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Real Aspects of the Moduli Space of Genus Zero Stable Maps

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Abstract

We show that the moduli space of genus zero stable maps is a real projective variety if the target space is a smooth convex real projective variety. We show that evaluation maps, forgetful maps are real morphisms. We analyze the real part of the moduli space.

Key Words: Moduli space of genus zero stable maps, real variety, real structure.

1. Introduction

We call a projective variety V as a *real* projective variety if V has an anti-holomorphic involution τ on the set of complex points $V(\mathbb{C})$. By a *real structure* on V, we mean an anti-holomorphic involution τ. The *real part* of (V, τ) is the locus which is fixed by τ.

In the following paragraph, readers can find the definitions of the moduli space of stable maps and various maps defined on it in [3].

Let's assume that X is a convex real projective variety. We show the following:

- The moduli space $\overline{M}_k(X, \beta)$ of k-pointed genus 0 stable maps is a real projective variety.

- Let $\overline{M}_k$ be the Deligne-Mumford moduli space of k-pointed genus 0 curves. The forgetful maps $F_n : \overline{M}_n(X, \beta) \rightarrow \overline{M}_{n-1}(X, \beta)$, $F : \overline{M}_k(X, \beta) \rightarrow \overline{M}_k$ are real

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morphisms(i.e., morphisms which commute with the anti-holomorphic involution on the domain and that on the target), where $n \geq 1$, $k \geq 3$.

- Let $ev_i : \overline{M}_k(X, \beta) \to X$ be the i-th evaluation map. Then, ev_i is a real map.
- Let $\mathbb{CP}^n$ have the real structure from the complex conjugation involution. Let X be a real projective variety such that the imbedding i which decides the real structure on X has a non-empty intersection with $\mathbb{RP}^n \subset \mathbb{CP}^n$. Let $M_k(X, \beta)$, $k \geq 3$, be the moduli space of k-pointed genus 0 stable maps with a smooth domain curve. Then, each point in the real part $M_k(X, \beta)^{re}$ of the moduli space represents a real stable map having marked points on the real part of the domain curve.

This paper is organized as follows. In Sec. 2, we show that the moduli space is a real projective variety. We prove that straightforwardly, based on the explicit tangent space splitting calculation. In Sec. 3, we show that the forgetful maps, and the evaluation map are real maps. Also, we do real part analysis when $k \geq 3$.

Theorem 2.2 is the main theorem in this paper. Real part analysis, done in sec. 3, shows that the studies of the intersection theoretic properties on the real part of $\overline{M}_k(X, \beta)$ are important for real enumerative applications. The main results of this paper are similar to those in [10]. The main Theorem in [10] and in this paper is based on the proof which shows that the defined involution on the *complex* moduli space of stable maps is an anti-holomorphic involution, when the target space X is a real convex projective variety. However, the practical methods of proofs are different.

The real version of the Gromov-Witten invariants are defined in [11], when the target space is a rational projective surface. Different from the Gromov-Witten invariant defined on $\overline{M}_k(X, \beta)$, the real version of the Gromov-Witten invariants are local invariants.

Relevant theory which considers the global minimum bound of real enumerative problems has been developed by J-Y Welschinger in [21], [22]. The Gromov-Witten invariant in the real world with Quantum Schubert calculus has been widely studied by F. Sottile. See [16], [17], [18], [19], [20].

Convention:

- The real structure on $\mathbb{CP}^1$, defined by the standard complex conjugation involution, and the real structure of the target space X will be always denoted by s, t respectively.

- Let C be an arithmetic genus 0 curve. Let $\pi : \widetilde{C} := \mathbb{CP}^1_1 \cup \ldots \cup \mathbb{CP}^1_l \to C$ be a normalization map. We will denote the irreducible component in $\widetilde{C}$ by $\mathbb{CP}^1_{q(p)}$, either if it contains $\pi^{-1}(p)$ where p is a non-singular point in C, or if it contains p where p is any point in $\widetilde{C}$.

2. Real Aspects of the moduli spaces

The following Lemma is well-known. See [6, 2.3], [4, sec.10], [13, 4.1]. However, the author couldn't find the proof. Thus, we include the proof.

The tangent space calculations done in Lemma 2.1, Theorem 2.1, are from repeated K-group calculations of vector spaces based on the simple homological algebraic fact (cf. Proposition 2.11. in [2]). The K-group we consider is the Grothendieck group of K_0(point) because the tangent space is calculated pointwise. The way to express the tangent space of the Deligne-Mumford moduli space $\overline{M}_k$ in the proof of Lemma 2.1 is somewhat different from the conventional one. However, they are equivalent in the K-theoretic point of view. The alternative expression is taken because it allows us to easily relate the underlying real structure of the pointed curve with the anti-holomorphic structure on the Deligne-Mumford moduli space. We include the details of the proof for this alternative expression. Note that each element in $\overline{M}_k \setminus M_k$ represents a singular curve having only nodal singularities. Different from the Deligne-Mumford moduli space of pointed higher genus curves, singular curves in the Deligne-Mumford moduli space of pointed genus zero curves are trees. Therefore, the number of singular points (i.e., gluing points) is exactly one less than the number of irreducible components.

