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KO-groups of Bounded Flag Manifolds

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Abstract

We exhibit an appropriate suspension of bounded flag manifolds as a wedge sum of Thom complexes of associated complex line bundles. We use the existence of such a splitting to assist our computation of real and complex K-groups. Moreover, we compute the Sq^2-homology of bounded flag manifolds to make use of relevant Atiyah-Hirzebruch spectral sequence of KO-theory.

Key Words: Bounded flag manifolds, Sq^2-homology, KO-theory, toric variety, stably complex structure

1. Introduction

As explained by Buchstaber and Ray [2], the geometry of bounded flag manifolds plays an important role in complex cobordism, namely that they generate the double cobordism ring Ω_*^{DU}. These objects were originally constructed by Bott and Samelson, and were introduced into complex cobordism by Ray [7].

Bounded flag manifolds also fit into the settings of toric geometry. We showed in [3] that they are smooth projective toric varieties associated to fans arising from crosspolytopes.

By analogy with many stable splitting phenomena discovered in the 80s, we will carry out a programme of exhibiting an appropriate suspension of bounded flag manifolds as a wedge sum of Thom complexes of associated complex line bundles. We then use the existence of such a splitting to assist our computation of real and complex K-groups. More generally, Bahri and Bendersky[1] have announced a method for computing KO-groups of any toric manifold via the relevant Adam spectral sequence. Our first step overlaps with theirs in that we compute the Sq^2-homology of bounded flag manifolds.

We begin with introducing some notations. We follow combinatorial convention by writing $[n]$ for the set of natural numbers $\{1, 2, \ldots, n\}$, and an interval in the poset $[n]$

has the form $[a, b]$ for some $1 \leq a \leq b \leq n$ which consists of all k satisfying $a \leq k \leq b$. Throughout, $\omega_1, \ldots, \omega_{n+1}$ will denote the standard basis vectors in $\mathbb{C}^{n+1}$, and we write $\mathbb{C}_I$ and $\mathbb{C}P_I$ for the subspace spanned by the vectors $\{\omega_i : i \in I\}$ and the projectivization of $\mathbb{C}_I$ respectively, where $I \subset [n+1]$.

Definition 1.1 *A flag $U: 0 < U_1 < \ldots < U_n < \mathbb{C}^{n+1}$ is called* bounded *if $\mathbb{C}_{[i-1]} < U_i$ for each $1 \leq i \leq n$. The set of all bounded flags in $\mathbb{C}^{n+1}$ is called* bounded flag manifold, *which is an n-dimensional smooth complex manifold and will be denoted by $B(\mathbb{C}^{n+1})$ (or simply by B_n).*

As a consequence of the definition, each factor U_i of any bounded flag $U \in B(\mathbb{C}^{n+1})$ is of the form $\mathbb{C}_{[i-1]} \oplus L_i$, where L_i is a line in $\mathbb{C}_i \oplus L_{i+1}$ for $1 \leq i \leq n$, and $L_{n+1} = \mathbb{C}_{n+1}$. Therefore, we may display U as

$$U: 0 < L_1 < \mathbb{C}_1 \oplus L_2 < \ldots < \mathbb{C}_{[n-1]} \oplus L_n < \mathbb{C}^{n+1}. \tag{1.2}$$

We define maps q_i and $r_i : B(\mathbb{C}^{n+1}) \to \mathbb{C}P_{[i,n+1]}$ by letting $q_i(U) = L_i$ and $r_i(U) = L_i^{\perp}$, where $L_i^{\perp}$ is the orthogonal complement of L_i in $\mathbb{C}_i \oplus L_{i+1}$ for each $U \in B(\mathbb{C}^{n+1})$, and $1 \leq i \leq n$.

If $B(\mathbb{C}_{[n-k+1,n+1]})$ denotes the set bounded flags in $\mathbb{C}_{[n-k+1,n+1]}$, which we abbreviate to B_k, then there is a sequence of projections

$$B_n \xrightarrow{\pi_{n-1}} B_{n-1} \xrightarrow{\pi_{n-2}} \ldots \xrightarrow{\pi_2} B_2 \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_0} *,$$

each of which is the projection of a fiber bundle whose fibers isomorphic to $\mathbb{C}P^1$ and can be given for any $0 \leq k \leq n-1$ as follows: $\pi_k : B_k \to B_{k-1}$ maps each flag

$$U_k : 0 < L_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]}$$

in B_k to the flag

$$U_{k-1} : 0 < L_{n-k+2} < \mathbb{C}_{n-k+2} \oplus L_{n-k+3} < \ldots < \mathbb{C}_{[n-k+2,n-1]} \oplus L_n < \mathbb{C}_{[n-k+2,n+1]}$$

in B_{k-1}. There are two inclusions i_k^S and $i_k^N : B_{k-1} \to B_k$, which are given respectively by

$$i_k^S(U_{k-1}) := 0 < \mathbb{C}_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]},$$

and

$$i_k^N(U_{k-1}) := 0 < L_{n-k+2} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]}.$$

We consider complex line bundles η_i and $\eta_i^\perp$ over B_n, classified respectively by the maps q_{n-i+1} and r_{n-i+1} for every $1 \leq i \leq n$, and we set η_0 to be the trivial line bundle with fiber $\mathbb{C}_{n+1}$. We sometimes to refer them as the *associated line bundles* on B_n. It follows that

$$\eta_i \oplus \eta_i^\perp \oplus \eta_{i-1}^\perp \oplus \ldots \oplus \eta_1^\perp \cong \mathbb{C}_{[n-i,n+1]} \tag{1.3}$$

for every i. As detailed in [7], there is an isomorphism

$$\tau(B_n) \oplus \mathbb{R}^2 \cong \bigoplus_{i=0}^{n-1} \eta_i \oplus \mathbb{C}, \tag{1.4}$$

giving a stable complex structure on B_n. However, each B_n can be identified with the total space of the sphere bundle of $\eta_{n-1} \oplus \mathbb{R}$ over B_{n-1}, and the above U-structure extends over the associated 3-disk bundle; hence, B_n represents zero in the complex cobordism ring Ω_*^U.

