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On Locally pre-C^* -Algebra Structures in Locally m-Convex H^*-Algebras

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Abstract

We endow any locally m-convex H^*-algebra (E, τ) with a locally pre-C^*-topology weaker than τ. Examples and applications are provided.

Key words and phrases: Locally pre-C^*-algebra, locally m-convex H^*-algebra, Q-algebra, positive semi-definite inner product.

Introduction

A natural extension of the classical H^*-algebras of W. Ambrose ([1]) was considered in the general contex of locally convex algebras ([4]). In this case, algebras are not necessarily endowed with an algebra involution. Here we consider H^*-algebras in the spirit of F. F. Bonsall and J. Duncan (cf. [2], Definition 6., p. 182). We show that every locally multiplicatively convex H^*-algebra (*l.m.c.* H^*-algebra) $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ can be endowed with a weaker locally convex topology given by a family $(\|.\|_\lambda)_{\lambda\in\Lambda}$ of C^*-seminorms such that $|xy|_\lambda \leq \|x\|_\lambda |y|_\lambda$, for every $x, y \in E$ and $\lambda \in \Lambda$. If moreover $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is a Q-algebra, then $\left(E, (\|.\|_\lambda)_{\lambda\in\Lambda}\right)$ is (modulo a topological algebra isomorphism) topologically and algebraically isomorphic to a pre-C^*-algebra. This last algebra becomes (modulo a topological algebra isomorphism) a C^*-algebra if and only if $\left(E, (\|.\|_\lambda)_{\lambda\in\Lambda}\right)$ is pseudo-complete (i.e., if every bounded and closed idempotent disk is Banach). We also obtain

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that any unital $l.m.c.$ H^*-algebra which is a Q-algebra is actually isomorphic to the complex field C provided that $|e|_\lambda = 1$, for every $\lambda \in \Lambda$, where e is the unit of E. This result remains valid in "Hilbertizable" $l.m.c.$ algebras ($l.m.c.$ H-algebras). Finally, we introduce and study a class of $l.m.c.$ H-algebras which contains, in particular, a concrete example used in the theory of Sobolev spaces.

1. Preliminaries

A locally m-convex algebra ($l.m.c.a.$ in short) is a topological algebra (E, τ) whose topology τ is defined by a directed family $(|.|_\lambda)_{\lambda \in \Lambda}$ of submultiplicative seminorms. Such an algebra will usually be denoted by $\left(E, (|.|_\lambda)_{\lambda \in \Lambda}\right)$. If, in addition, E is endowed with an involution $x \longmapsto x^*$ such that $|x|_\lambda = |x^*|_\lambda$, for any $x \in E$, $\lambda \in \Lambda$, then $\left(E, (|.|_\lambda)_{\lambda \in \Lambda}\right)$ is called an $l.m.c.*$-algebra. A locally m-convex C^*-algebra ($l.m.c.$ C^*-algebra in short) is an $l.m.c.*$-algebra $\left(E, (|.|_\lambda)_{\lambda \in \Lambda}\right)$ such that $|x^* x|_\lambda = |x|_\lambda^2$, for any $x \in E$ and $\lambda \in \Lambda$. Let $\left(E, (|.|_\lambda)_{\lambda \in \Lambda}\right)$ be a complex unitary and complete $l.m.c.a.$ It is known that $\left(E, (|.|_\lambda)_{\lambda \in \Lambda}\right)$ is the projective limit of the normed algebras $\left(E_\lambda, |.|'_\lambda\right)$, where $E_\lambda = E/N_\lambda$ with $N_\lambda = \{x \in E : |x|_\lambda = 0\}$; and $|\overline{x}|'_\lambda = |x|_\lambda$. An element x of E is written $x = (x_\lambda)_\lambda = (\pi_\lambda(x))_\lambda$, where $\pi_\lambda : E \longrightarrow E_\lambda$ is the canonical surjection. The algebra $\left(E, (|.|_\lambda)_{\lambda \in \Lambda}\right)$ is also the projective limit of the Banach algebras $\widehat{E_\lambda}$, the completions of E_λ's. The norm in $\widehat{E_\lambda}$ will also be denoted by $|.|'_\lambda$ ([6, p. 88, Theorem 3.1] and/or [7, p. 20, Theorem 5.1]). In the case $\left(E, (|.|_\lambda)_{\lambda \in \Lambda}\right)$ is a $l.m.c.*$-algebra, each $\widehat{E_\lambda}$, $\lambda \in \Lambda$, becomes an involutive Banach algebra. Concerning involutive $l.m.c.a.$'s, the reader is refereed to [3]. In the sequel, all algebras are complex. The spectral radius will be denoted by ρ that is $\rho(x) = \sup\{|z| : z \in Spx\}$.

