\lambda\, x}E_j||_{L^{8/7}(\mathbb{R}\times[0,\,1])}. \end{aligned} \tag{3.47}$$

We show that the terms involving the $L^{8/7}$-norm of the error E_1, E_2 and E_3 in (3.47) tend to zero as $M \to \infty$.

We consider the case $x > 0$ and $\lambda > 0$. From (3.44) with $\beta > \lambda$ it follows that

$$\begin{aligned} ||e^{\lambda\, x}\, E_1||^{8/7}_{L^{8/7}(\mathbb{R}\times[0,\,1])} &= 3^{8/7}\int_0^1\int_M^{2M}|e^{\lambda\, x}\partial\phi_M\partial^2 u|^{8/7}dx\,dt \\ &\le c\int_0^1\int_M^{2M}\left|\frac{e^{\lambda\, x}}{M}\,\partial^2 u\right|^{8/7}dx\,dt \\ &\le c\int_0^1\int_M^{2M}e^{8\,\lambda\, x/7}e^{-\,8\,\beta\, x/7}dx\,dt \\ &= c\int_0^1\int_M^{2M}e^{-\frac{8}{7}(\beta-\lambda)\,x}\,dx\,dt \to 0 \quad \text{as} \quad M\to\infty. \end{aligned} \tag{3.48}$$

Thus taking the limits as $M \to \infty$ in (3.46) and using (3.48) we obtain (3.45). □

Lemma 3.6. *Suppose that*

$$u \in C([0,\,1];\; H^4(\mathbb{R})) \cap C^1([0,\,1];\; H^1(\mathbb{R}))$$

satisfies (1.1)–(1.2)*, rewriting the equation,*

$$\partial_t u + \eta\,\partial^3 u - i\,\alpha\,\partial^2 u - i\,|u|^2\,u + \delta\,\partial u = 0, \quad x,\, t \in \mathbb{R} \tag{3.49}$$

$$u(x,\,0) = u_0(x) \tag{3.50}$$

with

$$\operatorname{supp}\, u(x,\,0) \subseteq (-\infty,\, b].$$

Then, for any $\beta > 0$,

$$\sum_{j\le 2} |\partial^j u(x,\, t)| \le c_b\, e^{-\beta\, x}, \quad \text{for} \quad x > 0, \quad t \in [0,\, 1].$$

Proof. From Lemma 3.2. □

Theorem 3.7. *Let* $|\alpha| \le 3\,\eta$. *Suppose that* $u(x,\, t)$ *is a sufficiently smooth solution of the* (3.49)–(3.50). *If*

$$\operatorname{supp} u(\,.\,,\, t_j) \subseteq (-\infty,\, b), \qquad j = 1,\, 2.$$

or

$$\operatorname{supp} u(\,.\,,\, t_j) \subseteq (a,\, \infty), \qquad j = 1,\, 2.$$

then

$$u(x,\, t) \equiv 0.$$

Proof. Without loss of generality we assume that $t_1 = 0$ and $t_2 = 1$. Thus,

$$\operatorname{supp} u(\,.\,,\, 0) \subseteq (-\infty,\, b) \quad \text{and} \quad \operatorname{supp} u(\,.\,,\, 1) \subseteq (-\infty,\, b).$$

We will show that there exists a large number $R > 0$ such that

$$\operatorname{supp} u(\,.\,,\, t) \subseteq (-\infty,\, 2R], \quad \forall\, t \in [0,\, 1].$$

Then the result will follow from Theorem 2.3.

Let $\mu \in C_0^\infty(\mathbb{R})$ be a nondecreasing function such that

$$\mu(x) = \begin{cases} 0, & x \le 1 \\ 1, & x \ge 2, \end{cases}$$

and $0 \le \mu(x) \le 1,\ \forall\, x \in \mathbb{R}$. For each $R \ne 0$ we define $\mu_R(x) = \mu\left(\frac{x}{R}\right)$, i. e.,

$$\mu_R(x) = \begin{cases} 0, & x \le R \\ 1, & x \ge 2\,R. \end{cases}$$

Let $u_R(x,\,t) = \mu_R(x)\,u(x,\,t)$, then $u_R(x,\,t) \in C_0^\infty(\mathbb{R})$, since $(\mu_R\,u) \in C^\infty(\mathbb{R})$, and moreover

$$\text{supp}\; u_R = \text{supp}\;(\mu_R\,u) \subseteq \text{supp}\;\mu_R \cap \text{supp}\;u \subseteq \text{supp}\;\mu_R.$$

