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Steepness in Natural Exponential Families

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Abstract

The present paper studies and develops the notion of steepness in multivariate natural exponential families. Let $F = \{P(m, F); m \in M_F\}$ be a multidimensional natural exponential family parameterized by its domain of the means M_F and let $\overline{m}$ be an element of ∂M_F the means domain boundary. A necessary and sufficient condition for the variance function V_F is established so that the family F be steep at $\overline{m} \in \partial M_F$. Some characteristic properties of a steep family are given. Also, we investigate the asymptotic behaviour of a steep family F at $\overline{m}$.

Key Words: Convex, natural exponential family, face, means domain, steep, variance function.

1. Introduction

Natural exponential families represent an important class of distributions in probability and statistical theory. Various interesting works have been devoted to the theory of natural exponential families in the previous few years. Such works are contained in Brown (1986), Barndorff-Nielsen (1978) and in Letac (1992). It is well known that the natural exponential families are characterized by their variance functions (Tweedie (1947)). A number of papers have been devoted to the classification of natural exponential families by their variance functions. Several classifications have been attained (see, Morris (1982); Letac and Mora (1990), Hassairi (1992) and Casalis (1994)). In this context the variance function of a natural exponential family (N.E.F) appears as the most appropriate tool and so it has received a great deal of attention in the statistical theory.

Let $F = F(\mu) = \{P(m, F);\ m \in M_F\}$ be a multidimensional natural exponential family on $\mathbb{R}^d$, generated by a positive measure μ, parameterized by its domain of the means M_F and let V_F be its variance function. It's well known that all members of the N.E.F F have the same support. If $cs(\mu)$ denotes the closed convex hull of the support of μ, then we have always $M_F \subset \mathrm{int}(cs(\mu))$. Actually, in most of the practical cases, these two sets are not equal. There are examples in $\mathbb{R}^d$ where M_F is not even a convex set (see Example 3.1). But in one dimension, M_F is always an open interval, since its definition implies that M_F is an open connected set. Barndorff-Nielsen (1978) has defined the notion of steepness in natural exponential families as follows: a N.E.F F is called steep if $\mathrm{int}(cs(\mu)) = M_F$. The steepness enables one to apply the convex analysis methods to the convex supports of N.E.Fs. The present paper is devoted to the study of local steepness in multivariate N.E.Fs. Let $\overline{m}$ be an element of the boundary ∂M_F of M_F, the family $F(\mu) = F$ is called steep at $\overline{m}$ if $\overline{m} \in \partial M_F \cap \partial cs(\mu)$. In particular, F is steep if it's steep at all points of the boundary ∂M_F of M_F. So, the local steepness of a full N.E.F is a very essential property for the study of the maximum likelihood estimation problem. Also, a natural problem within this approach is to identify points of the means domain's boundary with those of the convex support's boundary. We investigate the effect of the steepness of the N.E.F F at $\overline{m}$ on the behaviour of the family F around $\overline{m}$ by using its variance function. In the one dimensional cases, Jørgensen et al. (1994) or Letac Mora (1990) have shown that if F is a N.E.F with means domain $M_F =]a, b[$, then F is steep if and only if $V_F(a) = V_F(b) = 0$. Here, we generalize this result to multivariate dimensions. So, we show that if the variance function extends continuously at $\overline{m} \in \partial M_F$, then F is steep at $\overline{m}$ if and only if $V_F(\overline{m})$ is a degenerated matrix. In other words, $\mathrm{rank}(V_F(\overline{m})) < d$.

Now, let H be an exposed face of $cs(\mu)$, If $\mu(H) > 0$, then $\partial M_F \cap H \neq \emptyset$. This means that F is steep at all points of $\partial M_F \cap H$.

The outline in this paper is as follows: in section 2, we give a review of natural exponential families. In section 3, we state our main results and give some illustrating examples.

2. Natural Exponential Families

Let us begin with some definitions and notations which have become traditional in statistics. For more details, we refer to Letac (1992).