Lemma 2.1 *Let* $\mathbf{c} := [(C, a_1, \ldots, a_k)]$ *be a point in* $\overline{M}_k$. *Let* $\pi : \widetilde{C} := \mathbb{CP}^1_1 \cup \ldots \cup \mathbb{CP}^1_l \to C$ *be a normalization map, where* $\mathbb{CP}^1_i$ *is biholomorphic to* $\mathbb{CP}^1$. *Let* $g_1, \ldots, g_r$, $r = l - 1$, *be singular points in* C. *Let's denote two points in* $\widetilde{C}$ *corresponding to* $\pi^{-1}(g_i)$ *by* g_i^1, g_i^2.

Let $\overline{\mathbf{c}}$ *be the point in* $\overline{M}_k$ *represented by the pointed curve* $(\overline{C}, s(a_1), \ldots, s(a_k))$ *which satisfies the following:*

Let $\overline{\pi} : \widetilde{\overline{C}} := \mathbb{CP}^1_1 \cup \ldots \cup \mathbb{CP}^1_l \to \overline{C}$ *be a normalization map.*

- *If* $\widetilde{g}_1, \ldots, \widetilde{g}_r$ *are singular points on* $\overline{C}$, *then,* $\widetilde{g}_i^1 \in \mathbb{CP}^1_{q(g_i^1)}$, $\widetilde{g}_i^2 \in \mathbb{CP}^1_{q(g_i^2)}$ *in* $\overline{\pi}^{-1}(\widetilde{g}_i)$ *are* $s(g_i^1), s(g_i^2)$.

- *$s(a_i)$ is the point in $\mathbb{CP}^1_{q(a_i)}$ conjugate to a_i by the real structure s on $\mathbb{CP}^1_{q(a_i)}$.*

The involution, $I : \mathbf{c} \mapsto \overline{\mathbf{c}}$, defines a real structure on the genus zero Deligne-Mumford moduli space $\overline{M}_k$. $\overline{M}_k$ is a real projective variety.

Proof. It is well-known that the tangent space $T_{\mathbf{c}}\overline{M}_k$ at $\mathbf{c}$ is $Ext^1(\Omega^1_C(a_1+\ldots+a_k), \mathcal{O}_C)$. Thus,

$$T_{\mathbf{c}}\overline{M}_k = \bigoplus_{i=1}^{l} H^1(\mathbb{CP}^1_i, T\mathbb{CP}^1_i(-\sum_j q(a_{j,i}) - \sum_{\alpha,\beta} g_\alpha^{\beta,i})) \oplus \bigoplus_{i=1}^{r} T_{g_i^1}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{g_i^2}\mathbb{CP}^1_{q(g_i^2)}, \tag{1}$$

where $a_{j,i} \in \{a_1, \ldots, a_k\}$ such that $q(a_{j,i}) \in \mathbb{CP}^1_i$, and $g_\alpha^{\beta,i} \in \pi^{-1}(g_\alpha) \bigcap \mathbb{CP}^1_i$, where $g_\alpha \in \{g_1, \ldots, g_r\}$

$$\begin{aligned} &= \bigoplus_{i=1}^{k} T_{a_i}\mathbb{CP}^1_{q(a_i)} \oplus \bigoplus_{i=1,\ldots,r}^{j=1,2} T_{g_i^j}\mathbb{CP}^1_{q(g_i^j)} \ominus (\bigoplus_{i=1}^{l} H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i)) \\ &\qquad \oplus \bigoplus_{i=1}^{r} T_{g_i^1}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{g_i^2}\mathbb{CP}^1_{q(g_i^2)}. \end{aligned} \tag{2}$$

(1) comes from the following local to global spectral sequence (cf. [8], p. 99):

$$\begin{aligned} 0 \;\to\; & H^1(C, \underline{Ext}^0_C(\Omega^1_C(a_1+\ldots+a_k), \mathcal{O}_C)) \;\to \\ \to\; & Ext^1(\Omega^1_C(a_1+\ldots+a_k), \mathcal{O}_C) \;\to\; H^0(C, \underline{Ext}^1_C(\Omega^1_C(a_1+\ldots+a_k), \mathcal{O}_C)) \;\to\; 0. \end{aligned} \tag{3}$$