2. Pre-C^*-algebra structures in $l.m.c.$ H^*-algebras

The notion of locally convex H^*-algebras was introduced in [4] as a natural extension of the classical H^*-algebras of W. Ambrose ([1]). Here, we consider the case where the algebra is complete and it is endowed with a continuous involution.

Definition 2.1 *A locally m-convex H^*-algebra (l.m.c. H^*-algebra in short) is a complete l.m.c.$*$-algebra $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ on which is defined a family $(\langle .,. \rangle_\lambda)_{\lambda\in\Lambda}$ of positive semi-definite pseudo-inner products such that the following properties hold for all x, y, $z \in E$ and $\lambda \in \Lambda$:*

(i) $|x|_\lambda^2 = \langle x, x\rangle_\lambda$,

(ii) $\langle xy, z\rangle_\lambda = \langle y, x^* z\rangle_\lambda$,

(iii) $\langle yx, z\rangle_\lambda = \langle y, zx^*\rangle_\lambda$.

Remark 2.2 For every $\lambda \in \Lambda$, the quotient space $E_\lambda = E/N_\lambda$ is an inner product space under $\langle x_\lambda, y_\lambda\rangle_\lambda = \langle x, y\rangle_\lambda$. The underlying Banach-space $\widehat{E_\lambda}$ is a Hilbert space. Moreover, the involutive Banach algebra $\left(\widehat{E_\lambda}, \|.\|_\lambda\right)$ is an H^*-algebra ([2], Definition 6, p. 182). Thus the algebra $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is the projective limit of the Banach H^*-algebras $\left(\widehat{E_\lambda}, \|.\|_\lambda\right)$ ([4, p. 455, Theorem 2.3]).

Consider an *l.m.c.* H^*-algebra E. Since $*$ is an involution (Definition 2.1), E is *proper,* namely $lan\,(E) = \{0\}$, where $lan\,(E) = \{x \in E : xE = \{0\}\}$ is the left annihilator of E, (see [4: p. 452, Theorems 1.2 and 1.3; see also the comments before Theorem 1.2]). Hence [ibid, p. 455, Theorem 2.3] each $\widehat{E_\lambda}$, $\lambda \in \Lambda$, is proper, namely, $lan\left(\widehat{E_\lambda}\right) = \{0\}$, for every $\lambda \in \Lambda$. In this case,

$$Rad\widehat{E_\lambda} = \left\{x \in \widehat{E_\lambda} : x^* x = 0\right\} = \{0\}$$

by [2, lemma 9. p. 183]. Thus

$$Rad\ E = \bigcap_\lambda \pi_\lambda^{-1}\left(Rad\ \widehat{E_\lambda}\right) = \{0\}$$

(see [7, p. 29, Proposition 7.3]).

Proposition 2.3 *Let $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ be an l.m.c. H^*-algebra. Then E can be endowed with an l.m.c. C^*-topology defined by a family of seminorms $(\|.\|_\lambda)_{\lambda\in\Lambda}$ such that*

(1) $\|x\|_\lambda \leq |x|_\lambda$*; $x \in E$, $\lambda \in \Lambda$,*

(2) $|xy|_\lambda \leq \|x\|_\lambda |y|_\lambda$*; $x, y \in E$, $\lambda \in \Lambda$.*

Proof. Let $\mathcal{B}(E)$ be the involutive algebra of all bounded linear operators on E. For $a \in E$, we define the mapping $L_a : E \longrightarrow E$ by $L_a(b) = ab$, for all $b \in E$. For every $\lambda \in \Lambda$, we have $|L_a(b)|_\lambda = |ab|_\lambda \leq |a|_\lambda |b|_\lambda$ and therefore

$$|L_a|_\lambda = \sup \{|ab|_\lambda : |b|_\lambda \leq 1\} \leq |a|_\lambda .$$

Hence

$$|L_a|_\lambda \leq |a|_\lambda , \ a \in E, \ \lambda \in \Lambda.$$

Thus L_a is bounded. Now consider the mapping $L : E \longrightarrow \mathcal{B}(E)$ defined by $L(a) = L_a$. It is easy to verify that L is a faithful $*$-representation.