Let $(\theta, x) \longmapsto \langle\theta, x\rangle$ be the canonical scalar product on $\mathbb{R}^d$.
For a positive measure μ on $\mathbb{R}^d$, we denote

$$L_\mu : \mathbb{R}^d \longrightarrow]0, +\infty[: \theta \longmapsto \int_{\mathbb{R}^d} \exp\langle\theta, x\rangle \mu(dx),$$

$$D(\mu) = \{\theta \in \mathbb{R}^d; \quad L_\mu(\theta) < +\infty\},$$

$$\Theta(\mu) = \operatorname{int}(D(\mu)),$$

$$k_\mu = \log L_\mu,$$

where L_μ and k_μ are, respectively, the Laplace transform and the cumulant generating function of μ.

The set $\mathcal{M}(\mathbb{R}^d)$ is now defined as the set of positive measures μ such that μ is not concentrated on an affine hyperplane and $\Theta(\mu)$ is not empty. To each μ in $\mathcal{M}(\mathbb{R}^d)$ and θ in $\Theta(\mu)$, we associate the following probability distribution:

$$P(\theta, \mu)(dx) = \exp(\langle\theta, x\rangle - k_\mu(\theta))\mu(dx).$$

The set of probabilities

$$F = F(\mu) = \{P(\theta, \mu)(dx); \ \theta \in \Theta(\mu)\}$$

is called the natural exponential family (N.E.F) generated by μ. Of course, μ and μ' in $\mathcal{M}(\mathbb{R}^d)$ are such that $F(\mu) = F(\mu')$ if and only if there exists (a, b) in $\mathbb{R}^d \times \mathbb{R}$ such that

$$\mu'(dx) = \exp(\langle a, x\rangle + b)\mu(dx).$$

Since μ is in $\mathcal{M}(\mathbb{R}^d)$, then k_μ is analytic and strictly convex map on $\Theta(\mu)$, so that its differential

$$k'_\mu : \theta \longmapsto k'_\mu(\theta) = \int_{\mathbb{R}^d} x P(\theta, \mu)(dx)$$

defines a diffeomorphism from $\Theta(\mu)$ to its image M_F. This image is called the means domain of F. Let $\psi_\mu : M_F \mapsto \Theta(\mu)$ be the inverse function of k'_μ, and, for m in M_F,

$P(m,F) = P(\psi_\mu(m),\mu)$. Then we obtain a new parameterization of F by its means domain, so we have

$$F = \{P(m,F);\ m\ \in M_F\}.$$

The covariance matrix $V_F(m)$ of the probability measure $P(m,F)$ is symmetric and is positive definite. It can be written as a function of the mean parameter m:

$$V_F(m) = k_\mu^{''}(\psi_\mu(m)) = (\psi_\mu^{'}(m))^{-1}.$$

The map defined on M_F by $m \longmapsto V_F(m)$ is called the variance function of the family F and it's real analytic on M_F. The importance of the variance function stems from the fact that it characterizes the family F. Indeed, V_F characterizes the N.E.F in the following sense: if F_1 and F_2 are two N.E.Fs whose variance functions coincide on a nonempty open set of $M_{F_1} \cap M_{F_2}$, then $F_1 = F_2$.

Now, we examine the influence of an affine transformation on the elements of a N.E.F F. Let φ be in the affine group of $\mathbb{R}^d$, i.e., $x \mapsto \varphi(x) = a(x) + b$, where b is in $\mathbb{R}^d$ and a is in the linear group $GL(\mathbb{R}^d)$. The following facts are easily checked:

$$\varphi(F) = F(\varphi(\mu))$$

$$M_{\varphi(F)} = \varphi(M_F)$$

$$V_{\varphi(F)}(m) = aV_F(\varphi^{-1}(m))^t a; \quad \text{for all } m \in M_{\varphi(F)}.$$

The closure, interior and affine hull of a set C in $\mathbb{R}^d$ are denoted, respectively, by $\overline{C}$, $\text{int}(C)$ and $aff(C)$.

Let $cs(\mu)$ be the closed convex hull of the support of μ. The exposed faces H of $cs(\mu)$ are represented as the intersection of $cs(\mu)$ with a supporting hyperplane (see Rockafellar (1970), page 162). That means $H = cs(\mu) \cap \{x \in \mathbb{R}^d;\ \langle u, x-b\rangle = 0\}$, where $b \in cs(\mu)$ and $u \in \mathbb{R}^d$ is an exterior normal vector on H, (i.e., $\langle u;\ x-b\rangle \leq 0$ for each vector x in $cs(\mu)$).

