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On the Spectral Properties of the Regular Sturm-Liouville Problem with the Lag Argument for Which its Boundary Conditions Depends on the Spectral Parameter

Mehmet Bayramoğlu, Kevser Özden Köklü, Oya Baykal

Abstract

In this paper, the asymptotic expression of the eigenvalues and eigenfunctions of the Sturm-Liouville equation with the lag argument

$$y^{''}(t) + \lambda^2 y(t) + M(t)y(t - \Delta(t)) = 0$$

and the spectral parameter in the boundary conditions

$$\lambda y(0) + y^{'}(0) = 0$$
$$\lambda^2 y(\pi) + y^{'}(\pi) = 0$$
$$y(t - \Delta(t)) = y(0)\varphi(t - \Delta(t)), \qquad t - \Delta(t) < 0$$

has been founded in a finite interval, where M(t) and $\Delta(t) \geq 0$ are continuous functions on $[0, \pi]$, $\lambda > 0$ is a real parameter, $\varphi(t)$ is an initial function which is satisfied with the condition $\varphi(0) = 1$ and continuous in the initial set.

Key Words: Lag argument, Eigenvalue, Eigenfunction, Asymptotic expression

In this paper, we investigate the asymptotic behaviour of positive eigenvalues and corresponding eigenfunctions of the Sturm-Liouville equation with lag argument which have eigenvalues at both ends of the finite interval in the boundary conditions. Note that too few studies related to examination of the eigenvalues and eigenfunctions of regular Sturm-Liouville problem with eigenvalue in the boundary conditions have been published [1-3]. In studies [1],[3], this kind of problem was examined assuming an eigenvalue exist in one tip point of interval. Study [2] is only an abstract for proceeding. This study is the detailed case of [2]. Generally, boundary value problems with eigenvalues in the boundary conditions can found in the problems of mathematical physics [6]. We consider a boundary value problem of the following form:

$$y^{''}(t) + \lambda^2 y(t) + M(t)y(t - \Delta(t)) = 0 \tag{1}$$

$$\lambda y(0) + y^{'}(0) = 0 \tag{2}$$

$$\lambda^2 y(\pi) + y^{'}(\pi) = 0 \tag{3}$$

$$y(t - \Delta(t)) = y(0)\varphi(t - \Delta(t)), \qquad t - \Delta(t) < 0, \tag{4}$$

where M(t) and $\Delta(t) \geq 0$ are continous functions on $[0, \pi]$, $\lambda > 0$ is a real parameter, $\varphi(t)$ is an initial function which is satisfied with the condition $\varphi(0) = 1$ and continuous in the initial set $E_0 = \{t - \tau(t) : t - \tau(t) < 0, t > 0\} \bigcup \{0\}$.

1. Let $\omega(t, \lambda)$ be a solution of Eq. (1) and satisfy condition (4) and the following conditions

$$\omega(0, \lambda) = 1, \qquad \omega^{'}(0, \lambda) = -\lambda. \tag{5}$$

According to Theorem 1.2.1 in [5], it can be shown that there is a unique solution to Eq. (1) defined on $[0, \pi]$ and satisfied with initial conditions (4), (5).

If we change Eq. (1) and initial conditions (4), (5) to an equivalent integral equation, we obtain

$$\omega(t, \lambda) = \cos \lambda t - \sin \lambda t - \frac{1}{\lambda} \int_0^t M(\tau) \sin \lambda(t - \tau)\omega(\tau - \Delta(\tau), \lambda)d\tau \,. \tag{6}$$

We now consider that $\omega(\tau - \Delta(\tau), \lambda) \equiv \varphi(\tau - \Delta(\tau))$ while $\tau - \Delta(\tau) < 0$ in the integration operation according to (4).

We thus have the following theorem.

Theorem 1: The boundary value problem (1)–(4) can only have simple eigenvalues.

Proof : Let $\tilde{\lambda}$ be an eigenvalue of the problem (1)–(4) and $\tilde{y}(t,\tilde{\lambda})$ be an eigenfunction corresponding to this eigenvalue. According to (2) and (5), we have the following equality:

$$W(\tilde{y}(0,\tilde{\lambda}),\omega(0,\tilde{\lambda})) = \begin{vmatrix} \tilde{y}(0,\tilde{\lambda}) & 1 \\ \tilde{y}^{'}(0,\tilde{\lambda}) & -\lambda \end{vmatrix} = 0.$$