See [7] for some further details. Terms $\bigoplus_{i=1}^k T_{a_i}\mathbb{CP}^1_{q(a_i)}$, $\bigoplus_{i=1,\ldots,r}^{j=1,2} T_{g_i^j}\mathbb{CP}^1_{q(g_i^j)}$, $\bigoplus_{i=1}^l H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i)$ in (2) come from the long exact sequence of sheaf cohomology induced from the following short exact sequence of sheaves:

$$0 \to T\mathbb{CP}^1_i(-\sum_j q(a_{j,i}) - \sum_{\alpha,\beta} g_\alpha^{\beta,i}) \to T\mathbb{CP}^1_i \to \bigoplus_j T_{q(a_{j,i})}\mathbb{CP}^1_i \oplus \bigoplus_{\alpha,\beta} T_{g_\alpha^{\beta,i}}\mathbb{CP}^1_i \to 0.$$

Signs on the summations are from the K-group calculation by using Proposition 2.11. in [2]. Note that the rank of $H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i(-\sum_j q(a_{j,i}) - \sum_{\alpha,\beta} g_\alpha^{\beta,i}))$ is zero due to the stability condition.

The tangent space $T_{\overline{\mathbf{c}}}\overline{M}_k$ at $\overline{\mathbf{c}}$ is:

$$\bigoplus_{i=1}^{k} T_{s(a_i)}\mathbb{CP}^1_{q(a_i)} \oplus \bigoplus_{i=1,\dots,r}^{j=1,2} T_{s(g_i^j)}\mathbb{CP}^1_{q(g_i^j)} \tag{4}$$

$$\ominus(\bigoplus_{i=1}^{l} H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i)) \oplus \bigoplus_{i=1}^{r} T_{s(g_i^1)}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{s(g_i^2)}\mathbb{CP}^1_{q(g_i^2)}.$$

The actual expression of the tangent space splitting depends on the pointed curve representing the point $\mathbf{c}$. However, one can observe the following. Let $(C', \sigma(a_1), \dots, \sigma(a_k))$ represent the point $\mathbf{c}$ in $\overline{M}_k$, where σ is an element in $\mathrm{Aut}(\mathbb{CP}^1)$. Then, $(\overline{C'}, s \circ \sigma(a_1), \dots, s \circ \sigma(a_k))$ represents $\overline{\mathbf{c}}$.

Let v be an element in $H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i)$. Let's denote $v \mid_x$ be the value of v at x. Then, $\bar{v}$ defined by $\bar{v} \mid_{s(x)} := ds(v \mid_x)$ is an element in $H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i)$. The differential ds of the real structure s induces the anti-holomorphic involution, $v \mapsto \bar{v}$, on $H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i)$. The anti-holomorphic involutions on other components in (2) to (4) are obviously induced by the differential ds on each component. Thus, the differential $dI \mid_{\mathbf{c}}$: (2) $\mapsto$ (4) at $\mathbf{c}$ is an anti-holomorphic involution. I is an anti-holomorphic involution. □

Remark 2.1 We can also prove the Lemma2.1 as follows. The Deligne-Mumford moduli space is originally defined over $\mathbb{Z}$. (See [14, III.3].) So, it is defined over any field. The $\mathbb{C}$-scheme Deligne-Mumford moduli space can be obtained by a scalar extension from the $\mathbb{R}$-scheme Deligne-Mumford moduli space. That is, the $\mathbb{C}$-scheme Deligne-Mumford moduli space is a complexification $\overline{M}_k^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ of the $\mathbb{R}$-scheme Deligne-Mumford moduli space $\overline{M}_k^{\mathbb{R}}$. Thus, it has a canonical anti-holomorphic involution. (See [15, p4, (1.4) Proposition].) It is easily seen that the anti-holomorphic involution in Lemma 2.1 is identical to the corresponding canonical involution in this Remark.

We calculate the tangent space on the moduli space $\overline{M}_k(X, \beta)$ of k-pointed genus zero stable maps. Theorem 2.1 is proven in symplectic category by taking the different methods of calculations in [12] when f is an immersion on each irreducible component. Intuitive interpretations of the calculational results are seen in [12].

Theorem 2.1 *Let X be a convex projective variety.*
Let $\mathbf{f} := [(f, C, a_1, \ldots, a_k)]$ be a point in $\overline{M}_k(X, \beta)$ such that β is non-trivial. Let $\pi : \widetilde{C} := \mathbb{CP}^1_1 \cup \ldots \cup \mathbb{CP}^1_l \to C$ be a normalization map, where $\mathbb{CP}^1_m$ is biholomorphic to $\mathbb{CP}^1$. Let $g_1, \ldots, g_r$ be singular points on C, $r := l-1$. Let's denote elements in $\pi^{-1}(g_n)$ by g_n^1, g_n^2. Let N_m be the normal sheaf induced from a morphism $df_m : T\mathbb{CP}^1_m \to TX$.
(i) Suppose $f_m := f \mid_{\mathbb{CP}^1_m}$ is non-trivial, $m = 1, \ldots, l$. Then, the tangent space $T_{\mathbf{f}}\overline{M}_k(X, \beta)$ at $\mathbf{f}$ is