3. Main Results

In this work, we study the notion of local steepness in N.E.Fs. We explore the properties of any steep N.E.F at a point $\overline{m}$ of the boundary ∂M_F of M_F. We give the relation between the asymptotic behaviour of F and the steepness property at $\overline{m} \in \partial M_F$.

In what follows, we consider a N.E.F $F = F(\mu)$ on $\mathbb{R}^d$ generated by a positive measure $\mu \in \mathcal{M}(\mathbb{R}^d)$. Without loss of generality, we assume that the generating measure $\mu \in F$.

Definition 3.1 *Let $\overline{m} \in \partial M_F$. The N.E.F family $F = F(\mu)$ is called steep at $\overline{m}$ if $\overline{m} \in \partial M_F \cap \partial cs(\mu)$.*

According to the definition in Barndorff-Nielsen (1978), the family F is steep if it's steep at all points of the boundary ∂M_F of the means domain M_F. In the following theorem, we study the asymptotic behaviour of a steep N.E.F $F = F(\mu)$ at a point $\overline{m} \in \partial M_F$. So, we investigate the behaviour of ψ_μ around $\overline{m}$. It's essential because of the link between the variance function V_F and ψ_μ.

Theorem 3.1 *Let $F = F(\mu)$ be a N.E.F and let $\overline{m} \in \partial M_F$. Then the following statements are equivalents.*

i) The family F is steep at $\overline{m}$.

ii) $\lim\limits_{m \to \overline{m}} \|\psi_\mu(m)\| = +\infty$.

For the proof of this theorem we shall need the following lemma.

Lemma 3.2 *Let $(m_n)_{n\in\mathbb{N}}$ be a sequence in M_F converging to $\overline{m} \in \partial M_F$. Suppose that $\lim\limits_{n\to+\infty} \psi_\mu(m_n) = \overline{\theta}$. Then $\overline{\theta} \in D(\mu)\backslash\Theta(\mu)$.*

Proof of Lemma 3.2 Let $\theta \in \Theta(\mu)$ and let $\theta_n = \psi_\mu(m_n)$. Obviously $\overline{\theta} \notin \Theta(\mu)$. By way of contradiction, suppose that $\overline{\theta} \in \overline{\Theta(\mu)}\backslash D(\mu)$.

Define the following maps

$$\varphi_n : [0,1] \longrightarrow \mathbb{R};\ \lambda \mapsto \varphi_n(\lambda) = k_\mu((1-\lambda)\theta_n + \lambda\theta)$$

and

$$\varphi :]0,1] \longrightarrow \mathbb{R};\ \lambda \mapsto \varphi(\lambda) = k_\mu((1-\lambda)\overline{\theta} + \lambda\theta).$$

By the Rolle's formula there exists $\lambda_n \in]0,1[$ such that

$$\varphi_n(1) - \varphi_n(0) = {\varphi'}_n(\lambda_n).$$

Since ${\varphi'}_n$ is strictly increasing on $[0,1]$, we have

$$\varphi_n(0) = \varphi_n(1) - {\varphi'}_n(\lambda_n) < \varphi_n(1) - {\varphi'}_n(0).$$

Hence

$$k_\mu(\theta_n) < k_\mu(\theta) - \langle {k'}_\mu(\theta_n), \theta - \theta_n \rangle.$$

As $\overline{\theta} \notin D(\mu)$, Fatou lemma implies that

$$\lim_{n \to +\infty} k_\mu(\theta_n) = +\infty$$

and this contradicts the fact that the sequence $(k_\mu(\theta) - \langle {k'}_\mu(\theta_n), \theta - \theta_n \rangle)_n$ is bounded. Thus we have the desired result. $\square$

Proof of Theorem 3.1 $(ii) \Longrightarrow (i)$. By way of contradiction, suppose that $\overline{m} \notin \partial cs(\mu)$. Let (m_n) be a sequence in M_F converging to $\overline{m}$ such that $\lim\limits_{n \to +\infty} \|\psi_\mu(m_n)\| = +\infty$ and $\lim\limits_{n \longrightarrow +\infty} \dfrac{\psi_\mu(m_n)}{\|\psi_\mu(m_n)\|} = \overline{\theta}$.