We let $x_1, \ldots, x_n \in H^2(B_n; \mathbb{Z})$ denote the respective first Chern classes of $\eta_1, \ldots, \eta_n$.

Theorem 1.5 *[2] The integral cohomology ring $H^*(B_n)$ is generated by $x_1, \ldots, x_n$, and these are subject only to the relations $x_1^2 = 0$ and $x_i^2 = x_i x_{i-1}$ for each $2 \leq i \leq n$ and for all $n > 0$.*

2. Stable Splitting and Sq^2-Homology of Bounded Flags

Let $\xi = \{E, p, B, \mathbb{C}^n\}$ be an n-dimensional complex vector bundle over a CW-complex B. We let $D(\xi)$ denote the associated disk bundle consisting of vectors of length at most 1 in each fiber, while $S(\xi)$ denotes the associated sphere bundle. We then set

$$T\xi := D(\xi)/S(\xi).$$

The space $T\xi$ is called the *Thom space* or the *Thom complex* of ξ (see [9]). Alternatively, the Thom complex $T\xi$ can be constructed as

$$T\xi \cong \mathbb{C}P(\mathbb{C} \oplus \xi)/\mathbb{C}P(\xi), \tag{2.6}$$

where $\mathbb{C}P(\xi)$ is the space obtained by projectivizing each fiber. In particular, if ξ is a line bundle, it then follows that $\mathbb{C}P(\xi)$ is homeomorphic to the base space B so that $T\xi \cong \mathbb{C}P(\mathbb{C} \oplus \xi)/B$. Furthermore, the *Thom class* of ξ is defined to be the element (up to sign) $t \in H^n(D(\xi), S(\xi))$ such that $j^*(t)$ is a generator of $H^n(D^n, S^{n-1})$, where $j\colon (D^n, S^{n-1}) \to (D(\xi), S(\xi))$ is the inclusion of the fiber over some point. In this way, we obtain the *Thom isomorphism*

$$\Phi^*\colon H^i(B) \to H^{i+n}(D(\xi), S(\xi)), \quad \Phi^*(z) := p^*(z) \cup t \quad \text{for all } i \in \mathbb{Z}.$$

In the case of bounded flag manifolds, it follows from (2.6) that the Thom complex $T\eta_{k-1}$ of each η_{k-1} is of the form

$$T\eta_{k-1} \cong \mathbb{C}P(\mathbb{C} \oplus \eta_{k-1})/\mathbb{C}P(\eta_{k-1}) \cong B_k/B_{k-1}. \tag{2.7}$$

Therefore, there is a cofibre sequence

$$B_{k-1} \xrightarrow{i_k} B_k \xrightarrow{q_k} T\eta_{k-1}, \tag{2.8}$$

where $i_k\colon B_{k-1} \to B_k$ is either of the inclusions i_k^S or i_k^N, and $q_k\colon B_k \to T\eta_{k-1}$ is the quotient map. We then have a short exact sequence:

$$0 \to H^{2j}(T\eta_{k-1}) \xrightarrow{q_k^*} H^{2j}(B_k) \underset{i_k^*}{\overset{\pi_k^*}{\leftrightarrows}} H^{2j}(B_{k-1}) \to 0, \tag{2.9}$$

for each $1 \leq j \leq k-1$ and

$$0 \to H^{2k}(T\eta_{k-1}) \underset{\cong}{\xrightarrow{q_k^*}} H^{2k}(B_k) \to 0. \tag{2.10}$$

Since the composition $\pi_k \circ i_k$ is the identity map on B_{k-1}, where $\pi_k\colon B_k \to B_{k-1}$ is the projection map, then (2.9) splits as abelian groups, that is, the map

$$\Psi\colon H^{2j}(B_{k-1}) \oplus H^{2j}(T\eta_{k-1}) \to H^{2j}(B_k)$$

given by $\Psi := \pi_k^* + q_k^*$ is an isomorphism of free abelian groups.

Remark 2.11 *Of course, Ψ is not normally multiplicative. For example, recall that the Thom complex of the canonical line bundle $\eta_1 \to B_1 = \mathbb{C}P^1$ can be identified with $\mathbb{C}P^2$. Therefore, we have a cofibre sequence $B_1 \to B_2 \to \mathbb{C}P^2$, while it is obvious that the cohomology ring of B_2 is not isomorphic to that of the direct sum of B_1 and $\mathbb{C}P^2$.*

We now suspend (2.8), and consider the pinch map $\gamma\colon \Sigma B_k \to \Sigma B_k \vee \Sigma B_k$. We write ψ_k for the composite

$$\Sigma B_k \xrightarrow{\gamma} \Sigma B_k \vee \Sigma B_k \xrightarrow{\Sigma\pi_k \vee \Sigma q_k} \Sigma B_{k-1} \vee \Sigma T\eta_{k-1} \tag{2.12}$$

for each $1 \leq k \leq n$, and note that ψ_k^* is an isomorphism of cohomology groups

$$\psi_k^*\colon H^*(\Sigma B_{k-1}) \oplus H^*(\Sigma T\eta_{k-1}) \cong H^*(\Sigma B_k). \tag{2.13}$$

Theorem 2.14 *The map $\psi_k\colon \Sigma B_k \to \Sigma B_{k-1} \vee \Sigma T\eta_{k-1}$ is a homotopy equivalence.*

Proof. It easily follows from the definitions that the spaces ΣB_k and $\Sigma B_{k-1} \vee \Sigma T\eta_{k-1}$ are simply connected finite CW-complexes. Therefore, applying Whitehead's theorem [11] to (2.13), we obtain the desired result. □

Repeated application of Theorem 2.14 yields the following:

Theorem 2.15 *For each $1 \leq k \leq n$, there is a homotopy equivalence*

$$\Psi_k\colon \Sigma B_k \simeq \Sigma T\eta_0 \vee \ldots \vee \Sigma T\eta_{k-1}. \tag{2.16}$$

We note that, if $t_k \in H^2(T\eta_k;\mathbb{Z})$ is the Thom class, then it satisfies $t_k^2 = t_k c_1(\eta_k)$, and the class x_{k+1} is the pullback of t_k to $H^2(B_{k+1};\mathbb{Z})$ for any $1 \leq k \leq n$.