From the above inequality we have that $\text{supp}\;u_R \subseteq (-\infty,\,2\,R]$. Using that u is a sufficiently smooth function (see [6]) and Lemma 3.6, we can apply Lemma 3.5 to $u_R(x,\,t)$ for R sufficiently large. Thus

$$\begin{aligned}
&\{\partial_t + \eta\,\partial^3\}[\mu_R \cdot u] \\
&\quad= \mu_R\,\{\partial_t u + \eta\,\partial^3 u\} + 3\,\partial\mu_R \cdot \partial^2 u + 3\,\partial^2\mu_R \cdot \partial u + \partial^3\mu_R \cdot u \\
&\quad= \mu_R\,\{i\,\alpha\,\partial^2 u + i\,|u|^2\,u - \delta\,\partial u\} + 3\,\partial\mu_R \cdot \partial^2 u + 3\,\partial^2\mu_R \cdot \partial u + \partial^3\mu_R \cdot u \\
&\quad= \mu_R \cdot V_1 + 3\,\partial\mu_R \cdot \partial^2 u + 3\,\partial^2\mu_R \cdot \partial u + \partial^3\mu_R \cdot u \\
&\quad= \mu_R \cdot V_1 + F_1 + F_2 + F_3 \\
&\quad= \mu_R \cdot V_1 + F_R,
\end{aligned} \tag{3.51}$$

where $V_1(x,\,t) = i\,\alpha\,\partial^2 u + i\,|u|^2\,u - \delta\,\partial u$. Then, by using Lemma 3.5 and (3.51)

$$\begin{aligned}
&||e^{\lambda\,x}\,\mu_R \cdot u||_{L^8(\mathbb{R}\times[0,\,1])} \\
&\quad\le c_0\,||e^{\lambda\,x}\{\partial_t + \eta\,\partial^3\}\,\mu_R \cdot u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \\
&\quad\le c_0\,||e^{\lambda\,x}\,\mu_R \cdot V_1 + e^{\lambda x} F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \\
&\quad\le c_0\,||e^{\lambda\,x}\mu_R \cdot V_1||_{L^{8/7}(\mathbb{R}\times[0,\,1])} + c_0\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])}.
\end{aligned} \tag{3.52}$$

We estimate the $||e^{\lambda x}\, \mu_R \cdot V_1||_{L^{8/7}(\mathbb{R}\times[0,\,1])}$ term.

$$
\begin{aligned}
&||e^{\lambda x}\, \mu_R \cdot V_1||_{L^{8/7}(\mathbb{R}\times[0,\,1])}\\
&\quad= \left(\int_0^1 \int_{\mathbb{R}} |e^{\lambda x}\, \mu_R \cdot V_1|^{8/7}\, dx\, dt\right)^{8/7}\\
&\quad= \left(\int_0^1 \int_{x>R} |e^{\lambda x}\, \mu_R \cdot V_1|^{8/7}\, dx\, dt\right)^{8/7}\\
&\quad\leq \left(\int_0^1 \left[\int_{x\geq R} |e^{\lambda x}\, \mu_R|^{8}\, dx\right]^{1/7} \left[\int_{x\geq R} |V_1|^{8/6}\, dx\right]^{6/7} dt\right)^{7/8}\\
&\quad= \left(\int_0^1 \left[\int_{x\geq R} |e^{\lambda x}\, \mu_R|^{8} dx\right]^{1/7} \left[\int_{x\geq R} |V_1|^{4/3}\, dx\right]^{6/7} dt\right)^{7/8}\\
&\quad= \left(\int_0^1 \left[\int_{\mathbb{R}} |e^{\lambda x} \mu_R|^{8}\, dx\right]^{1/7} \left[\int_{x\geq R} |V_1|^{4/3}\, dx\right]^{6/7} dt\right)^{7/8}\\
&\quad\leq \left(\int_0^1 \int_{\mathbb{R}} |e^{\lambda x} \mu_R|^{8}\, dx\, dt\right)^{1/8} \left(\int_0^1 \int_{x\geq R} |V_1|^{4/3}\, dx\, dt\right)^{3/4}\\
&\quad= ||e^{\lambda x} \mu_R||_{L^{8}(\mathbb{R}\times[0,\,1])}\, ||V_1||_{L^{4/3}(\{x\geq R\}\times[0,\,1])}
\end{aligned}
$$

then

$$
c_0\, ||e^{\lambda x}\, \mu_R \cdot V_1||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \leq c_0\, ||e^{\lambda x}\, \mu_R||_{L^{8}(\mathbb{R}\times[0,\,1])}\, ||V_1||_{L^{4/3}(\{x\geq R\}\times[0,\,1])}. \tag{3.53}
$$