Now, we start by showing that $\lambda\overline{\theta} \in \Theta(\mu)$ for any $\lambda \geq 0$.

Indeed, since $\lim_{n \to +\infty} \|\psi_\mu(m_n)\| = +\infty$, then there exists $n_o \in \mathbb{N}$, such that $\forall n \geq n_o$, one has $0 \leq \frac{\lambda}{\|\psi_\mu(m_n)\|} \leq 1$.

As, the generating measure $\mu \in F$ then $0 \in \Theta(\mu)$. Furthermore, $\Theta(\mu)$ is a convex set then, $\forall n \geq n_o$, one has

$$\frac{\lambda}{\|\psi_\mu(m_n)\|} \psi_\mu(m_n) \in \Theta(\mu).$$

By letting $n \to +\infty$, we infer that

$$\{\lambda\overline{\theta} \ \ ; \ \ \lambda \geq 0 \ \} \subset \overline{\Theta(\mu)}.$$

The fact that $0 \in \Theta(\mu)$ and $\Theta(\mu)$ is an open convex set imply

$$\{\lambda\overline{\theta} \ \ ; \ \ \lambda \geq 0 \ \} \subset \Theta(\mu).$$

Now, let $\bar{\mu}$ be the image probability measure of μ by the map $\phi : x \longmapsto \langle \bar{\theta}, \overline{m} - x \rangle$. Since $\overline{m} \notin \partial cs(\mu)$, then $\overline{m} \in \text{int}(cs(\mu))$.
Therefore $0 = \phi(\overline{m}) \in \phi(\text{int}(cs(\mu))) = \text{int}(cs(\bar{\mu}))$; because ϕ is an affine map.

Let $\lambda \geq 0$, we have

$$L_{\bar{\mu}}(-\lambda) = \int_{\mathbb{R}^d} e^{-\lambda\langle\bar{\theta},\overline{m}-x\rangle}\mu(dx) < +\infty,$$

because $\lambda\bar{\theta} \in \Theta(\mu)$. This gives $k_{\bar{\mu}}(-\lambda) = -\lambda\langle\bar{\theta},\overline{m}\rangle + k_\mu(\lambda\bar{\theta})$. Taking the differential with respect to λ, we directly obtain,

$$k_{\bar{\mu}}^{'}(-\lambda) = \langle\bar{\theta},\overline{m}\rangle - \langle k_\mu^{'}(\lambda\bar{\theta}),\bar{\theta}\rangle \tag{3.1}$$

and

$$k_{\bar{\mu}}^{''}(-\lambda) = \langle k_\mu^{''}(\lambda\bar{\theta})\bar{\theta},\bar{\theta}\rangle.$$

Now, we will prove that $M_{F(\bar{\mu})} \subset]0,+\infty[$.

In fact, since k_μ is strictly convex on $\Theta(\mu)$ and for all integer $n \geq n_o$, $\frac{\lambda}{\|\psi_\mu(m_n)\|} \in [0,1]$ then

$$\langle k_\mu^{'}(\frac{\lambda\psi_\mu(m_n)}{\|\psi_\mu(m_n)\|}), \frac{\psi_\mu(m_n)}{\|\psi_\mu(m_n)\|}\rangle \leq \langle k_\mu^{'}(\psi_\mu(m_n)), \frac{\psi_\mu(m_n)}{\|\psi_\mu(m_n)\|}\rangle = \langle m_n, \frac{\psi_\mu(m_n)}{\|\psi_\mu(m_n)\|}\rangle.$$

Taking limits in both sides, one gets

$$\langle k_\mu^{'}(\lambda\bar{\theta}),\bar{\theta}\rangle \leq \langle\overline{m},\bar{\theta}\rangle. \tag{3.2}$$

