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On Cartan Spaces with (α, β)-metric

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Abstract

É. Cartan [2] has originally introduced a Cartan space, which is considered as dual of Finsler space. H. Rund [10], F. Brickell [1] and others studied the relation between these two spaces. The theory of Hamilton spaces was introduced and studied by R. Miron ([8] , [9]). He proved that Cartan space is a particular case of Hamilton space. T. Igrashi ([5], [6]) introduced the notion of the (α, β)-metric in Cartan spaces and obtained the metric tensor and the invariants ρ and r which characterize the special classes of Cartan spaces with (α, β)-metric. This paper presents a study of Cartan spaces with (α, β)-metric admitting h-metrical d-connection. We prove the conditions for these spaces to be locally Minkowski and conformally flat.

Key Words: Cartan space, (α, β)-metric, h-metrical d-connection, Conformally flat.

1. Introduction

Let M be a real smooth manifold and (T^*M, π, M) its cotangent bundle. Let $C^n = (M, K(x,p))$, where $K : T^*M \to R$ is a scalar function which is differentiable on $T^*\widetilde{M} = T^*M - \{0\}$, and is homogeneous on the fibres of T^*M. The hessian of K^2, i.e. $g^{ij}(x,p) = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j K^2$, where $\dot{\partial}^i = \frac{\partial}{\partial p_i}$, is positively defined on $T^*\widetilde{M}$. Here C^n is called the Cartan space and the functions $K(x,p)$ and $g^{ij}(x,p)$ are called, respectively, the fundamental function and the metric tensor of the Cartan space C^n.

The reciprocal $g_{ij}(x,p)$ of $g^{ij}(x,p)$ is given by $g_{ij}(x,p)g^{ij}(x,p) = \delta^k_j$, where $g_{ij}(x,p)$

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and $g^{ij}(x,p)$ are both symmetric and homogeneous of order 0 in p_j.

A Cartan space $C^n = (M, K)$ is said to be with (α, β)-metric if $K(x,p)$ is a function of the variables $\alpha(x,p) = (a^{ij}\, p_i\, p_j)^{\frac{1}{2}}$, $\beta(x,p) = p_i\, b^i(x)$, where $a^{ij}(x)$ is a Riemannian metric and $b^i(x)$ is a vector field depending only on x. Clearly K must satisfy the conditions imposed to the fundamental function of a Cartan space.

In this paper, we consider the Cartan spaces with (α, β)-metric admitting h-metrical d-connection in section 2 and their conformal change in section 3. The fundamental tensor $g^{ij}(x,p)$ and its reciprocal $g_{ij}(x,p)$ of the Cartan space $C^n = (M, K(\alpha, \beta))$ are given by [6] the relation

$$g^{ij} = \rho a^{ij} + \rho b^i b^j + \rho_{-1}\left(b^i p^j + b^j p^i\right) + \rho_{-2} p^i p^j, \tag{1.1}$$

where ρ, ρ_0, ρ_{-1}, ρ_{-2} are the invariants given by

$$\begin{aligned} \rho = \frac{1}{2}\alpha^{-1} K_\alpha,\ \rho_{-1} = \frac{1}{2}\alpha^{-1} K_{\alpha\beta},\ \rho_{-2} = \frac{1}{2}\alpha^{-2}(K_{\alpha\alpha} - \alpha^{-1} K_\alpha) \\ \rho_0 = \frac{1}{2} K_{\beta\beta}, \end{aligned} \tag{1.2}$$

and

$$g_{ij} = \sigma a_{ij} - \sigma_0 b_i b_j + \sigma_{-1}(b_i p_j + b_j p_i) + \sigma_{-2} p_i p_j \tag{1.3}$$

where

$$\sigma = \frac{1}{\rho},\ \sigma_0 = \frac{\rho_0}{\rho\,\tau},\ \tau = \sigma + \sigma_0 B^2 + \rho_{-1}\beta,\ \sigma_{-1} = \frac{\rho_{-1}}{\rho\tau},\ \sigma_{-2} = \frac{\rho_{-2}}{\rho\tau}, \tag{1.4}$$