$$\bigoplus_{m=1}^{l} H^0(\mathbb{CP}^1_m, N_m) \oplus \bigoplus_{i=1,\ldots,k} T_{a_i}\mathbb{CP}^1_{q(a_i)} \oplus \Big(\bigoplus_{n=1,\ldots,r} T_{g_n^1}\mathbb{CP}^1_{q(g_n^1)} \otimes T_{g_n^2}\mathbb{CP}^1_{q(g_n^2)}\Big) \oplus$$

$$\oplus \bigoplus_{n=1,\ldots,r}^{j=1,2} T_{g_n^j}\mathbb{CP}^1_{q(g_n^j)} \ominus \Big(\bigoplus_{n=1}^{r} T_{f(g_n)}X\Big).$$

(ii) We may assume the following by reordering if necessary:
a. $f_i := f \mid_{\mathbb{CP}^1_i}$ is trivial if $i = 1, \ldots, m$, and is non-trivial if $i = m+1, \ldots, l$.
b. If $i = 1, \ldots, h$, then, g_i joins the irreducible components on which the restriction of f is non-trivial. If $i = h+1, \ldots, r$, then g_i joins the irreducible components such that f is trivial on one of the components or both components. Then, the tangent space $T_{\mathbf{f}}\overline{M}_k(X, \beta)$ at $\mathbf{f}$ is

$$\bigoplus_{i=m+1}^{l} H^0(\mathbb{CP}^1_i, N_i) \oplus \bigoplus_{i=1,\ldots,k} T_{a_i}\mathbb{CP}^1_{q(a_i)} \oplus \Big(\bigoplus_{i=1,\ldots,r} T_{g_i^1}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{g_i^2}\mathbb{CP}^1_{q(g_i^2)}\Big) \oplus$$

$$\oplus \bigoplus_{i=1,\ldots,r}^{j=1,2} T_{g_i^j}\mathbb{CP}^1_{q(g_i^j)} \ominus \Big(\bigoplus_{i=1}^{h} T_{f(g_i)}X\Big) \ominus \bigoplus_{i=1}^{m} H^0(\mathbb{CP}^1_i, T\mathbb{CP}^1_i).$$

Proof. We will use the following index notations throughout the proof of (i):

$i = 1, \ldots, k$	the index for marked points;
m or $m' = 1, \ldots, l$	the index for irreducible components;
$n = 1, \ldots, l-1$	the index for gluing points;
$j = 1, 2$	the upper index (with the lower index n);
	for pregluing points in $\pi^{-1}(g_i)$.

To make the notations simpler, we will use the following notations:

$$
\begin{aligned}
a_i(m) \quad &= a_i \quad \text{if } a_i \in \mathbb{CP}^1_m \\
&= \emptyset \quad \text{if } a_i \notin \mathbb{CP}^1_m; \\
g^j_n(m) \quad &= g^j_n \quad \text{if} \quad g^j_n \in \mathbb{CP}^1_m \\
&= \emptyset \quad \text{if} \quad g^j_n \notin \mathbb{CP}^1_m; \\
T_\emptyset \mathbb{CP}^1_m \quad &:= \emptyset.
\end{aligned}
$$

For example, if $a_1(1) = \emptyset$, $a_2(1) = a_2$, then $a_1(1) + a_2(1) = a_2$ and $T_{a_1(1)}\mathbb{CP}^1_1 \bigoplus T_{a_2(1)}\mathbb{CP}^1_1 = T_{a_2}\mathbb{CP}^1_1$. Obviously, $\sum_{i,m} a_i(m) = \sum_i a_i$.

As a convention, if we don't specify the range of the indices, e.g., a_i, $i = 1, 2$, then we always consider all possible indexes. That is, a_i means a_i, $i = 1, \ldots, k$.

The tangent space at $\mathbf{f}$ is the hyperext group $Ext^1(f^*\Omega^1_X \to \Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C)$.

From the long exact sequence associated with the hyperext group $Ext^1(f^*\Omega^1_X \to \Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C)$ (cf. [3, p285]):

$$
\begin{aligned}
0 \;\to\; Hom(\Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C) \;\to\; H^0(C, f^*TX) \to \\
\to Ext^1(f^*\Omega^1_X \to \Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C) \to \\
\to Ext^1(\Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C) \to 0,
\end{aligned}
$$

we get the following tangent space splitting at $\mathbf{f}$:

$$
\ominus Hom(\Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C) \oplus H^0(C, f^*TX) \oplus Ext^1(\Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C). \qquad (5)
$$

We will calculate each term's splitting first.