Formulae (3.1) and (3.2) imply that $M_{F(\bar{\mu})} \subset]0,+\infty[$. Therefore $k_{\bar{\mu}}$ is strictly increasing on $]-\infty,0]$, implying that $k_{\bar{\mu}}(-\lambda) \leq k_{\bar{\mu}}(0) = 0$. Hence, for any $\lambda \geq 0$, one has $L_{\bar{\mu}}(-\lambda) \leq 1$. Using again Fatou lemma, we obtain

$$\int_{\mathbb{R}} \lim_{\lambda\longrightarrow+\infty} e^{-\lambda x}\bar{\mu}(dx) \leq 1.$$

Necessarily, $x \geq 0 \quad \bar{\mu}\ a.\ s.$ So, $cs(\bar{\mu}) \subset [0,+\infty[$, contradicting the fact that $0 \in \mathrm{int}(cs(\bar{\mu}))$.

$(i) \Longrightarrow (ii)$. By way of contradiction. Let (m_n) be a sequence in M_F, converging to $\overline{m} \in \partial M_F \cap \partial cs(\mu)$, and suppose that

$$\theta_n = \psi_\mu(m_n) \xrightarrow[n\longmapsto+\infty]{} \overline{\theta}.$$

From Lemma 3.2, it follows that $\overline{\theta} \in D(\mu)$.

Let H be an exposed face of $cs(\mu)$ containing $\overline{m}$ and let u be an exterior normal vector which exposes H. Up to translation, there is no loss of generality in assuming that $0 \in aff(H)$. Therefore

$$M_F \subset cs(\mu) \subset \{x \in \mathbb{R}^d \ ; \ \langle x, u \rangle \leq 0\}$$

and

$$H = cs(\mu) \cap \{x \in \mathbb{R}^d \ ; \ \langle x, u \rangle = 0\}.$$

So, it is clear that $\forall n \in \mathbb{N}$,

$$\langle k_\mu^{'}(\theta_n), u \rangle < 0.$$

Again, by using Fatou lemma and the fact that $k_\mu^{'}(\theta_n) \underset{n \longrightarrow +\infty}{\longrightarrow} \overline{m}$, we obtain

$$0 \leq -\int_{\mathbb{R}^d} \langle x, u \rangle e^{\langle \overline{\theta}, x \rangle - k_\mu(\overline{\theta})} \mu(dx) \leq - \lim_{n \longrightarrow +\infty} \langle k_\mu^{'}(\theta_n), u \rangle = 0.$$

Necessarily,

$$\langle x, u \rangle = 0 \quad \mu \ \ a. \ \ s.$$

Hence μ is concentrated on the affine hyperplane containing 0 and with exterior normal vector u, contradicting the fact that $\mu \in \mathcal{M}(\mathbb{R}^d)$. □

Remark: In the statistical literature, the N.E.F $F = F(\mu)$ is called regular if $\Theta(\mu) = D(\mu)$. It is clear that the regular families are steep. In fact, it suffices to apply Lemma 3.2. Of course, there are steep families which are not regular.

Definition 3.2 *Let F be a N.E.F on $\mathbb{R}^d$ and let C be a subset of the boundary ∂M_F. F is called steep in C if it's steep at all points $\overline{m}$ of C.*

The following theorem generalizes the one dimensional result due to Jørgensen and all (1994) or Letac Mora (1990), asserting that if F is a real N.E.F with means domain $M_F =]a, b[$, then F is steep at a if and only if $V_F(a) = 0$. So, an algebraic property concerning the variance function V_F can be extended to multidimensional cases as follows.

Theorem 3.3: *Let F be a N.E.F on $\mathbb{R}^d$. Suppose that the variance function V_F extends continuously at $\overline{m} \in \partial M_F$. Then F is steep at $\overline{m}$ if and only if rank$(V_F(\overline{m})) < d$.*

Proof. $(\Longleftarrow)$ By hypothesis, $\operatorname{rank}(V_F(\overline{m})) < d$. Then there exists $u \in \mathbb{R}^d \setminus \{0\}$ such that $V_F(\overline{m})u = 0$. Suppose that F is not steep at $\overline{m}$ and according to Theorem 3.1, there exists a sequence (m_n) in M_F converging to $\overline{m}$ such that $\lim_{n\longrightarrow+\infty} \psi_\mu(m_n) = \overline{\theta}$. By using Lemma 3.2, we have $\overline{\theta} \in D(\mu)$. We know that the variance function is defined as