According to theorem 2.2.2 given in [5], $\tilde{y}(t,\tilde{\lambda})$ and $\omega(t,\tilde{\lambda})$ are linear dependent on $[0,\pi]$. Hence, we have the results that $\omega(t,\tilde{\lambda})$ is an eigenfunction of (1)–(4) and the eigenfunctions corresponding to $\tilde{\lambda}$ are linearly dependent to each others. Thus $\omega(t,\lambda)$ not only satisfies the boundary condition at the left point but is also a nontrivial solution of Eq. (1). Using $\omega(t,\lambda)$ in (3), we obtain the characteristic of Eq. (7):

$$F(\lambda) \equiv \lambda^2\omega(\pi,\lambda) + \omega^{'}(\pi,\lambda) = 0. \tag{7}$$

According to Theorem 1, eigenvalues set of the boundary value problem (1)–(4) coincide with the set of real roots of Eq. (7).

The following representations are assumed

$$\Delta_0 = \max_{t\in[0,\pi]} \Delta(t), \qquad M_\pi = \int_0^\pi |M(\tau)|d\tau \ ,$$

and $\varphi(t)$ is extended to interval $[-\Delta_0, 0]$ continuously and assumed that

$$\varphi_0 = \max_{t\in[-\Delta_0,0]} |\varphi(t)|.$$

Lemma 1: Let $\lambda \geq 2M_\pi$. Then, for solution $\omega(t,\lambda)$ of Eq.(6),

$$|\omega(t,\lambda)| \leq \max\{2\sqrt{2};\varphi_0\}, \qquad (-\Delta_0 \leq t \leq \pi) \tag{8}$$

is satisfied.

Proof : Let $B_\lambda = \max|\omega(t,\lambda)|$. In that case, from (6) according to (4) for every $\lambda > 0$, one of the following inequalities is provided :

$$B_\lambda \leq \sqrt{2} + \frac{1}{\lambda} B_\lambda M_\pi$$
$$B_\lambda \leq \sqrt{2} + \frac{1}{\lambda} \varphi_0 M_\pi$$

While $\lambda \geq 2M_\pi$ for both of two inequalities, we obtain the following inequality.

$$B_\lambda \leq \max\{2\sqrt{2}; \varphi_0\}.$$

Then, according to (4), inequality (8) is obtained.

Theorem 2 : The boundary value problem (1)–(4) has infinite number of positive eigenvalues.

Proof : If we derive expression (6) with respect to t, we obtain the following equality:

$$\begin{aligned} \omega^{'}(t,\lambda) &= -\lambda \sin \lambda t - \lambda \cos \lambda t \\ &- \int_0^t M(\tau) \cos \lambda (t-\tau) \omega(\tau - \Delta(\tau), \lambda) d\tau. \end{aligned} \tag{9}$$

Using (6) and (9) in (7), we obtain the equation

$$\begin{aligned} \lambda^2 \Big(\cos \lambda \pi &- \sin \lambda \pi - \frac{1}{\lambda} \int_0^\pi M(\tau) \sin \lambda (\pi - \tau) \omega(\tau - \Delta(\tau), \lambda) d\tau \Big) \\ &- \lambda \sin \lambda \pi - \lambda \cos \lambda \pi \\ &- \int_0^\pi M(\tau) \cos \lambda (\pi - \tau) \omega(\tau - \Delta(\tau), \lambda) d\tau = 0. \end{aligned} \tag{10}$$

If both of sides of this equation are divided by λ, then

$$\lambda(\cos \lambda\pi - \sin \lambda\pi) + O(1) = 0$$

or

$$\lambda \sin(\frac{\pi}{4} - \lambda\pi) + O(1) = 0 \tag{11}$$

is obtained. Denoting by $\lambda = \mu - \frac{1}{4}$, Eq. (11) can be written as

$$\mu \sin \pi\mu + O(1) = 0. \tag{12}$$

It is obvious that Eq. (12) has infinite number of roots at large values of μ (see [4] or [5]).

2. Let us examine the asymptotic behaviour of positive eigenvalues and corresponding eigenfunctions. It is assumed that parameter λ is a big enough. According to (8),

$$\omega(t, \lambda) = O(1) \tag{13}$$

is provided in interval $[-\Delta_0, \pi]$. According to a theorem related to the derivative with respect to parameter (Theorem 1.4.1 in [5]), while $|\lambda| < \infty$, $\omega'_\lambda(t, \lambda)$ and its continuity exist in $0 \leq t \leq \pi$. In $[-\Delta_0, 0]$, $-\Delta_0 \leq t \leq 0$, $\omega(t, \lambda) \equiv \varphi(t)$ and $\omega'_\lambda(t, \lambda) \equiv 0$ while λ is arbitrary.