The standard fact we will use in the following calculations is $Hom(\Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C)$, $\underline{Ext}^0(\Omega^1_C(a_1 + \ldots a_k), \mathcal{O}_C)$ is the sheaf of derivations one gets from the pushforward of the sheaf of vector fields on $\widetilde{C} := \mathbb{CP}^1_1 \cup \ldots \cup \mathbb{CP}^1_l$ vanishing at the inverse images g^j_n of the node in C and the marked points a_i. (cf. [8, p100])

For $\ominus Hom(\Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C)$ term, we use the short exact sequences of sheaves:

$$0 \to T\mathbb{CP}^1_m(-\sum_i a_i(m) - \sum_{j,n} g^j_n(m)) \to T\mathbb{CP}^1_m \to$$
$$\to \bigoplus_i T_{a_i(m)}\mathbb{CP}^1_m \oplus \bigoplus_{j,n} T_{g^j_n(m)}\mathbb{CP}^1_m \to 0 \quad (6)$$

to get the following K-group equation:

$$\begin{aligned} &Hom(\Omega^1_C(\sum_i a_i), \mathcal{O}_C) = H^0(C, TC(-\sum_i a_i)) \\ &= H^0(C, \pi_*(T\widetilde{C}(-\sum_i a_i - \sum_{j,n} g^j_n))) \\ &= H^0(\widetilde{C}, T\widetilde{C}(-\sum_i a_i - \sum_{j,n} g^j_n)) \\ &= \bigoplus_m H^0(\mathbb{CP}^1_m, T\mathbb{CP}^1_m(-\sum_i a_i(m) - \sum_{j,n} g^j_n(m))) \\ &= \bigoplus_m [H^0(\mathbb{CP}^1_m, T\mathbb{CP}^1_m) \ominus (\bigoplus_i T_{a_i(m)}\mathbb{CP}^1_m) \\ &\quad \ominus (\bigoplus_{j,n} T_{g^j_n(m)}\mathbb{CP}^1_m) \oplus H^1(\mathbb{CP}^1_m, T\mathbb{CP}^1_m(-\sum_i a_i(m) - \sum_{j,n} g^j_n(m)))] \end{aligned}$$

by (6).

For $H^0(C, f^*TX)$, we use the short exact sequence of sheaves

$$0 \to f^*TX \to \bigoplus_m f^*_m TX \to \bigoplus_n T_{f(g_n)}X \to 0,$$

to get the K-group equation

$$H^0(C, f^*TX) = \bigoplus_m H^0(\mathbb{CP}^1_m, f^*_m TX) \ominus \bigoplus_n T_{f(g_n)}X,$$

because $H^1(C, f^*TX)$ vanishes by Lemma 10 in [5].

For $Ext^1(\Omega^1_C(a_1 + \ldots + a_k), \mathcal{O}_C)$, we use the exact sequence from the local to global spectral sequence in (3) to get

$$
\begin{aligned}
&Ext^1(\Omega^1_C(a_1+\ldots+a_k),\mathcal{O}_C)\\
&= H^1(C,\underline{Ext}^0(\Omega_C(\sum_i a_i),\mathcal{O}_C)) \oplus H^0(C,\underline{Ext}^1(\Omega_C(\sum_i a_i),\mathcal{O}_C)\\
&= H^1(C,\pi_*(T\widetilde{C}(-\sum_i a_i - \sum_{i,n} g^j_n))) \oplus H^0(C,\underline{Ext}^1(\Omega_C(\sum_i a_i),\mathcal{O}_C))\\
&= \bigoplus_m [H^1(\mathbb{CP}^1_m, T\mathbb{CP}^1_m(-\sum_i a_i(m) - \sum_{n,j} g^j_n(m)))]\\
&\quad \oplus \bigoplus_{m,m',n} [(T_{g^1_n(m)}\mathbb{CP}^1_m)\otimes(T_{g^2_n(m')}\mathbb{CP}^1_{m'})].
\end{aligned}
$$

The result follows by putting all terms to (5), and simplify further by K-group calculations, with the long exact sequence of sheaf cohomology induced from the following short exact sequence of sheaves:

$$0 \to T\mathbb{CP}^1_m \to f^*_m TX \to N_m \to 0.$$

The proof of (ii) is very similar to the proof of (i). We employ the fact that if f_α is trivial, then $H^0(\mathbb{CP}^1_\alpha, f^*_\alpha TX)$ is isomorphic to $T_{f_\alpha(\mathbb{CP}^1_\alpha)}X$. □

For the proof of Theorem 2.2, it is enough to show that the involution defined by $[(f,C,a_1,\ldots,a_k)] \mapsto [(\overline{f},\overline{C},s(a_1),\ldots,s(a_k))]$ on the moduli space $\overline{M}_k(X,\beta)$ is an anti-holomorphic involution (cf. [15, p4, (1.4) Proposition]). $\overline{M}_k(X,\beta)$ is a normal projective variety. It has orbifold singularities. So, we show that the defined involution is an anti-holomorphic involution with local chart before the local quotient by a finite group action and then show there is a canonical conjugate group action on the conjugate local chart around the conjugate point.