$\langle V_F(m_n)u, u\rangle = \int_{\mathbb{R}^d} \langle x - m_n, u\rangle^2 e^{\langle \psi_\mu(m_n), x\rangle - k_\mu(\psi_\mu(m_n))} \mu(dx).$

By tending $n \longrightarrow +\infty$, and applying Fatou Lemma, we obtain

$0 \leq \int_{\mathbb{R}^d} \langle x - \overline{m}, u\rangle^2 e^{\langle \overline{\theta}, x\rangle - k_\mu(\overline{\theta})} \mu(dx) \leq \langle V_F(\overline{m})u, u\rangle = 0.$

Therefore

$$\langle x - \overline{m}, u\rangle = 0 \quad \mu \ \ a. \ \ s.$$

Thus, μ is concentrated on the affine hyperplane containing $\overline{m}$ and with exterior normal vector u, contradicting the fact that $\mu \in \mathcal{M}(\mathbb{R}^d)$.

$(\Longrightarrow)$ Suppose the contrary, this means that $V_F(\overline{m})$ is positive definite. Hence, the map $m \longmapsto (V_F(m))^{-1} = \psi'_\mu(m)$ is continuous at $\overline{m}$. Therefore, ψ_μ extends continuously at $\overline{m}$. Then, from Theorem 3.1, we obtain a contradiction. □

As a consequence one can check easily that the quadratic N.E.Fs (Casalis (1994)) are steep families.

The following example is due to J. Del Castillo (1994). This example is also described in Letac (1992), pages 19–20.

Example 3.1 *Let μ be the image measure, in $\mathbb{R}^2$, of $\nu(dx) = e^{-x}\mathbf{1}_{[0,+\infty[}(x)dx$ by $x \longrightarrow (x, -\frac{x^2}{2})$. Then*

$$M_F = \{(x,y) \in \mathbb{R}^2 \ \ ; \ \ -x^2 < y < -\frac{x^2}{2} \text{ and } x > 0\}$$

and

$$\operatorname{int}(cs(\mu)) = \{(x,y) \in \mathbb{R}^2 \ \ ; \ \ y < -\frac{x^2}{2} \text{ and } x > 0\} \supsetneq M_F.$$

Obviously, M_F is not convex.

The family F is steep in $\{(x, -\frac{x^2}{2}) \ ; \ x \geq 0\}$ but not steep in $\{(x, -x^2) \ ; \ x > 0\}$.

Let H be an exposed face of $cs(\mu)$. The following result gives a sufficient condition on the generator measure μ so that $\partial M_F \cap H \neq \emptyset$. In this case, the N.E.F is steep in $\partial M_F \cap H$.

Proposition 3.4 *Let H be an exposed face of $cs(\mu)$ and let u be an exterior normal vector on H such that $\mu(H) > 0$. Then, for any $\theta \in \Theta(\mu)$, one has*

i) $\theta + nu \in \Theta(\mu),\ \forall n \in \mathbb{N}$.

ii) $\partial M_F \cap H \neq \emptyset$.

Proof. i) Up to translation, we assume that

$$H = cs(\mu) \cap \{x \in \mathbb{R}^d \ \ ;\ \langle u, x\rangle = 0\}$$

and

$$cs(\mu) \subset \{x \in \mathbb{R}^d \ \ ;\ \langle u, x\rangle \leq 0\}.$$

Observe that, for $\beta \in D(\mu)$, $L_\mu(\beta + nu) \leq L_\mu(\beta) < +\infty$. Hence, $\beta + nu \in D(\mu)$. Now, let $\theta \in \Theta(\mu)$. Then there exists an open ball $B(\theta, r)$, with center θ and radius $r > 0$, such that $B(\theta, r) \subset \Theta(\mu)$. Therefore

$$B(\theta, r) + nu \subset D(\mu).$$

As $B(\theta, r) + nu$ is an open set then $B(\theta, r) + nu \subset \Theta(\mu)$.