In order to compute KO-groups of bounded flag manifolds, we will make use of the relevant Atiyah-Hirzebruch spectral sequences. By a theorem of Thomas [10], some of the differentials in this spectral sequence can be related to Steenrod squares. Moreover, in our case, the E_3-term will turn out to be so-called Sq^2-homology of the bounded flag manifold. Therefore, to assist such computation, we will determine these homology groups in advance.

Since $H^*(B_n;\mathbb{Z}_2)$ is concentrated in even dimensions,

$$\ldots \to H^{2k-2}(B_n;\mathbb{Z}_2) \xrightarrow{Sq^2} H^{2k}(B_n;\mathbb{Z}_2) \to \ldots$$

is a chain complex because $Sq^2Sq^2 = Sq^3Sq^1 = 0$. The homology of this chain complex is said to be the Sq^2 *homology* of the bounded flag manifold and denoted by $H_*(B_n;Sq^2)$. Our main task is now to prove the following theorem.

Theorem 2.17 *The homology group $H_{2k}(B_n;Sq^2)$ is trivial for all $n \geq 1$ and $k > 1$.*

We divide the proof of this theorem into several steps. Let $y_1, \dots, y_n$ be the generators of the group $H^2(B_n;\mathbb{Z}_2)$ so that they satisfy the relation

$$y_i^2 = y_i y_{i-1} \text{ for all } i = 2, \dots, n \text{ and } y_1^2 = 0. \tag{2.18}$$

If $y_{i_1} \dots y_{i_k}$ is a monomial in $H^{2k}(B_n;\mathbb{Z}_2)$, we denote it simply by y_I, where $I = \{i_1, \dots, i_k\}$. In this way, we get a bijection between the non-zero monomials in $H^{2k}(B_n;\mathbb{Z}_2)$ and the elements of the set $\mathcal{D}_k^n$ consisting of all subsets of $[n]$ with k-elements. Furthermore, we denote by $\mathcal{C}_k^n$, the $\mathbb{Z}_2$-vector space generated by the set $\mathcal{D}_k^n$; hence, $\mathcal{C}_k^n$ is an isomorphic copy of $H^{2k}(B_n;\mathbb{Z}_2)$. The idea behind replacing $H^{2k}(B_n;\mathbb{Z}_2)$ with $\mathcal{C}_k^n$ is just to simplify the notation.

Each non-empty set $I \in \mathcal{D}_k^n$ can be uniquely written as

$$I = [a_1, b_1] \cup \dots \cup [a_t, b_t], \tag{2.19}$$

where $b_{i-1} + 1 < a_i$ for any $2 \leq i \leq t$. For any $k > 1$, we define a map $\mathcal{S}q^2 \colon \mathcal{C}_{k-1}^n \to \mathcal{C}_k^n$ by

$$\mathcal{S}q^2(I) := \sum_{i=1}^{t} (b_i - a_i + 1)[a_1, b_1] \cup \dots \cup [a_i - 1, b_i] \cup \dots \cup [a_t, b_t] \pmod 2 \tag{2.20}$$

for each $I \in \mathcal{D}_{k-1}^n$ with the conventions that

- if $a_1 = 1$, then the first term in the sum is deleted,
- $\mathcal{S}q^2$ maps the empty set to itself,

and for an arbitrary sum $I_1 + \dots + I_l \in \mathcal{C}_{k-1}^n$, we insist that

$$\mathcal{S}q^2(I_1 + \dots + I_l) = \mathcal{S}q^2(I_1) + \dots + \mathcal{S}q^2(I_l).$$

Example 2.21 *Consider the set $I = \{1, 3, 4, 7, 9\} \in \mathcal{D}_5^n$ for some $n \geq 9$. Then, we may write I as $I = [1, 1] \cup [3, 4] \cup [7, 7] \cup [9, 9]$, and by definition,*

$$\begin{aligned} \mathcal{S}q^2(I) =& 2[1, 1] \cup [2, 4] \cup [7, 7] \cup [9, 9] + [1, 1] \cup [3, 4] \cup [6, 7] \cup [9, 9] \\ &+ [1, 1] \cup [3, 4] \cup [7, 7] \cup [8, 9] \\ =& [1, 1] \cup [3, 4] \cup [6, 7] \cup [9, 9] + [1, 1] \cup [3, 4] \cup [7, 9] \\ =& \{1, 3, 4, 6, 7, 9\} + \{1, 3, 4, 7, 8, 9\}. \end{aligned}$$

Definition 2.22 *If $y_{i_1}\dots y_{i_k}$ is a monomial in $H^{2k}(B_n;\mathbb{Z}_2)$ such that $\mathcal{S}q^2(I) = I_1 + \dots + I_s$, then we define $y_{\mathcal{S}q^2(I)} := y_{I_1} + \dots + y_{I_s}$.*

Lemma 2.23 *For any $I = [a_1,b_1] \cup \dots \cup [a_t,b_t]$, the Steenrod square Sq^2 maps the monomial y_I to $y_{\mathcal{S}q^2(I)}$, i.e. $Sq^2(y_I) = y_{\mathcal{S}q^2(I)}$.*

Proof. We will proceed by induction on t. When $t = 1$, let $I = [i, i+k-1] \in \mathcal{D}^n_k$ for some $n \geq 1$ and $k > 1$. Then, by using the relation (2.18) together with the Cartan formula, we have

$$Sq^2(y_I) = y_{[i-1,i+k-1]} = y_{\mathcal{S}q^2(I)}.$$

Assume that the claim holds for $t-1$ so that $Sq^2(y_J) = y_{\mathcal{S}q^2(J)}$ for all $J = [c_1,d_1] \cup \dots \cup [c_{t-1}, d_{t-1}]$. For a given $I = [a_1,b_1] \cup \dots \cup [a_t,b_t] \in \mathcal{D}^n_k$, we define $J := I \backslash [a_t,b_t]$. Now, it is easy to verify that if $\mathcal{S}q^2(J) = \sum_{j=1}^{t-1} J_j$, then

$$\mathcal{S}q^2(I) = J_1 \cup [a_t,b_t] + \dots + J_{t-1} \cup [a_t,b_t] + (b_t - a_t + 1)J \cup [a_t - 1, b_t].$$