(1) We introduce a family $(\|.\|_\lambda)_{\lambda\in\Lambda}$ of seminorms in E defined by $\|a\|_\lambda = |L_a|_\lambda$. The algebra $\left(E, (\|.\|_\lambda)_{\lambda\in\Lambda}\right)$ is locally m-convex. Since $\mathcal{B}(E)$ is an *l.m.c.* C^*-algebra, we have obviously $\|x\|_\lambda = \|x^*\|_\lambda$ and $\|x^*x\|_\lambda = \|x\|_\lambda^2$. Moreover, $\|x\|_\lambda \leq |x|_\lambda$; for all $x \in E$ and $\lambda \in \Lambda$.

(2) For every $x, y \in E$ and $\lambda \in \Lambda$, we have

$$|xy|_\lambda = |L_x(y)|_\lambda \leq |L_x|_\lambda |y|_\lambda = \|x\|_\lambda |y|_\lambda .$$

This completes the proof. □

Proposition 2.4 *Let $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ be an l.m.c. H^*-algebra which is a Q-algebra. Then $\left(E, (\|.\|_\lambda)_{\lambda\in\Lambda}\right)$ is topologically and algebraically isomorphic to a pre-C^*-algebra.*

Proof. Since $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is a Q-algebra, there is $\lambda_0 \in \Lambda$ such that $\rho(x) \leq |x|_{\lambda_0}$ for every $x \in E$ ([8, p. 551, Corollary 4.1]). Using (2) of Proposition 2.3**,** we obtain

$$\rho(xy) \leq \|y\|_{\lambda_0} |x|_{\lambda_0} ; \ x, y \in E$$

([6, p.100, Corollary 6.1]). Writing this for $y = x^k$, with $k = 1, 2,...,$ and using submultiplicativity of $\|.\|_{\lambda_0}$, it follows that $\rho(x) \leq \|x\|_{\lambda_0}$ for every $x \in E$. Now, for every $x \in E$, we get

$$\|x\|_{\lambda_0}^2 \leq \sup_{\lambda\in\Lambda} \|x\|_\lambda^2 = \sup_{\lambda\in\Lambda} \|x^*x\|_\lambda = \rho(x^*x) \leq \|x\|_{\lambda_0}^2 .$$

Thus the topology of $\left(E, (\|.\|_\lambda)_{\lambda\in\Lambda}\right)$ is equivalent to that introduced by the pre-C^*-norm

$$\|x\|_{\lambda_0} = \sup_{\lambda\in\Lambda} \|x\|_\lambda \,; x \in E.$$

This completes the proof. □

Remark 2.5 In the previous proposition, the algebra $\left(E, (\|.\|_\lambda)_{\lambda\in\Lambda}\right)$ is topologically and algebraically isomorphic to a C^*-algebra under the weakest completion notion. More precisely, one has that $\left(E, (\|.\|_\lambda)_{\lambda\in\Lambda}\right)$ is a pseudo-complete algebra if and only if $\left(E, \|.\|_{\lambda_0}\right)$ is a C^*-algebra.

Proposition 2.6 *Let $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ be an l.m.c. H^*-algebra. If E has a unit element e such that $|e|_\lambda = 1$, for every $\lambda \in \Lambda$, then E is the diagonal of a product whose factors are all isomorphic to C. If moreover $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is a Q-algebra, then it is isomorphic to C.*

Proof. The algebra $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is a projective limit of the H^*-algebras $\overset{\wedge}{E_\lambda}$. Since E is unital, $\overset{\wedge}{E_\lambda}$ is so ([6, p. 91, Theorem 4.1]). Hence, by a result of Hirschfeld ([5]), the algebra $\overset{\wedge}{E_\lambda}$ is isomorphic to C, for every $\lambda \in \Lambda$. But, a projective limit whose factors are equal and the relative morphisms all reduce to the identity map is exactly the diagonal of the product of its factors. Now, if moreover $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is a Q-algebra, then

$$\|x\| = \sup \{|xy|_\lambda : |y|_\lambda \leq 1\}$$

is a Banach algebra norm such that

$$\|x\| \leq \|x\|_\lambda \,;\ x \in E,\ \lambda \in \Lambda$$

by (2) of Proposition 2.3. It follows from proposition 2.4 that $\|.\| \leq \|.\|_{\lambda_0} = \sup_{\lambda\in\Lambda} \|.\|_\lambda$. But $|.|_\lambda \leq \|.\|$ since E is unital, hence $\|.\| = \|.\|_{\lambda_0} = |.|_\lambda$, for every $\lambda \in \Lambda$. Thus E is a unital Banach H^*-algebra and so it is isomorphic to C, by a result of Hirschfeld ([5]). This completes the proof. □