Theorem 2.2 *Let X be a convex real projective variety. Then, the moduli space $\overline{M}_k(X,\beta)$ of stable maps is a real projective variety whose real structure comes from the involution defined by $[(f,C,a_1,\ldots,a_k)] \mapsto [(t\circ f\circ s,\overline{C},s(a_1),\ldots,s(a_k))]$, where the notations C and $\overline{C}$ are the same as in Lemma 2.1.*

Proof. Let $\overline{\mathbf{f}}$ be a point in $\overline{M}_k(X,\beta)$ represented by $(\overline{f},\overline{C},s(a_1),\ldots,s(a_k))$, where $\overline{f}(z) := t\circ f\circ s(z)$. Let $H : \overline{M}_k(X,\beta) \to \overline{M}_k(X,\beta)$ be the involution defined by $\mathbf{f} \mapsto \overline{\mathbf{f}}$. The Theorem follows if we show that H is an anti-holomorphic involution.

Let β be non-trivial. Let's suppose that f is non-trivial on every component. From Theorem2.1 (i), we know that the tangent space at $\mathbf{f} := [(f, C, a_1, \dots, a_k)]$ is

$$\bigoplus_{i=1}^{l} H^0(\mathbb{CP}^1_i, N_i) \oplus \bigoplus_{i=1,\dots,k} T_{a_i}\mathbb{CP}^1_{q(a_i)} \oplus (\bigoplus_{i=1,\dots,r} T_{g_i^1}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{g_i^2}\mathbb{CP}^1_{q(g_i^2)}) \oplus$$

$$\oplus \bigoplus_{i=1,\dots,r}^{j=1,2} T_{g_i^j}\mathbb{CP}^1_{q(g_i^j)} \ominus (\bigoplus_{i=1}^{r} T_{f(g_i)}X).$$

Let $\overline{N}_i$ be the normal sheaf induced from the morphism $d\overline{f}_i : T\mathbb{CP}^1_i \to TX$, where $\overline{f}_i(z) := t \circ f_i \circ s(z)$. The tangent space at $\overline{\mathbf{f}}$ is:

$$\bigoplus_{i=1}^{l} H^0(\mathbb{CP}^1_i, \overline{N}_i) \oplus \bigoplus_{i=1,\dots,k} T_{s(a_i)}\mathbb{CP}^1_{q(a_i)} \oplus (\bigoplus_{i=1,\dots,r} T_{s(g_i^1)}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{s(g_i^2)}\mathbb{CP}^1_{q(g_i^2)}) \oplus$$

$$\oplus \bigoplus_{i=1,\dots,r}^{j=1,2} T_{s(g_i^j)}\mathbb{CP}^1_{q(g_i^j)} \ominus (\bigoplus_{i=1}^{r} T_{t\circ f(g_i)}X).$$

Each term in the tangent space splitting at $\mathbf{f}$, $\overline{\mathbf{f}}$ is a complex vector space, such that

$$\bigoplus_{i=1,\dots,k} T_{a_i}\mathbb{CP}^1_{q(a_i)} \overset{dH}{\mapsto} \bigoplus_{i=1,\dots,l} T_{s(a_i)}\mathbb{CP}^1_{q(a_i)} \tag{7}$$

$$\bigoplus_{i=1,\dots,r} T_{g_i^1}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{g_i^2}\mathbb{CP}^1_{q(g_i^2)} \overset{dH}{\mapsto} \bigoplus_{i=1,\dots,r} T_{s(g_i^1)}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{s(g_i^2)}\mathbb{CP}^1_{q(g_i^2)} \tag{8}$$

$$\bigoplus_{i=1,\dots,r}^{j=1,2} T_{g_i^j}\mathbb{CP}^1_{q(g_i^j)} \overset{dH}{\mapsto} \bigoplus_{i=1,\dots,r}^{j=1,2} T_{s(g_i^j)}\mathbb{CP}^1_{q(g_i^j)}. \tag{9}$$

It is obvious that dH in (7), (8), (9) is the anti-holomorphic involution induced by the real structure of a complex conjugation map on $\mathbb{CP}^1$:

$$\bigoplus_{i=1}^{r} T_{f(g_i)}X \overset{dH}{\mapsto} \bigoplus_{i=1}^{r} T_{t\circ f(g_i)}X \tag{10}$$

Clearly, dH in (10) is the anti-holomorphic involution induced by the real structure t on the target space X.