Then, it follows from the induction assumption that

$$\begin{aligned} Sq^2(y_I) &= Sq^2(y_J \cdot y_{[a_t,b_t]}) \\ &= Sq^2(y_J) \cdot y_{[a_t,b_t]} + y_J \cdot Sq^2(y_{[a_t,b_t]}) \\ &= y_{\mathcal{S}q^2(J)} \cdot y_{[a_t,b_t]} + y_J \cdot y_{\mathcal{S}q^2([a_t,b_t])} \\ &= (y_{J_1} \cdot y_{[a_t,b_t]} + \dots + y_{J_{t-1}} \cdot y_{[a_t,b_t]}) + y_J \cdot y_{\mathcal{S}q^2([a_t,b_t])} \\ &= y_{\mathcal{S}q^2(I)}. \end{aligned}$$

□

Proposition 2.24 *Let $J \in \mathcal{D}^n_k$ be given, where $k > 1$. If $\mathcal{S}q^2(J) = \emptyset$, then there exists $I \in \mathcal{D}^n_{k-1}$ such that $\mathcal{S}q^2(I) = J$.*

Proof. Assume that $\mathcal{S}q^2(J) = \emptyset$ for some $J \in \mathcal{D}^n_k$, where $J = [a_1,b_1] \cup \dots \cup [a_t,b_t]$. Then, it follows from the definition of $\mathcal{S}q^2$ that either

(i) $a_1 = 1$ and $b_j - a_j$ is odd for all $j = 2,\dots,t$, or

(ii) $a_1 > 1$ and $b_j - a_j$ is odd for all $j = 1,\dots,t$,

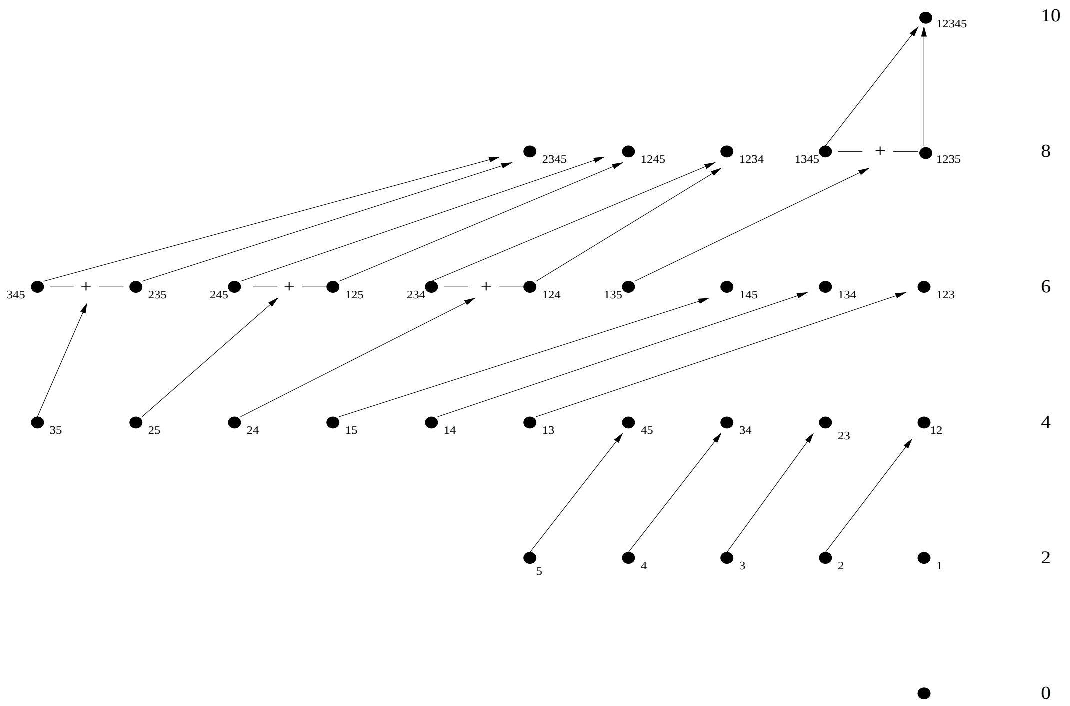

Figure 1. Sq^2 connections for B_5

According to above situations, if we define $I \in \mathcal{D}^n_{k-1}$ by either

(i) $I := [a_1, b_1] \cup \ldots \cup [a_j, b_j] \backslash \{b_j - 1\} \cup \ldots [a_t, b_t]$ for some $j = 2, \ldots, t$, or

(ii) $I := [a_1, b_1] \cup \ldots \cup [a_j, b_j] \backslash \{b_j - 1\} \cup \ldots [a_t, b_t]$ for some $j = 1, \ldots, t$,

respectively, it is easy to verify that $\mathcal{S}q^2(I) = J$. □

Example 2.25 *Let $J \in \mathcal{D}^n_7$ be given by $J = \{1, 3, 4, 5, 6, 8, 9\}$ for some $n \geq 9$ which we can write as $J = [1, 1] \cup [3, 6] \cup [8, 9]$. Then,*

$$\begin{aligned}\mathcal{S}q^2(J) =& (3 + 1)[1, 1] \cup [2, 6] \cup [8, 9] + (1 + 1)[1, 1] \cup [3, 6] \cup [7, 9] \\ =& \varnothing.\end{aligned}$$

Following the proof of Proposition 2.24, the image of $I_1 := [1,1] \cup [3,4] \cup [6,6] \cup [8,9]$ *or* $I_2 := [1,1] \cup [3,6] \cup [9,9]$ *is exactly* J.

If I, J are two elements in $\mathcal{D}_k^n$, then we define $d(I,J)$, "*the difference of* I *by* J", to be the integer $|(I \backslash I \cap J)|$.