Remark 2.7 The result of Proposition 2.6 remains true in *l.m.c.* H-algebras $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ in the sense that $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is a complete *l.m.c.a.* with the property that $(|.|_\lambda)_{\lambda\in\Lambda}$ arises from a family $(\langle .,.\rangle_\lambda)_{\lambda\in\Lambda}$ of positive semi-definite pseudo-inner products such that $|x|_\lambda^2 = \langle x,x\rangle_\lambda$, for all $x \in E$ and $\lambda \in \Lambda$.

Scholium 2.8 Notice that the algebras (*l.m.c.* H-algebras) considered in Remark 2.7 have also been considered in [4, p. 456, Definition 3.1], even *without completeness* and "m", called therein "pseudo-H-algebras".

3. The structure of the *l.m.c.* H-algebra $L^2_\Omega(R)$

In the sequel, Ω will denote a family of measurable non negative and locally integrable functions ω in R, such that

$$\omega^{-1} * \omega^{-1} \leq \omega^{-1}, \tag{1}$$

we will consider the space $L^2_\omega(R)$ of all equivalence classes (under equality almost evrey-where) f such that $|f|^2\omega$ is a Lebesgue integrable function on R, where the same symbol f is used to denote both a function and its equivalente class. $L^2_\omega(R)$ endowed with the norm

$$|f|_\omega = \left(\int_R |f(t)|^2 \omega(t)dt\right)^{\frac{1}{2}},$$

becomes a Banach space. If f and g are complex functions in R, their convolution $f * g$ is defined by

$$(f*g)(x) = \int_R f(x-y)g(y)dy,$$

provided that the integral exists for all (or at least for almost all) $x \in R$. We will also consider the space

$$L^2_\Omega(R) = \left\{ f : R \longrightarrow C : \ |f|^2\omega \in L^1(R), \text{ for every } \omega \in \Omega \right\}$$

endowed with the topology τ defined by the norms $(|.|_\omega)_{\omega\in\Omega}$. Then we have the following proposition.

Proposition 3.1 *The space* $\left(L^2_\Omega\left(R\right), \left(|.|_\omega\right)_{\omega\in\Omega}\right)$ *endowed with convolution as the product is an l.m.c. H-algebra.*

Proof. We first prove that $\left(L^2_\Omega\left(R\right), \left(|.|_\omega\right)_{\omega\in\Omega}\right)$ is an *l.m.c.* algebra. Since the algebra $\mathcal{K}(R)$ of continuous complex-valued functions with compact support is dense in $\left(L^2_\Omega\left(R\right), \left(|.|_\omega\right)_{\omega\in\Omega}\right)$, it suffices to show that

$$|f * g|_\omega \leq |f|_\omega\, |g|_\omega\, ; \quad f, g \in \mathcal{K}(R).$$

If $f, g \in \mathcal{K}(R)$ and $h \equiv f * g$, then writing

$$|h(x)| = \left| \int_R f(x-y)g(y) \left| \frac{\omega(x-y)\omega(y)}{\omega(x-y)\omega(y)} \right|^{\frac{1}{2}} dy \right|$$

and using Cauchy-Schwarz inequality, we obtain

$$|h(x)| \leq \left(\int_R |f(x-y)|^2\, \omega(x-y)\, |g(y)|^2\, \omega(y) dy \right)^{\frac{1}{2}} W^{\frac{1}{2}}(x),$$

where $W = \omega^{-1} * \omega^{-1}$. It follows that

$$\begin{aligned} \left| \textstyle\int_R |h(x)|^2\, W^{-1}(x) dx \right| &\leq \textstyle\int_R |f(x-y)|^2\, \omega(x-y) dx \int_R |g(y)|^2\, \omega(y) dy \\ &\leq |f|^2_\omega\, |g|^2_\omega\, . \end{aligned}$$