ii) Let $\pi_o(dx) = \frac{1}{\mu(H)}\mathbf{1}_H(x)\mu(dx)$ be the probability measure concentrated on H (i,e the conditional probability of μ given H). By using dominated convergence, we obtain

$$\lim_{n\longmapsto+\infty} L_\mu(\theta + nu) = \int_H e^{\langle\theta,x\rangle}\mu(dx) \text{ and } \lim_{n\longmapsto+\infty} L'_\mu(\theta + nu) = \int_H xe^{\langle\theta,x\rangle}\mu(dx).$$

Recall that $k'_\mu(\theta + nu) = \frac{L'_\mu(\theta+nu)}{L_\mu(\theta+nu)}$ and it immediately yields

$$\lim_{n\longmapsto+\infty} k'_\mu(\theta + nu) = \frac{L'_{\pi_o}(\theta)}{L_{\pi_o}(\theta)} = k'_{\pi_o}(\theta) \in \partial M_F.$$

On the other hand, π_o is concentrated on H; then $k'_{\pi_o}(\theta) \in H$.

Consequently,

$$k'_{\pi_o}(\Theta(\mu)) \subset \partial M_F \cap H.$$

Thus, $\partial M_F \cap H \neq \emptyset$ and the proof of Proposition 3.4 is complete. □

We conclude the following example which illustrates our results.

Example 3.2 *Consider a stable law ν_α with parameter $\alpha \in]1,2[$, concentrated on the real line, such its cumulant function is defined on $D(\nu_\alpha) =]-\infty, 0]$ by $k_{\nu_\alpha}(\theta) = -(-\theta)^\alpha$. The N.E.F $F(\nu_\alpha)$ is not steep because $M_{F(\nu_\alpha)} =]0, +\infty[\subsetneq cs(\nu_\alpha) = \mathbb{R}$. Let $F = F(\mu)$ be the N.E.F on $\mathbb{R}^2$ generated by the following probability measure*

$$\mu(dx, dy) = \frac{1}{2}[\nu_\alpha(dx) \otimes \delta_0(dy) + \delta_{(0,1)}(dx, dy)],$$

where $\delta_{(0,1)}$ denotes the Dirac measure at $(0,1)$. We check easily that

$$M_F =]0, +\infty[\times]0, 1[\subsetneq \mathrm{int}(cs(\mu)) = \mathbb{R} \times]0, 1[$$

and, $\forall (m_1, m_2) \in M_F$,

$$\psi_\mu(m_1, m_2) = \left(-[\frac{m_1}{\alpha(1-m_2)}]^{\frac{1}{\alpha-1}} \ , \ \log(\frac{m_2}{1-m_2}) - [\frac{m_1}{\alpha(1-m_2)}]^{\frac{\alpha}{\alpha-1}} \ \right).$$

So, the family F is not steep at all points of the boundary of the means domain. However M_F is a convex set.

Also, observe that

$$\lim_{m_1 \to 0} \psi_\mu(m_1, m_2) = (0 \ , \ \log(\frac{m_2}{1-m_2}) \ ).$$

Hence, by applying Theorem 3.1 the N.E.F is not steep in $\{0\} \times]0, 1[$.

Let $H = \mathbb{R} \times \{0\}$ be an exposed face of $cs(\mu)$. It's easy to see that $\mu(H) > 0$. Thus, this example shows that the N.E.F F is steep in $\partial M_F \cap H = [0, +\infty[\times \{0\}$ but not in $]-\infty, 0[\times \{0\}$.

The calculation of the variance function leads to

$$V_F(m_1, m_2) = \begin{pmatrix} V_{11}(m_1, m_2) & m_1 m_2 \\ -m_1 m_2 & m_2(1-m_2) \end{pmatrix},$$

where $V_{11}(m_1, m_2) = \frac{m_1^2 m_2}{1-m_2} + \alpha(1-\alpha)(1-m_2)[\frac{m_1}{\alpha(1-m_2)}]^{\frac{\alpha-2}{\alpha-1}}$. Note that the variance function extends continuously to $C =]0, +\infty[\times \{0\}$. Obviously, for all $\overline{m} \in C$, $V_F(\overline{m})$ is a degenerated matrix and F is steep in C (see Theorem 3.3).

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