Proposition 2.26 *Given* $\sum_{j\in\mathcal{A}} J_j \in \mathcal{C}_k^n$*, where* $\mathcal{A}$ *is an arbitrary index set, such that*

$$\mathcal{S}q^2(\sum_{j\in\mathcal{A}} J_j) = \emptyset, \tag{2.27}$$

then there exist $\sum_{i\in\mathcal{B}} I_i \in \mathcal{C}_{k-1}^n$ *for which*

$$\mathcal{S}q^2(\sum_{i\in\mathcal{B}} I_i) = \sum_{j\in\mathcal{A}} J_j, \tag{2.28}$$

where $\mathcal{B} \subset \mathcal{A}$.

Proof. Firstly, by consideration of Proposition 2.24, we may assume in (2.27) that $\mathcal{S}q^2(J_j) \neq \emptyset$ for all $j \in \mathcal{A}$. The equation (2.27) implies that

$$\sum_{j\in\mathcal{A}} \mathcal{S}q^2(J_j) = \emptyset.$$

Suppose that $\mathcal{S}q^2(J_j) = \sum_{s=1}^{t_j} J_{j,s}$ for each $j \in \mathcal{A}$, where we write the sum over all (j,s) such that $J_{j,s} \neq \emptyset$. Therefore

$$\sum_{j\in\mathcal{A}} \mathcal{S}q^2(J_j) = \sum_{j\in\mathcal{A}} \sum_{s=1}^{t_j} J_{j,s} = \emptyset.$$

If we define $\mathcal{U}(j,s) := \{(e,f)\colon J_{j,s} = J_{e,f}\}$ for any $j \in \mathcal{A}$ and $1 \leq s \leq t_j$, then it follows that the number of elements in $\mathcal{U}(j,s)$ must be even, since we are working over $\mathbb{Z}_2$. It is also clear from the definitions that if $(e,f) \in \mathcal{U}(j,s)\backslash\{(j,s)\}$, then $d(J_j, J_e) = 1$. Pick any (e,f) in $\mathcal{U}(j,s)$ different than (j,s) and define d_j and d_e to be the elements of J_j and J_e respectively such that $d_j, d_e \notin J_j \cap J_e$, and set

$$\mathcal{U}(j) := \{e \in \mathcal{A}\colon (e,f) \in \mathcal{U}(j,s) \quad \text{for some} \quad s \text{ and } f\}.$$

Now, let $I_j := J_j \backslash \{d_j\}$, then $\mathcal{S}q^2(I_j) = \sum_{e\in\mathcal{U}(j)} J_e$, which may be obtained easily from the facts that if $I_j = [a_1, b_1] \cup \ldots \cup [a_t, b_t]$, then we have $d_e < b_t$ and $I_j \cup \{d_e\} = J_e$

for all $e \in \mathcal{U}(j)$. Finally, we define $\mathcal{B}$ to be the subset of $\mathcal{A}$ for which $j, e \in \mathcal{B}$ whenever $\mathcal{U}(j) \cap \mathcal{U}(e) = \emptyset$. Thus,

$$\mathcal{S}q^2(\sum_{i\in\mathcal{B}} I_i) = \sum_{i\in\mathcal{B}} \mathcal{S}q^2(I_i) = \sum_{j\in\mathcal{A}} J_j,$$

from which we deduce Equation (2.28). This completes the proof. □

Example 2.29 *Let* $J_1 + \ldots + J_4 \in \mathcal{C}_5^n$ *be given as follows:* $J_1 = \{1,2,4,7,10\}$*,* $J_2 = \{2,3,4,7,10\}$*,* $J_3 = \{2,4,6,7,10\}$ *and* $J_4 = \{2,4,7,9,10\}$*. Then,*

$$\begin{aligned}
&\mathcal{S}q^2(J_1 + \ldots + J_4) = \mathcal{S}q^2(J_1) + \ldots + \mathcal{S}q^2(J_4)\\
&= [1,4]\cup[7,7]\cup[10,10] + [1,2]\cup[4,4]\cup[6,7]\cup[10,10] + [1,2]\cup[4,4]\cup[7,7]\cup[9,10]\\
&+ [1,4]\cup[7,7]\cup[10,10] + [2,4]\cup[6,7]\cup[10,10] + [2,4]\cup[7,7]\cup[9,10]\\
&+ [1,2]\cup[4,4]\cup[6,7]\cup[10,10] + [2,4]\cup[6,7]\cup[10,10] + [2,2]\cup[4,4]\cup[6,7]\cup[9,10]\\
&+ [1,2]\cup[4,4]\cup[7,7]\cup[9,10] + [2,4]\cup[7,7]\cup[9,10] + [2,2]\cup[4,4]\cup[6,7]\cup[9,10]\\
&= \emptyset.
\end{aligned}$$

We see that $\mathcal{U}(1) = \mathcal{U}(2) = \mathcal{U}(3) = \mathcal{U}(4) = \{1,2,3,4\}$*. Therefore, we define* $I := J_1\backslash\{1\}$*, which is equal to* $[2,2]\cup[4,4]\cup[7,7]\cup[10,10]$*, and*

$$\begin{aligned}
\mathcal{S}q^2(I) &= [1,2]\cup[4,4]\cup[7,7]\cup[10,10]\\
&+ [2,4]\cup[7,7]\cup[10,10]\\
&+ [2,2]\cup[4,4]\cup[6,7]\cup[10,10]\\
&+ [2,2]\cup[4,4]\cup[7,7]\cup[9,10]\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Proof. [***Proof of Theorem 2.17***] Combining Lemma 2.23, Propositions 2.24 and 2.26, we see that $ker(Sq^2) = Im(Sq^2)$ for any $k > 1$, which completes the proof. □

Theorem 2.17 also allows us to compute the Sq^2 homology of the Thom space of the associated line bundle η_i over B_i for any $i \geq 1$.