$$\bigoplus_{i=1}^{l} H^0(\mathbb{CP}^1_i, N_i) \overset{dH}{\mapsto} \bigoplus_{i=1}^{l} H^0(\mathbb{CP}^1_i, \overline{N}_i). \tag{11}$$

Each normal sheaf N_i is the direct sum of the locally free sheaf NB_i and skyscraper sheaves supported by critical points of f. Similar to the case (10), the restriction of dH to each skyscraper sheaf is the anti-holomorphic involution induced by the real structure t on the target space X. Normal bundles NB_i, $\overline{NB}_i$ split into line bundles on $\mathbb{CP}^1$ by splitting principle. By considering Weil divisors characterizing each line bundle and the definition of $\overline{f}_i$, one can check that $\overline{NB}_i$ is a conjugate bundle for the bundle NB_i. Thus, the restriction of dH to $H^0(\mathbb{CP}^1_i, NB_i)$ is an anti-holomorphic involution.

The general case considered in Theorem2.1 (ii) can be easily proven by repeating the same arguments we did above, that is, by checking the componentwise anti-holomorphicity of dH.

Let's assume that $(f', C', b_1, \ldots, b_k)$ represents $\mathbf{f}$. Then, there exists the element $\sigma \in \mathrm{Aut}(\mathbb{CP}^1)$ such that $f = f' \circ \sigma$ and $b_i = \sigma(a_i)$. $\overline{\mathbf{f}}$ is represented by $(\overline{f}, \overline{C}, s(a_1), \ldots, s(a_k))$. Note that $\overline{f}' \circ \overline{\sigma} = t \circ f' \circ s \circ s \circ \sigma \circ s = t \circ f' \circ \sigma \circ s = t \circ f \circ s = \overline{f}$, $s(b_i) = s \circ \sigma \circ s(s(a_i))$. One can check that $s \circ \sigma \circ s$ is also an element in $\mathrm{Aut}(\mathbb{CP}^1)$. Thus, it shows that $(t \circ f' \circ s, \overline{C}', s(b_1), \ldots, s(b_k))$ represents $\overline{\mathbf{f}}$. This implies that the map H is an anti-holomorphic involution on the local charts, independent of the actual choice of the chosen pointed stable map representing $\mathbf{f}$.

Let G be a finite group acting on the local chart $\mathcal{O}_{\mathbf{h}}$ around $\mathbf{h}$ such that $G \times \mathcal{O}_{\mathbf{h}} \to \mathcal{O}_{\mathbf{h}}, (g, \mathbf{f}) \mapsto g \cdot \mathbf{f}$. Let $\overline{\mathcal{O}}_{\overline{\mathbf{h}}}$ be the conjugate local chart around $\overline{\mathbf{h}}$, i.e., $\overline{\mathcal{O}}_{\overline{\mathbf{h}}} := \{\overline{\mathbf{f}} \mid \mathbf{f} \in \mathcal{O}_{\mathbf{h}}\}$. Then, there is a canonical conjugate group action of G defined by $g \cdot \overline{\mathbf{f}} := \overline{g \cdot \mathbf{f}}$

Thus, H is an anti-holomorphic involution on the orbifold $\overline{M}_k(X, \beta)$.

Let's assume that β is trivial. Then, the moduli space $\overline{M}_k(X, \beta)$ is isomorphic to the complex manifold $\overline{M}_k \times X$. It is obvious the defined involution is an anti-holomorphic involution. □

Remark 2.2 Araujo-Kollar constructed the moduli space of stable maps on any Noetherian scheme in [1, sec. 10]. It is interesting to see whether the variety developed by the complexification of the moduli space defined over $\mathbb{R}$ is isomorphic to the moduli space of stable maps defined over $\mathbb{C}$ or not. Note that the complexification as a variety doesn't need to have any meaning as a moduli space. Nevertheless, an element in the real part of the variety gotten by the complexification uniquely corresponds to a real point in the moduli space constructed over $\mathbb{R}$.

Let's consider the degree 2 maps from $\mathbb{CP}^1$ to $\mathbb{CP}^1(:= \mathbb{C} \cup \{\infty\})$, defined by $z \mapsto z^2$ and $z \mapsto -z^2$. Then, they are represented by two distinct real points in the moduli space constructed over $\mathbb{R}$. But they are represented by one point in the real part of the complex moduli space $\overline{M}_0(\mathbb{CP}^1, 2 \cdot [\text{line}])$, because they are equivalent maps. Thus, the complexification of the moduli space of degree 2 maps over $\mathbb{R}$ cannot be isomorphic to the complex moduli space $\overline{M}_0(\mathbb{CP}^1, 2 \cdot [\text{line}])$.