Lemma 2: The following equality is satified in $[-\Delta_0, \pi]$, and

$$\omega'_\lambda(t, \lambda) = O(1). \tag{14}$$

Proof: Let us derive the Eq. (6) with respect to λ and consider Eq. (13). Therefore

$$\begin{aligned} \omega'_\lambda(t, \lambda) &= -t \sin \lambda t - t \cos \lambda t \\ &+ \frac{1}{\lambda^2} \int_0^t M(\tau) \sin \lambda(t-\tau) \omega(\tau - \Delta(\tau), \lambda) d\tau \\ &- \frac{1}{\lambda} \int_0^t M(\tau) \sin \lambda(t-\tau) \omega'_\lambda(\tau - \Delta(\tau), \lambda) d\tau \\ &- \frac{1}{\lambda} \int_0^t (t-\tau) M(\tau) \cos \lambda(t-\tau) \omega(\tau - \Delta(\tau), \lambda) d\tau \end{aligned}$$

$$\omega_\lambda{}'(t, \lambda) = -\frac{1}{\lambda} \int_0^t M(\tau) \sin \lambda(t-\tau) \omega'_\lambda(\tau - \Delta(\tau), \lambda) d\tau + K(t, \lambda) \tag{15}$$

$(|K(t,\lambda)| \leq K_0, \quad K_0 = \text{const.})$ is obtained. Let $C_\lambda = \max\limits_{t\in[-\Delta_0,\pi]} |\omega_\lambda^{'}(t,\lambda)|$. That C_λ exists is shown by the derivative being continuous in $[-\Delta_0, \pi]$. From (15), we then obtain that

$$C_\lambda \leq \frac{1}{\lambda} M_\pi C_\lambda + K_0.$$

Now assume $\lambda > 2M_\pi$. Hence $C_\lambda \leq 2K_0$ and it is obvious that the asymtotic statement (14) is satisfied. n is assumed to be a large enough natural number. If inequality $|n-\lambda| < \frac{1}{4}$ is satisfied, then we say the number μ^2 is in proximity to the number n^2.

Theorem 3: Let n be a natural number. Having large values of n the boundary value problem (1)–(4) has only unique eigenvalue in proximity of number n^2.

Proof: Consider the expression

$$\begin{aligned}
- \quad & \frac{1}{\lambda^2}\int_0^\pi M(\tau)\sin\lambda(\pi-\tau)\omega(\tau-\Delta(\tau),\lambda)d\tau - \sin\lambda\pi - \cos\lambda\pi - \\
- \quad & \frac{1}{\lambda}\int_0^\pi M(\tau)\cos\lambda(\pi-\tau)\omega(\tau-\Delta(\tau),\lambda)d\tau
\end{aligned}$$

that is indicated by O(1) in (11). Differentiating the formulas (13), (14) with respect to λ, it can be shown that the derivative of last expression is bounded at large values of λ . It is directly seen that the roots of Eq. (11) are in proximity of integers at large values of μ. We now show that Eq. (11) has only unique roots in proximity of n at large values of n.

Consider the function

$$F(\mu) = \mu\sin\mu\pi + O(1).$$

The derivative of this function

$$F^{'}(\mu) = \sin\mu\pi + \mu\pi\cos\mu\pi + O(1)$$

is not equal to zero at large values of n and μ is in proximity of n. Therefore, according to Rolle Thorem, we have shown that Eq. (11) has only unique root in proximity of n at large values of n.

Equation (11) enables one to obtain the asymptotic formula for eigenvalues of boundary value problem (1)–(4).

We assume n is big enough and then let denote eigenvalues in proximity of n^2 of boundary value problem (1)–(4) with $\lambda_n = (\mu_n - \frac{1}{4})^2$. Substituting $\mu_n = n + \delta_n$ in expression $\mu \sin \mu\pi + O(1) = 0$, we have

$$(n+\delta_n)|\sin(n+\delta_n)\pi| = (n+\delta_n)|\sin \delta_n \pi| = O(1).$$

Then $\sin \delta_n \pi = O(\frac{1}{n})$ and thus $\delta_n = O(\frac{1}{n})$ is obtained for large values of n. Therefore,

$$\mu_n = n + O(\frac{1}{n}). \tag{16}$$

Formula (16) also enables one to find asymptotic behaviour of the eigenfunctions of boundary value problem (1)–(4). From (6), according to (13), formula

$$\omega(t,\lambda) = \sqrt{2}\sin\left(\frac{\pi}{4} - \lambda t\right) + O(\frac{1}{\lambda^2}) \tag{17}$$

is obtained. It is seen that obtained asymptotic formula coincide with asymptotic behaviour of the eigenvalues and the eigenfunctions of classic Sturm-Liouville problem[4].