Corollary 2.30 *The group* $H_{2k}(T\eta_i; Sq^2)$ *is trivial for all* i *and* $k \geq 1$*.*

Proof. This follows from the existence of the homotopy equivalence

$$\psi_{i+1}\colon \Sigma B_{i+1} \to \Sigma B_i \vee \Sigma T\eta_i \tag{2.31}$$

for all $i \geq 1$ and $k > 1$. When $k = 1$, it is easy to see the map $Sq^2\colon H^2(T\eta_i;\mathbb{Z}_2) \to H^4(T\eta_i;\mathbb{Z}_2)$ is an injection. □

3. KO-Groups of Bounded Flags

Throughout, we will consider the spectra KO and K representing real and complex K-theory respectively. There are natural transformations: complexification $c\colon KO^*(X) \to K^*(X)$, realification $r\colon K^*(X) \to KO^*(X)$, and conjugation
$-\colon K^*(X) \to K^*(X)$. The formulas

$$r \cdot c = 2\colon KO^*(X) \to KO^*(X),$$
$$c \cdot r = 1 + -\colon K^*(X) \to K^*(X),$$

are well known, where c is a ring homomorphism, but r is not. The coefficient rings $K_*(S^0)$ and $KO_*(S^0)$ are given as follows.

$$KO_* \cong \mathbb{Z}[e, x, y, y^{-1}]/\{2e, e^3, e \cdot x, x^2 - 4y\} \quad \text{and} \quad K_* \cong \mathbb{Z}[z, z^{-1}], \tag{3.32}$$

where z is represented by the complex Hopf bundle over S^2, and e, x and y are represented by the real Hopf bundle over S^1, the symplectic Hopf bundle over S^4, and the canonical bundle over S^8, respectively.

We recall that $\eta_1, \ldots, \eta_n$ are the associated line bundles over B_n with the first Chern classes $x_1, \ldots, x_n$ respectively, described as in Section 1 . In order to compute the real and complex K-theory of B_n, we first recall Theorem 2.15, the stable splitting of B_n, which is possible after one-suspension:

$$B_n \simeq \bigvee_{i=0}^{n-1} T\eta_i. \tag{3.33}$$

It is important to note that in the above decomposition, we consider each η_i over B_i rather than B_n for every $1 \leq i \leq n-1$, while η_0 is the trivial line bundle over a point, whose Thom complex may be identified with $\mathbb{C}P^1$. If we recall the Thom isomorphism $\Phi\colon H^{2k}(B_i;\mathbb{Z}) \to H^{2k+2}(T\eta_i;\mathbb{Z})$ for $1 \leq k \leq n$ and the fact that B_i is a toric variety[3] arising from an i-crosspolytope, we see that the group $H^{2k+2}(T\eta_i;\mathbb{Z})$ is of rank h_k when $k \geq 1$, where $h = (h_0, \ldots, h_i)$ is the h-vector of the i-crosspolytope given by $h_k = \binom{i}{k}$ for any $0 \leq k \leq i$. Generators are given by $t_i x_I$, where $I \subseteq [i]$ is any subset of cardinality k .

Firstly, we compute the complex K-theory. Let γ_i be the element in $\widetilde{K}^2(B_n)$ such that $z\gamma_i = \eta_i - 1 \in \widetilde{K}^0(B_n)$ for each $1 \leq i \leq n$, and let θ_n be the line bundle over $T\eta_n$ such that $c_1(\theta_n) = t_n$ in $H^2(T\eta_n;\mathbb{Z})$; then $t_n^K = z^{-1}(\theta_n - 1)$ is the Thom class in $K^2(T\eta_n)$. Then the corresponding Thom isomorphism expresses $K^*(T\eta_n)$ as a free module over $K^*(B_n)$ on generators 1 and t_n^K.

Proposition 3.34 *The multiplicative structure of $K^*(T\eta_n)$ is determined by $(t_n^K)^2 = t_n^K\gamma_n$.*

Proof. The construction of t_n^K ensures that

$$ch(t_n^K) = (e^{ut_n} - 1)/u = t_n(e^{ux_n} - 1)/x_n u,$$

since $t_n^2 = t_n x_n$, where $H_*(K) \cong \mathbb{Q}[u, u^{-1}]$. Therefore,

$$\begin{aligned} ch((t_n^K)^2) = (ch(t_n^K))^2 &= (e^{ut_n} - 1) \cdot t_n(e^{ux_n} - 1)/x_n u^2, \\ &= (e^{ut_n} - 1) \cdot (e^{ux_n} - 1)/u^2, \\ &= ch(t_n^K) \cdot ch(\gamma_n). \end{aligned}$$

On the other hand, the Atiyah-Hirzebruch spectral sequence for $T\eta_n$ collapses, since $T\eta_n$ has a cell-decomposition concentrated in even dimensions; the Chern character ch is therefore monic and the result follows. □

Theorem 3.35 *For each $n \geq 1$, the complex K-theory $K^*(B_n)$ of bounded flag manifold is given by*

$$K^*(B_n) \cong K_*[\gamma_1, \ldots, \gamma_n]/(\gamma_i^2 - \gamma_i\gamma_{i-1},\ \ \gamma_1^2). \tag{3.36}$$

Proof. This follows from (3.33) and Proposition 3.34. □

In order to compute the KO-groups of B_n, we recall that the E_2 and E_∞ terms of the Atiyah-Hirzebruch spectral sequence of $\widetilde{K}O$-theory are given by

$$E_2^{p,q} \cong H^p(X; \widetilde{K}O_q(S^0)), \tag{3.37}$$

$$E_\infty^{p,q} \cong G_p\widetilde{K}O^{p+q}(X) = F_p^{p+q}(X)/F_{p+1}^{p+q}(X), \tag{3.38}$$

where $F_p^m(X) = \mathrm{Ker}[\widetilde{K}O^m(X) \to \widetilde{K}O^m(X^{p-1})]$ and X^{p-1} is the $(p-1)$-skeleton of X. As for the differentials $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$, it follows from Theorem 4.2 of [10] that

$$\begin{aligned} d_2^{p,-8t} &= Sq^2 \circ \rho\colon H^p(X;\mathbb{Z}) \to H^{p+2}(X;\mathbb{Z}_2), \\ d_2^{p,-8t-1} &= Sq^2\colon H^p(X;\mathbb{Z}_2) \to H^{p+2}(X;\mathbb{Z}_2), \end{aligned} \tag{3.39}$$

where ρ is the mod 2 reduction map.