and where $B^2 = b^i b_i$.
The Cartan tensor C^{ijk} is given by

$$\begin{aligned} C^{ijk} = -\frac{1}{2}[r_{-1} b^i b^j b^k + \{\rho_{-1} a^{ij} b^k + \rho_{-2} a^{ij} p^k + r_{-2} b^i b^j p^k + \\ r_{-3} b^i p^j p^k + i/j/k\} + r_{-4} p^i p^j p^k], \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} r_{-1} = \frac{1}{2} K_{\beta\beta\beta},\ r_{-2} = \frac{1}{2}\alpha^{-1} K_{\alpha\beta\beta},\ r_{-3} = \frac{1}{2}\alpha^{-2}(K_{\alpha\alpha\beta} - \alpha^{-1} K_{\alpha\beta}) \\ r_{-4} = \frac{1}{2}\alpha^{-3}\{K_{\alpha\alpha\alpha} - 3\alpha^{-1} K_{\alpha\alpha} + 3\alpha^{-2} K_\alpha\}. \end{aligned} \tag{1.6}$$

Let ':' denote the covariant differentiation with respect to Christoffel symbols γ^i_{jk} constructed from a_{ij}. Since $a^{ij}{}_{:k} = 0$ and $p_{i:k} = 0$, if $b^i{}_{:k} = 0$, then $g^{ij}{}_{:k} = 0$. Using the Christoffel symbols $\Gamma^i_{jk}(p) = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$ constructed from $g_{ij}(x,p)$, we can define canonical N-connection

$$N_{ij} = \Gamma^k_{ij}p_k - \frac{1}{2}\Gamma^k_{hr}p_k p^r \dot{\partial}^h g_{ij}. \tag{1.7}$$

We consider the canonical d-connection

$$D\Gamma = (N_{jk}, H^i_{jk}, C^{jk}_i) \tag{1.8}$$

where

$$H^i_{jk} = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}). \tag{1.9}$$

The d-tensor field of type (2,1) C^{jk}_i is given by

$$C^{jk}{}_i = -\frac{1}{2}g_{ir}\,\dot{\partial}^r g^{jk} = g_{ir}C^{rjk}, \tag{1.10}$$

Let $'|k'$ denote the h-covariant differentiation with respect to $D\Gamma$.

Definition 1.1 *A d-connection $D\Gamma$ of a Cartan space C^n with (α,β)-metric is called the h-metrical d-connection if it satisfies the conditions*

- *h-deflection tensor $D_{ij}(= p_{i|j}) = 0$;*
- *$\alpha^{ij}{}_{|k} = 0$;*
- *$g^{ij}{}_{|k} = 0$.*

2. Cartan Spaces with (α,β)-metric admitting h-metrical d-connection

If the connection $D\Gamma$ is h-metrical, then $\alpha_{|h} = 0$, from which we get that

$$0 = K_{|h} = \alpha_{|h}K_\alpha + \beta_{|h}K_\beta = \beta_{|h}K_\beta$$

and

$$\beta_{|h} = b^i{}_{|h}p_i = 0 \tag{2.1}$$

From (1.1), we have

$$g^{ij}{}_{|k} = b^{i}{}_{|k}(\rho_0 b^j + \rho_{-1} p^j) + b^{j}{}_{|k}(\rho_0 b^i + \rho_{-1} p^i) = 0$$

Transvecting the above with p_i, and by virtue of (2.1) we get

$$b^{j}{}_{|k}(\rho_0 \beta + \rho_{-1}\alpha) = 0,$$

which gives $b^{j}{}_{|k} = 0$.

Now from $a^{ij}{}_{|k} = 0$, we can get $H^i_{jk} = \gamma^i_{jk}$. Hence we have

$$b^{i}{}_{:k} = 0, \tag{2.2}$$

and also the curvature tensor D^i_{hjk} of $D\Gamma$ coincides with the curvature tensor R^i_{hjk} of Riemannian connection $R\Gamma = (\gamma^i_{jk}, \gamma^i_{jk} y_i, 0)$.

If $R^i_{hjk} = 0$ then $D^i_{hjk} = 0$. Thus we have the following proposition.

Proposition 2.1 *A Cartan space C^n with (α, β) metric admitting a h-metrical d-connection is locally flat if and only if the associated Riemannian space is locally flat.*

If the connection $D\Gamma$ is h-metrical, then $g^{ij}{}_{|h} = 0$, $\alpha_{|h} = 0$, $a^{ij}{}_{|h} = 0$, $b^{k}{}_{|h} = 0$, $p^{k}{}_{|h} = 0$, from which we get $r_{-1|h} = 0$, $r_{-2|h} = 0$, $r_{-3|h} = 0$ and $r_{-4|h} = 0$.