Due to the differences of the categories and equivalence relations, the complexification of the moduli space defined over $\mathbb{R}$ as a variety is not always isomorphic to the moduli space defined over $\mathbb{C}$. Recall Remark 2.1 for the Deligne-Mumford moduli space.

3. Real Properties of the Moduli space

Proposition 3.1 *The i-th evaluation map ev_i is a real map.*

Proof. It is enough to show that ev_i commutes with the real structure H on $\overline{M}_k(X, \beta)$, t on X. See [9, p107, 4.7.(c)]. Let $\mathbf{f} := [(f, C, a_1, \ldots, a_k)]$. Then, $H(\mathbf{f}) := [(t \circ f \circ s, s(a_1), \ldots, s(a_k))]$. Thus, $t \circ f \circ s(s(a_i)) = ev_i(H(\mathbf{f})) = t(ev_i(\mathbf{f})) = t(f(a_i))$. This commutation relation is independent of the pointed stable map representing the point $\mathbf{f} \in \overline{M}_k(X, \beta)$. Thus, the Proposition follows. □

One can show the following Proposition as we proved Proposition 3.1. Its proof is left to readers.

Proposition 3.2 *The forgetful maps, $\overline{M}_k(X, \beta) \to \overline{M}_{k-1}(X, \beta)$, $\overline{M}_k(X, \beta) \to \overline{M}_k$, are real maps.*

Proposition 3.3 *Let $\mathbb{CP}^n$ have the real structure from a complex conjugation involution. Let X be a real projective variety such that the imbedding i which decides the real structure of X intersects with the real part $\mathbb{RP}^n$ of $\mathbb{CP}^n$. If $k \geq 3$, then the real part of*

$M_k(X,\beta)$ (the locus in $\overline{M}_k(X,\beta)$ whose domain curve is smooth) consists of stable maps $[(f,\mathbb{CP}^1,a_1,\dots,a_k)]$ such that a_i is in $\mathbb{RP}^1(\subset \mathbb{CP}^1)$ and f is a real map, i.e., $f\circ s = t\circ f$.

Proof. It is well-known that the real part of the Deligne-Mumford moduli space M_k (before the compactification) consists of curves whose marked points are on the real part $\mathbb{RP}^1$ of the domain curve $\mathbb{CP}^1$. (See [6, sec. 2.3].) If $\mathbf{f} := [(f,\mathbb{CP}^1,a_1,\dots,a_k)]$ represents a point in the real part of $M_k(X,\beta)$, then Proposition 3.2 asserts that $\mathbf{f}$ is represented by a map $(f,\mathbb{CP}^1,a_1,\dots,a_k)$, where a_i is on the real part $\mathbb{RP}^1$ in $\mathbb{CP}^1$. And there exists $\sigma\in \mathrm{Aut}(\mathbb{CP}^1)$ such that $f = t\circ f\circ s\circ\sigma$, $a_i=\sigma(a_i)$. Since $k\geq 3$, σ is an identity map. It shows that f is a real map. □

Let's assume that the target space X has the same real structure stated in Proposition 3.3. Then, of course, the type of stable maps described in Proposition 3.3 are in the real part of $M_k(X,\beta)$. But, the real part analysis of the whole moduli space $\overline{M}_k(X,\beta)$ for all k is subtle. Let's illustrate with some examples.

- If $k=0$, then some non-real maps are in the real part of the moduli space. For example, the non-real map $f:\mathbb{CP}^1\to\mathbb{CP}^2$ whose image curve is represented by the equation $x^2+y^2+z^2$ in $\mathbb{CP}^2$ is in the real part of the moduli space.

- On $\overline{M}_k(X,\beta)\setminus M_k(X,\beta)$, the stable maps all of whose gluing points are in the real part of the domain curve, and all of whose marked points are in the real part of each irreducible component are in the real part of the moduli space.

- The gluing points in the reducible domain curve don't have to be in the real part of the domain curve. Let $[(f,C,a_1)]$ be the element in $\overline{M}_1(\mathbb{CP}^1, 2\cdot[\text{line}])$ such that

  $*$ the normalization $\widetilde{C}$ of the domain curve C is $\mathbb{CP}^1_0\cup\mathbb{CP}^1_1\cup\mathbb{CP}^1_2$, where $\mathbb{CP}^1_i\cong\mathbb{CP}^1\cong\mathbb{C}\cup\{\infty\}$

  $*$ the point $0\in\mathbb{CP}^1_1$ is glued to the point $i\in\mathbb{CP}^1_0$ and the point $0\in\mathbb{CP}^1_2$ is glued to the point $-i\in\mathbb{CP}^1_0$

  $*$ $f\mid_{\mathbb{CP}^1_0}=0$, $f\mid_{\mathbb{CP}^1_1}=f\mid_{\mathbb{CP}^1_2}=$ identity map

The last example was given by Pierre Deligne to the author.

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