Theorem 3.40 *The groups $\widetilde{K}O^{2j+1}(T\eta_i)$ are trivial for any $i \geq 0$ and $j \in \mathbb{Z}$, except for $\widetilde{K}O^{8j-7}(T\eta_0)$, which is isomorphic to $\mathbb{Z}_2$.*

Proof. For $\widetilde{K}O^{-1}(T\eta_i) \cong \widetilde{K}O^7(T\eta_i)$, the E_2-term of the spectral sequence is given by $E_2^{p+7,-p} \cong H^{p+7}(T\eta_i;\widetilde{K}O_p(S^0))$, which is trivial except for $p \equiv 1 \pmod 8$, where $-7 < p \leq 2i-5$. Assume that $p = 8t+1$ for some $t \geq 0$. Then, by (3.39), we can replace the sequence

$$\cdots \to E_2^{8t+6,-8t} \xrightarrow{d_2^{8t+6,-8t}} E_2^{8t+8,-8t-1} \xrightarrow{d_2^{8t+8,-8t-1}} E_2^{8t+10,-8t-2} \to \cdots$$

with

$$\cdots \to H^{8t+6}(T\eta_i;\mathbb{Z}) \xrightarrow{Sq^2\rho} H^{8t+8}(T\eta_i;\mathbb{Z}_2) \xrightarrow{Sq^2} H^{8t+10}(T\eta_i;\mathbb{Z}_2) \to \cdots$$

By Corollary 2.30, it follows that $E_3^{8t+8,-8t-1} \cong 0$; hence the spectral sequence collapses to the E_3-term, and the result follows. The proofs of the other cases are similar to that of $\widetilde{K}O^{-1}(T\eta_i)$. On the other hand, since we may identify $T\eta_o$ with $\mathbb{C}P^1$, it follows from (3.32) that $\widetilde{K}O^{-7}(T\eta_0) \cong \mathbb{Z}_2$. □

Corollary 3.41 *The groups $\widetilde{K}O^{2j+1}(B_n)$ are trivial for any $j \in \mathbb{Z}$, except for $\widetilde{K}O^{8j-7}(B_n)$, which is isomorphic to $\mathbb{Z}_2$.*

Proof. The splitting given by (3.33) induces an isomorphism of groups

$$\widetilde{K}O^{2j+1}(B_n) \cong \bigoplus_{i=0}^{n-1} \widetilde{K}O^{2j+1}(T\eta_i), \tag{3.42}$$

for $j \in \mathbb{Z}$. Thus, the claim follows from Theorem 3.40. □

Theorem 3.43 *The groups $\widetilde{KO}^{2j}(T\eta_i)$ are free of rank 2^{i-1} for any $i \geq 1$ and $j \in \mathbb{Z}$.*

Proof. To show that these groups are free, we consider the fibration $U \to U/O$, and from the fact that $BO \times \mathbb{Z} = \Omega(U/O)$, we obtain the exact Bott sequence for any CW-complex X in the form

$$\cdots \to \widetilde{KO}^k(X) \to \widetilde{K}^k(X) \to \widetilde{KO}^{k+2}(X) \to \widetilde{KO}^{k+1}(X) \to \cdots . \tag{3.44}$$

Applying the sequence (3.44) for $T\eta_i$ and $k = -1$, we get the exact sequence

$$\cdots \to \widetilde{K}^{-1}(T\eta_i) \to \widetilde{KO}^1(T\eta_i) \to \widetilde{KO}^0(T\eta_i) \to \widetilde{K}^0(T\eta_i) \to \widetilde{KO}^2(T\eta_i) \to \cdots . \tag{3.45}$$

By Theorem 3.40, the group $\widetilde{KO}^1(T\eta_i) \cong \widetilde{KO}^{-7}(T\eta_i)$ is trivial for any $i \geq 1$; hence, $\widetilde{KO}^0(T\eta_i)$ is a free group. A similar argument will apply to the other cases.

To find the rank of these groups, we consider the related Atiyah-Hirzebruch spectral sequences. For example, for $\widetilde{KO}^0(T\eta_i)$, we have $E_2^{p,-p} \cong H^p(T\eta_i; \widetilde{KO}_p(S^0))$, which is trivial except for $p \equiv 0, 2, 4 \pmod 8$, where $0 < p \leq 2i + 2$.

(i) Let $p = 8t$ for some $t \geq 1$. Then, from the sequence

$$\cdots \to E_2^{8t-2,-8t+1} \xrightarrow{d_2^{8t-2,-8t+1}} E_2^{8t,-8t} \xrightarrow{d_2^{8t,-8t}} E_2^{8t+2,-8t-1} \to \cdots , \tag{3.46}$$

we have that $E_2^{8t-2,-8t+1} \cong 0$, since $\widetilde{KO}_{8t-1}(S^0) \cong \widetilde{KO}_7(S^0) \cong 0$. By (3.39), we can replace (3.46) with the sequence

$$0 \to H^{8t}(T\eta_i; \mathbb{Z}) \xrightarrow{Sq^2 \circ \rho} H^{8t+2}(T\eta_i; \mathbb{Z}_2) \to \cdots ,$$

from which we obtain that

$$E_3^{8t,-8t} \cong \mathrm{Ker}[Sq^2\rho\colon H^{8t}(T\eta_i; \mathbb{Z}) \to H^{8t+2}(T\eta_i; \mathbb{Z}_2)].$$

Since the differential $d_k\colon E_k^{p,-p} \to E^{p+k,-p-k+1}$ (total degree 1) is a zero map (compare to Theorem 3.40) for any $k \geq 3$, the group $E_3^{8t,-8t}$ will survive to $E_\infty^{8t,-8t}$. Moreover, the group $E_\infty^{8t,-8t}$ is isomorphic to h_{4t-1} copies of $\mathbb{Z}$ for each $t \geq 1$, where $h = (h_0, \ldots , h_i)$ is the h-vector of the i-crosspolytope.