But $\omega \leq W^{-1}$ by (1). Hence

$$\begin{aligned} |f * g|_\omega &= \left| \left(\textstyle\int_R |h(x)|^2\, \omega(x) dx \right)^{\frac{1}{2}} \right| \\ &\leq |f|_\omega\, |g|_\omega\, . \end{aligned}$$

It remains to show that $\left(L^2_\omega\left(R\right), |.|_\omega\right)$ is a Hilbertizable Banach algebra, for every $\omega \in \Omega$. If $f, g \in L^2_\omega\left(R\right)$, then $f\sqrt{\omega}$, $g\sqrt{\omega} \in L^2\left(R\right)$ and the inner product is defined by

$$\langle f, g \rangle_\omega = \int_R f(t)\overline{g(t)}\omega(t) dt.$$

It follows that the underlying Banach space of $\left(L^2_\omega\left(R\right), |.|_\omega\right)$ is a Hilbert space such that $|f|^2_\omega = \langle f, f \rangle_\omega$, for every $f \in L^2_\omega\left(R\right)$. This completes the proof. □

Remark 3.2 Associate to each $f \in L^2_\Omega(R)$ a function $f^\sharp \in L^2_\Omega(R)$ defined by $f^\sharp(x) = \overline{f(-x)}$, for every $x \in R$. Then $f \longmapsto f^\sharp$ is an algebra involution on $L^2_\Omega(R)$. The *l.m.c.* H-algebra $L^2_\Omega(R)$ endowed with the involution $f \longmapsto f^\sharp$ is not an *l.m.c.* H^*-algebra, otherwise, we will have, by ii) of Definition 2.1, that ω is a constant almost everywhere, for every $\omega \in \Omega$, a contradiction.

Remark 3.3 If $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \leq \omega_2$, then $L^2_{\omega_2}(R) \subset L^2_{\omega_1}(R)$. This implies that

$$\lim_{\omega \longleftarrow} L^2_\omega(R) = \bigcap_{\omega \in \Omega} L^2_\omega(R) = L^2_\Omega(R).$$

Concerning the global spectrum, we have

$$\mathcal{M}\left(L^2_\Omega(R)\right) = \lim_{\longrightarrow \omega} \mathcal{M}\left(L^2_\omega(R)\right)$$

by [6, p. 172, Lemma 6.3], where $\mathcal{M}\left(L^2_\Omega(R)\right)$ (resp. $\mathcal{M}\left(L^2_\omega(R)\right)$) denote the set of all non zero continuous characters of $L^2_\Omega(R)$ (resp. $L^2_\omega(R)$). It follows that

$$\mathcal{M}\left(L^2_\Omega(R)\right) = \bigcup_{\omega \in \Omega} \mathcal{M}\left(L^2_\omega(R)\right).$$

([6, p. 156, Lemma 5.1 and p. 172, Lemma 6.3]).

In the rest of this section, we consider a concrete example used in the theory of Sobolev spaces. For $s > \frac{1}{2}$, put

$$\omega_s(x) = \left(1 + |x|^2\right)^s \text{ and } \Omega = \left\{\omega_s : s > \frac{1}{2}\right\}.$$

By a simple calculation, the reader can prove that

$$\omega_s^{-1} * \omega_s^{-1} \leq c_s \ \omega_s^{-1}, \text{ for every } s > \frac{1}{2}, \qquad (1)$$

where c_s is a positive constant depending only on s. As in Proposition 3.1, we obtain

$$|f * g|_{\omega_s} \leq c_s \ |f|_{\omega_s} |g|_{\omega_s}; \quad f,\ g \in L^2_\Omega(R).$$

Therefore, without loss of generality, we may suppose that $\left(L^2_\Omega(R), \left(|.|_{\omega_s}\right)_{s > \frac{1}{2}}\right)$ is an *l.m.c.* H-algebra but not an *l.m.c.* H^*-algebra.

Remark 3.4 Since $\mathcal{K}(R)$ is dense in $L^1(R)$ and $\mathcal{K}(R) \subset L^2_\Omega(R) \subset L^1(R)$ for $s > \frac{1}{2}$, the global spectrum $\mathcal{M}\left(L^2_\Omega(R)\right)$, of $L^2_\Omega(R)$, is homeomorphic to R. Moreover, as in $L^1(R)$, for every non zero continuous character χ of $L^2_\Omega(R)$, there exists a unique $t \in R$ such that $\chi(f) = \widehat{f}(t)$, where $\widehat{f}$ is the Fourier transform of f.

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