We define $V_1(x,\,t) = i\,\alpha\,\partial^2 u + i\,|u|^2\, u - \delta\,\partial u \in L^q(\mathbb{R}\times[0,\,1])$ with $q \in [0,\,\infty)$. Now, we fix R so large such that

$$
c\, ||V_1||_{L^{4/3}(\{x\geq R\}\times[0,\,1])} \leq \frac{1}{2}
$$

then

$$
c\, ||e^{\lambda x}\, \mu_R \cdot V_1||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \leq \frac{1}{2}\, ||e^{\lambda x}\, \mu_R||_{L^{8}(\mathbb{R}\times[0,\,1])};
$$

this way

$$
||e^{\lambda x}\, \mu_R||_{L^{8}(\mathbb{R}\times[0,\,1])} \leq \frac{1}{2}\, ||e^{\lambda x}\, \mu_R||_{L^{8}(\mathbb{R}\times[0,\,1])} + c\, ||e^{\lambda x}\, F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])},
$$

hence

$$\frac{1}{2}\,||e^{\lambda\,x}\,\mu_R||_{L^8(\mathbb{R}\times[0,\,1])} \leq c\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])}$$

thus

$$||e^{\lambda\,x}\mu_R||_{L^8(\mathbb{R}\times[0,\,1])} \leq 2\,c\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])}. \tag{3.54}$$

To estimate the F_R term it suffices to consider one of the terms in F_R, say F_2, since the proofs for F_1, and F_3, are similar. We have that

$$\begin{aligned} F_R &= F_1 + F_2 + F_3 \\ &= 3\,\partial\mu_R\cdot\partial^2 u + 3\,\partial^2\mu_R\cdot\partial u + \partial^3\mu_R\cdot u \end{aligned}$$

and $\operatorname{supp} F_i,\,\subseteq[R,\,2R]$, $i = 1,\,2,\,3$. We estimate F_2 :

$$\begin{aligned} 2\,c\,||e^{\lambda\,x}\,F_2||_{L^{8/7}(\mathbb{R}\times[0,\,1])} &= 2\,c\left(\int_0^1\int_{\mathbb{R}}|e^{\lambda\,x}\,F_2|^{8/7}\,dx\,dt\right)^{7/8} \\ &= 2\,c\left(\int_0^1\int_R^{2\,R}|3\,e^{\lambda\,x}\,\partial^2\mu_R\cdot\partial u|^{8/7}\,dx\,dt\right)^{7/8} \\ &= 2\,\frac{c}{R^2}\left(\int_0^1\int_R^{2R}e^{\frac{8}{7}\,\lambda\,x}\,|\partial^2\mu\cdot\partial u|^{8/7}\,dx\,dt\right)^{7/8} \\ &\leq 2\,\frac{c}{R^2}\left(\int_0^1\int_R^{2R}e^{\frac{8}{7}\,\lambda\,x}\,|\partial u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}. \end{aligned}$$

Then

$$2\,c\,||e^{\lambda\,x}\,F_2||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \leq 2\,\frac{c}{R^2}\,e^{2\,\lambda\,R}\left(\int_0^1\int_R^{2R}|\partial u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}. \tag{3.55}$$

On the other hand,

$$||e^{\lambda\,x}\mu_R\cdot u||_{L^8(\mathbb{R}\times[0,\,1])} \geq \left(\int_0^1\int_{x>2R}e^{8\,\lambda\,x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8},$$

then

$$\begin{aligned}\left(\int_0^1 \int_{x>2R} e^{8\,\lambda\,x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8} &\leq ||e^{\lambda\,x}\mu_R\cdot u||_{L^8(\mathbb{R}\times[0,\,1])}\\ &\leq 2\,c\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])}\\ &\leq 2\,\frac{c}{R^2}\,e^{2\,\lambda\,R}\left(\int_0^1\int_R^{2R}|\partial u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8},\end{aligned}$$

hence,

$$\left(\int_0^1 \int_{x>2R} e^{8\,\lambda\,x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8} \leq c_0\left(\int_0^1\int_R^{2R}|\partial u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}.$$

This way we have using (3.54) and (3.55)

$$\begin{aligned}\left(\int_0^1 \int_{x>2R} e^{8\,\lambda\,x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8} &\leq ||e^{\lambda\,x}\,\mu_R\cdot u||_{L^8(\mathbb{R}\times[0,\,1])}\\ &\leq 2\,c\,||e^{\lambda\,x}\,F_2||_{L^{8/7}(\mathbb{R}\times[0,\,1])}\\ &\leq 2\,\frac{c}{R^2}\,e^{2\,\lambda\,R}\left(\int_0^1\int_R^{2R}|\partial u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8};\end{aligned}$$

then

$$\left(\int_0^1 \int_{x>2R} e^{8\,\lambda\,(x-2R)}|u(x,\,t)|^8\,dx\,dt\right)^{1/8} \leq 2\,\frac{c}{R^2}\left(\int_0^1\int_R^{2R}|\partial u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8};$$

and letting $\lambda \to \infty$ it follows that

$$u(x,\,t) \equiv 0 \qquad \text{for} \qquad x > 2R, \qquad t \in [0,\,1]$$

which yields the proof. □

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