3. Under some addition conditions, we will obtain more certain asymptotic expressions which depends on delaying.

The following lemma can be proved.

Lemma 3: Assume that the derivative functions $M^{'}(t)$ and $\Delta^{''}(t)$ exist, are bounded and $\Delta^{'}(t) \leq h < 2$. Then the following equations are satisfied:

$$\int_0^t M(\tau)\cos\lambda(2\tau - \Delta(\tau))d\tau = O(\frac{1}{\lambda})$$

$$\int_0^t M(\tau)\sin\lambda(2\tau - \Delta(\tau))d\tau = O(\frac{1}{\lambda}),$$

while $0 \leq t \leq \pi$.

Let $\Delta^{'}(t) \leq 1$ and $\Delta(0) = 0$. Therefore, the inequality

$$t - \Delta(t) \geq 0, \qquad (0 \leq t \leq \pi) \tag{18}$$

is obvious. According to (17) and (18), on $[0, \pi]$ we have

$$\omega(\tau - \Delta(\tau), \lambda) = \sqrt{2} \sin\big(\frac{\pi}{4} - \lambda(\tau - \Delta(\tau))\big) + O(\frac{1}{\lambda^2}). \tag{19}$$

Substitute this expression into (10), we then get

$$\begin{aligned}
& \lambda(\cos \lambda\pi - \sin \lambda\pi) - \sin \lambda\pi \int_0^\pi M(\tau) \cos \lambda\tau [\cos \lambda(\tau - \Delta(\tau)) \\
- & \sin \lambda(\tau - \Delta(\tau))] d\tau \\
+ & \cos \lambda\pi \int_0^\pi M(\tau) \sin \lambda\tau [\cos \lambda(\tau - \Delta(\tau)) - \sin \lambda(\tau - \Delta(\tau))] d\tau \\
- & \sin \lambda\pi - \cos \lambda\pi - \frac{\cos \lambda\pi}{\lambda} \int_0^\pi M(\tau) \cos \lambda\tau [\cos \lambda(\tau - \Delta(\tau)) \\
- & \sin \lambda(\tau - \Delta(\tau))] d\tau \\
- & \frac{\sin \lambda\pi}{\lambda} \int_0^\pi M(\tau) \sin \lambda\tau [\cos \lambda(\tau - \Delta(\tau)) \\
- & \sin \lambda(\tau - \Delta(\tau))] d\tau + O(\frac{1}{\lambda^2}) = 0.
\end{aligned} \tag{20}$$

Substitute the following identities into (20)

$$\begin{aligned}
\cos \lambda\tau \cos \lambda(\tau - \Delta(\tau)) &= \frac{1}{2}[\cos \lambda\Delta(\tau) + \cos \lambda(2\tau - \Delta(\tau))] \\
\sin \lambda\tau \cos \lambda(\tau - \Delta(\tau)) &= \frac{1}{2}[\sin \lambda\Delta(\tau) + \sin \lambda(2\tau - \Delta(\tau))] \\
\sin \lambda\tau \sin \lambda(\tau - \Delta(\tau)) &= \frac{1}{2}[\cos \lambda\Delta(\tau) - \cos \lambda(2\tau - \Delta(\tau))] \\
\cos \lambda\tau \sin \lambda(\tau - \Delta(\tau)) &= \frac{1}{2}[\sin \lambda(2\tau - \Delta(\tau)) - \sin \lambda\Delta(\tau)]
\end{aligned}$$

and denote

$$\left.\begin{aligned} A(t,\lambda,\Delta(\tau)) &= \tfrac{1}{2}\int\limits_0^t M(\tau)\sin\lambda\Delta(\tau)d\tau \\ B(t,\lambda,\Delta(\tau)) &= \tfrac{1}{2}\int\limits_0^t M(\tau)\cos\lambda\Delta(\tau)d\tau \end{aligned}\right\} \tag{21}$$

It is obvious that the functions $A(t,\lambda,\Delta(\tau))$ and $B(t,\lambda,\Delta(\tau))$ are bounded in $0 \le t \le \pi$, $0 < \lambda < \infty$. Considering Lemma 3, and with some additional processes, we obtain the following equality:

$$\sin\big(\frac{\pi}{4} - \lambda\pi\big)(\lambda + A(\pi,\lambda,\Delta(\tau)) - \cos\big(\frac{\pi}{4} - \lambda\pi\big)[B(\pi,\lambda,\Delta(\tau)) + 1] = O(\frac{1}{\lambda})$$