Hence from (1.5), (1.6) and (1.10), we have

$$C^{ij}{}_{k|h} = 0. \tag{2.3}$$

Definition 2.1 *A Cartan space C^n is a Berwald space if and only if $C^{ij}{}_{k|h} = 0$.*

Hence from (2.3) we have the following proposition.

Proposition 2.2 *A Cartan space with (α, β) metric admitting a h-metrical d-connection is a Berwald space.*

As it is well known [11] a locally Minkowski space is a Berwald space in which the curvature tensor vanishes.
Hence from Proposition (2.1) and Proposition (2.2), we have the following proposition.

Theorem 2.1 *A Cartan space with (α, β) metric admitting a h-metrical d-connection is locally Minkowski if and only if the associated Riemannian space is locally flat.*

3. Conformal Change of a Cartan Space

Let $C^n = (M, K)$ be an n-dimensional Cartan space with (α, β)-metric $K = K(\alpha, \beta)$. By a conformal change $\sigma : K \to \overline{K} : \overline{K}(\overline{\alpha}, \overline{\beta}) = e^{\sigma} K(\alpha, \beta)$, we have another Cartan space $\overline{C}^n = (M, \overline{K}(\overline{\alpha}, \overline{\beta}))$, where $\overline{\alpha} = e^{\sigma}\alpha$ and $\overline{\beta} = e^{\sigma}\beta$.

Putting $\alpha = (a^{ij}(x)p_ip_j)^{\frac{1}{2}}$ and $\beta = b^i(x)p_i$, we get $\overline{a}^{ij} = e^{2\sigma}a^{ij}$ and $\overline{b}^i = e^{\sigma}b^i$. Then the Christoffel symbols $\overline{\gamma}^i_{jk}$ constructed from $\overline{a}^{ij}$ are written as

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + B^i_{jk} \tag{3.1}$$

where $B^i_{jk} = \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}$, $\sigma^i = \sigma_j a^{ij}$.

Taking covariant derivative of $\overline{b}^i$ with respect to $\overline{\gamma}^i_{jk}$, we get

$$\overline{b}^i{}_{:k} = e^{\sigma}(b^i{}_{:k} + 2\sigma_k b^i + b^r\sigma_r\delta^i_k - \sigma^i b^r a_{rk}).$$

Transvecting the above by $\overline{b}^k$, and putting

$$M^i = \frac{1}{B^2}\left\{b^k b^i_{:k} - \frac{b^r{}_{:r}b^i}{n+4}\right\}, \tag{3.2}$$

we have $\sigma^i = \overline{M}^i - M^i$, from which we get $\sigma_i = \overline{M}_i - M_i$ Substituting this in (3.1) and putting

$D^i_{hj} = \gamma^i_{hj} + \delta^i_h M_j + \delta^i_j M_h - M^i a_{hj}$, we have

$$\overline{D}^i_{hj} = D^i_{hj}. \tag{3.3}$$

D^i_{hj} is a symmetric conformally invariant linear connection on M.

Thus we have the following proposition.

Proposition 3.1 *In a Cartan space with (α, β)-metric, there exists a conformally invariant symmetric linear connection D^i_{jk}.*

If we denote by D^i_{hjk} the curvature tensor of D^i_{jk}, we have from (3.3)

$$\overline{D}^i_{hjk} = D^i_{hjk}. \tag{3.4}$$

Since $b^i{}_{:k} = 0$, we have $M^i = 0$. Hence $D^i_{jk} = \gamma^i_{jk}$ and $D^i_{hjk} = R^i_{hjk}$. Thus we have the next proposition.

Proposition 3.2 *In a Cartan space C^n with (α, β)-metric admitting h-metrical d-connection, $M^i = 0$ and there exists a conformally invariant symmetric linear connection D^i_{jk} such that $D^i_{jk} = \gamma^i_{jk}$ and its curvature tensor $D^i_{hjk} = R^i_{hjk}$.*

If the associated Riemannian space (M, α) is locally flat ($R^i_{hjk} = 0$), then from (3.4) and Proposition (3.2), we have $\overline{D}^i_{hjk} = 0$, i.e. the space C^n is conformally flat.

Thus we conclude that.

Theorem 3.1 *A Cartan space $C^n = (M, K(\alpha, \beta))$ with (α, β) metric admitting h-metrical d-connection is conformally flat if and only if associated Riemannian space is locally flat.*

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