(ii) Let $p \equiv 2 \pmod 8$, then $p = 8t + 2$ for some $t \geq 0$. Then, from the sequence

$$\cdots \to E_2^{8t,-8t-1} \to E_2^{8t+2,-8t-2} \to E_2^{8t+4,-8t-3} \to \cdots ,$$

we see that $E_2^{8t+4,-8t-3} \cong 0$, and

$$E_3^{8t+2,-8t-2} \cong H^{8t+2}(T\eta_i;\mathbb{Z}_2)/\mathrm{Im}[Sq^2\colon H^{8t}(T\eta_i;\mathbb{Z}_2) \to H^{8t+2}(T\eta_i;\mathbb{Z}_2)],$$

which is a finite group for any $t \geq 0$.

(iii) Let $p \equiv 4 \pmod 8$, then $p = 8t+4$ for some $t \geq 1$. Then, from the sequence

$$\ldots \to E_2^{8t+2,-8t-3} \to E_2^{8t+4,-8t-4} \to E_2^{8r+6,-8t-5} \to \ldots,$$

we deduce that $E_3^{8t+4,-8r-4} \cong H^{8t+4}(T\eta_i;\mathbb{Z})$. Similar to the case (**i**), the group $E_3^{8t+4,-8t-4}$ will survive to E_∞ so that the group $E_\infty^{8t+4,-8t-4}$ is of rank h_{4t+1} for each $t \geq 1$.

As a conclusion, since all groups in our filtration are free, all the extension problems are trivial, and from the well-known formula

$$h_1 + h_3 + h_5 + \ldots = \binom{i}{1} + \binom{i}{3} + \binom{i}{5} + \ldots = 2^{i-1},$$

the group $\widetilde{KO}^0(T\eta_i)$ is isomorphic to $\mathbb{Z}^{2^{i-1}}$ for any $i \geq 1$. For the other cases, we can obtain the results in the same way as the proof of $j = 0$. □

Corollary 3.47 *(a)* $\widetilde{KO}^0(B_n) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^{2^{n-1}-1}$,

(b) $\widetilde{KO}^{-2}(B_n) \cong \widetilde{KO}^{-6}(B_n) \cong \mathbb{Z}^{2^{n-1}}$,

(c) $\widetilde{KO}^{-4}(B_n) \cong \mathbb{Z}^{2^{n-1}-1}$.

Proof. Once again, we apply to (3.33), from which we obtain an isomorphism

$$\widetilde{KO}^{2j}(B_n) \cong \bigoplus_{i=0}^{n-1} \widetilde{KO}^{2j}(T\eta_i), \tag{3.48}$$

for any $j \in \mathbb{Z}$. On the other hand, it follows from (3.32) that $\widetilde{KO}^0(\mathbb{C}P^1) \cong \mathbb{Z}_2$, $\widetilde{KO}^{-2}(\mathbb{C}P^1) \cong \widetilde{KO}^{-6}(\mathbb{C}P^1) \cong \mathbb{Z}$ and $\widetilde{KO}^{-4}(\mathbb{C}P^1) \cong 0$. Now, the claims follow from Theorem 3.43 and the formula

$$2^0 + 2^1 + 2^2 + \ldots + 2^{n-2} = 2^{n-1} - 1.$$

□

Let us explain what we have gained so far. We first recall that from the fibration

$$O/U \xrightarrow{f} BU \xrightarrow{r} BO, \tag{3.49}$$

we obtain the associated exact Bott sequence for B_n:

$$\ldots \to \widetilde{KO}^{-1}(B_n) \to \widetilde{KO}^{-2}(B_n) \xrightarrow{\chi} \widetilde{K}^0(B_n) \xrightarrow{r} \widetilde{KO}^0(B_n) \to \widetilde{KO}^7(B_n) \to \ldots,$$

which links the real and complex K-theory through the realification homomorphism r. Here, χ is induced by f and may be identified with $z^{-1} \cdot c$ by composing the complexification homomorphism with multiplication by z^{-1}.

When combined with Corollary 3.41, it reduces to a short exact sequence

$$0 \to \widetilde{KO}^{-2}(B_n) \xrightarrow{\chi} \widetilde{K}^0(B_n) \xrightarrow{r} \widetilde{KO}^0(B_n) \to 0. \tag{3.50}$$

Therefore, if $\mathcal{K}_n$ denotes the kernel of $r\colon \widetilde{K}^0(B_n) \to \widetilde{KO}^0(B_n)$, then it follows that $\mathcal{K}_n \cong \widetilde{KO}^{-2}(B_n) \cong \mathbb{Z}^{2^{n-1}}$ by Corollary 3.47. We note that since χ is a monomorphism in this case, the group $\mathcal{K}_n$ can be identified with the set of stably complex structures on B_n.

References

[1] A. Bahri and M. Bendersky. *The KO-theory of toric manifolds.* Transaction of American Mathematical Society, 352, 3: 1191-1202, 2000 .

[2] V.M. Buchstaber and N. Ray. *Flag manifolds and the Landweber-Novikov algebra.* Geometry and Topology, 2: 79-101, 1998.

[3] Y. Civan. *The Topology of Families of Toric Manifolds.* PhD Thesis, University of Manchester, 2001.

[4] Y. Civan and N. Ray. *Homotopy decompositions and real K-theory of Bott towers.* In preperation, University of Manchester, 2002.

[5] M. Fuji. *KO-groups of projective spaces.* Osaka Journal of Mathematics, 4: 141-149, 1967.

[6] H. Miller. *Stable splitting of Stiefel manifolds.* Topology, 24: 411-419, 1985.

[7] N. Ray. *On a construction in bordism theory.* Proceedings of the Edinburgh Mathematical Society, 29: 413-422, 1986.

[8] C.M. Riberio. *KO-groups of the Stiefel Manifolds $V_{n,4}$ and $V_{n,5}$.* Proceedings of the London Mathematical Society, 3: 182-208, 1989.

[9] R.M. Switzer. *Algebraic Topology- Homotopy and Homology.* Springer-Verlag, Berlin, 1975.

[10] E. Thomas. *Homotopy classification of maps by cohomology homomorphisms.* American Mathematical Society, 111: 138-151, 1964.

[11] G. Whitehead. *Elements of Homotopy Theory.* Springer-Verlag, Graduate Texts in Mathematics, Vol.61, New York, 1978.

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