Here, if we denote $\lambda = \frac{1}{4} + \mu$, we have

$$tg\mu\pi = \frac{1}{\frac{1}{4}+\mu}B(\pi,\frac{1}{4}+\mu,\Delta(\tau)) + O(\frac{1}{\mu^2}).$$

According to (16), assuming $\mu_n = n + \delta_n$, the following equality is obtained:

$$tg(n+\delta_n)\pi = tg\delta_n\pi = \frac{-1}{n}\Big[B(\pi,\frac{1}{4}+n,\Delta(\tau)) + 1\Big] + O(\frac{1}{n^2}).$$

Hence, for large values of n

$$\delta_n = \frac{-1}{n\pi}\Big[B(\pi,\frac{1}{4}+n,\Delta(\tau)) + 1\Big] + O(\frac{1}{n^2}).$$

Consequently the following asymptotic formula are obtained:

$$\mu_n = n - \frac{1}{n\pi}\Big[B(\pi,\frac{1}{4}+n,\Delta(\tau)) + 1\Big] + O(\frac{1}{n^2}) \tag{22}$$

$$\lambda_n = \frac{1}{4} + n + \frac{1}{n\pi}\Big[B(\pi,\frac{1}{4}+n,\Delta(\tau)) + 1\Big] + O(\frac{1}{n^2}). \tag{23}$$

Now let us find the certain asymptotic expression for the eigenfunctions. According to (19), from (6), the following expressions can be written:

$$\begin{aligned}\omega(t,\lambda) = \cos\lambda t - \sin\lambda t \quad &- \quad \frac{1}{\lambda}\int_0^t M(\tau)\sin\lambda(t-\tau)[\cos\lambda(\tau-\Delta(\tau)) \\ &- \quad \sin\lambda(\tau-\Delta(\tau))]d\tau + O(\frac{1}{\lambda^2})\end{aligned}$$

Note that

$$\sin\lambda(t-\tau)\cos\lambda(t-\tau) = \tfrac{1}{2}[\sin\lambda(t-\Delta(\tau)) + \sin\lambda(t-(2\tau-\Delta(\tau)))]$$
$$\sin\lambda(t-\tau)\sin\lambda(t-\Delta(\tau)) = \tfrac{1}{2}[\cos\lambda(t-(2\tau-\Delta(\tau))) - \cos\lambda(t-\Delta(\tau))].$$

From here, considering Eq. (21), Lemma 3 and doing some operations, we have

$$\begin{aligned}\omega(t,\lambda) \quad &= \quad \sqrt{2}\sin(\frac{\pi}{4}-\lambda t)\Big(1+\frac{1}{\lambda}B(t,\lambda,\Delta(\tau))\Big) \\ &+ \quad \frac{\sqrt{2}}{\lambda}\cos(\frac{\pi}{4}-\lambda t)A(t,\lambda,\Delta(\tau)) + O(\frac{1}{\lambda^2}).\end{aligned} \tag{24}$$

If λ is changed with λ_n, from (24)

$$\begin{aligned}U_n(t) \quad &= \quad \omega(t,\lambda_n) = \sqrt{2}\sin[\frac{\pi}{4}-(\frac{1}{4}+n)t] \quad \left(1-\frac{B(t,\frac{1}{4}+n,\Delta(\tau))}{n}\right) \\ &- \quad \frac{\sqrt{2}}{n\pi}\cos[\frac{\pi}{4}-(\frac{1}{4}+n)t] \quad \Big[A(t,\frac{1}{4}+n,\Delta(\tau))\pi \\ &+ \quad B(\pi,\frac{1}{4}+n,\Delta(\tau))t\Big] + O(\frac{1}{n^2})\end{aligned} \tag{25}$$

is obtained. Thus the following main theorem was proved.

Theorem 4: If derivatives $M^{'}(t)$ and $M^{''}(t)$ exist and are bounded, and $\Delta^{'}(t) \leq 1$, $\Delta(0) = 0$ are satisfied, then the asymptotic statements (23) and (25) are satisfied for eigenvalues and eigenfunctions of boundary value problem (1)–(4).

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Mehmet BAYRAMOĞLU, Kevser ÖZDEN KÖKLÜ
Received 17.01.2002
Yıldız Technical University,
Mathematical Engineering,
Department, Davutpaşa, İstanbul-TURKEY,
e-mail: ozkoklu@yildiz.edu.tr
Oya BAYKAL
Yıldız Technical University,
Department of Mathematics,
Davutpaşa, İstanbul-TURKEY
e-mail: baykal@yildiz.edu.tr