Turk J Math
26 (2002) , 305 – 316.

One Sided Banach Algebras

A. El Kinani, A. Najmi, M. Oudadess

Abstract

Many properties of two-sided algebras remain valid for one-sided algebras. Namely, any one sided Banach algebra is commutative modulo its Jacobson radical.

Key Words: Right-sidedness, two-sidedness, commutativity, Banach algebra.

Introduction

In [1], the authors have proceeded to a study of algebras said to be two-sided by E. Hille and R. S. Philips ([3]). We consider here the left (or right) sidedness, where the notions of two-sidedness and one-sidedness are distinct (Example I-3). Many algebraic properties of [1] are still true.

In the case of normed algebras, the one-sidedness is not inherited by a sub-algebra, nor by the completion of a normed algebra. About the structure of these algebras, every right-sided finite dimensional algebra A (and, more generally every, Artinian Banach algebra) is written as $A = Rad(A) \oplus \mathbf{C}^n$, where $Rad(A)$ is the (Jacobson) radical. It is two-sided if, and only if, $Rad(A)$ is two-sided. We examine the case of a right-sided Banach algebra A such that $Rad(A)$ is finite dimensional and $A/Rad(A)$ is a $B(\infty)$ direct sum of total matrix algebras. We prove also that a right-sided Banach algebra is commutative modulo the Jacobson radical like in the two-sided case ([1]). Some conditions for the converse to be true are equally given. For example, if $Rad(A)$ is right-sided and $A/Rad(A)$ is a

1991 *Mathematics Subject Classification:* 46H05, 46H20

C^*-algebra or an l_1-algebra, then A is right-sided. Recall that $RadA$ is the intersection of all regular right (or all regular left) ideals of A.

1. Algebraic properties

All algebras considered here are complex. In the sequel, we put $A^2 = \{xy : x, y \in A\}$. A zero-algebra is an algebra A such that $A^2 = \{0\}$. For every fixed $x \in A$, we write $Ann_d(x)$ for the right annihilator of x and B_x for an algebraic complementary of $Ann_d(x)$ in A.

Definition 1.1 . *A complex algebra A, is said to be right-sided if*

$$(\forall x, y \in A)(\exists u \in A) : xy = yu.$$

It is said to be left-sided if

$$(\forall x, y \in A)(\exists v \in A) : xy = vx.$$

Remark 1.2 . *(1) Let A be a right-sided algebra. Then, endowed with the reversed product, A is left-sided. Consequently, we will study only right-sided algebras.*

(2) From the definition, an algebra A is right-sided if, and only if, $Ax \subset xA$ for every $x \in A$. This is also equivalent to the existence of an application g, vanishing on

$$\bigcup_{s \in A} (Ann_d(s))$$

(called the function of right-sidedness) such that $xy = yg(x, y)$, for every $x, y \in A$.

Every two-sided algebra ([1]) is right-sided. We give now some examples of right-sided algebras that are not two-sided.

Example 1.3 . *Let $\{e_i : i \in N^*\}$ be a sequence of symbols such that*

(a) $e_i e_j = 0$ when $j \neq i + 1$; and $e_i e_{i+1} \neq 0$ for every i.

(b) $e_i e_{i+1} = 2e_{i+1} e_{i+2}$ for all $i \in N^*$,

(c) $e_i e_j e_k = 0$ *for all* $i, j, k \in N^*$.

Let A *be the algebra spanned by* $\{e_i : i \in N^*\}$. *It is associative, because* $A^3 = \{0\}$. *It is a right-sided algebra. For every* $x \in A$, *we have*

$$x = \lambda(x,0)e_1e_2 + \sum_{i=1}^{\infty} \lambda(x,i)e_i,$$

where just a finite number of coefficients $\lambda(x,i)$ *are different from zero. For* $x, y \in A$, *one has*

$$xy = \sum_{i=1}^{\infty} \lambda(x,i)\lambda(y,i+1)e_ie_{i+1} = \sum_{i=1}^{\infty} 2^{-i+1}\lambda(x,i)\lambda(y,i+1)e_1e_2.$$

If $xy \neq 0$, *there is* $i_0 \geq 0$ *such that* $\lambda(x,i_0)\lambda(y,i_0+1) \neq 0$. *The equation* $xy = yz$ *admits a solution* z *such that*

$$\begin{aligned} \lambda(z, i_0+2) &= 2^{-i_0+2}(\lambda(y,i_0+1))^{-1}\sum_{i=1}^{\infty} 2^{-i+1}\lambda(x,i)\lambda(y,i+1) \\ \lambda(z,i) &= 0, \text{ for } i \neq i_0+2. \end{aligned}$$

Then z *is written as* $z = \lambda(z, i_0{+}2)e_{i_0+2}$. *So* A *is right-sided. The algebra* A *is not left-sided because the equation* $e_1e_2 = xe_1$, *with the unknown* $x = \lambda(x,0)e_1e_2 + \sum_{i=1}^{\infty} \lambda(x,i)e_i$, *is equivalent to* $e_1e_2 = 0$; *and that contradicts (a).*

Example 1.4 . *Let* A *be a right but not two-sided algebra and* B *a two-sided one. Then the Cartesian product* $A \times B$ *is a right-sided, but not two-sided algebra.*

Now we give an interesting property of right ideals.

Proposition 1.5 . *Let* A *be a complex algebra. If* A *is right-sided, then every right ideal is two-sided. The converse is true in the unitary case.*

Proof. We will use (2) of Remark I-2. Let I be a right ideal of A. By $Ax \subset xA$ for every $x \in A$, we have $AI = \bigcup_{x\in I}(Ax) \subset \bigcup_{x \in I}(xA) = IA \subset I$. Now, suppose that A is unitary and every right ideal of A is two-sided. For every $x \in A$, we have

$Ax \subset (Ax)A = A(xA) \subset xA$. So A is right sided. □

Remark 1.6 *In proposition I-5, the existence of a unit in the converse is necessary as it is shown by the following example.*

Let

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbf{C} \right\}.$$

Then

$$Rad(A) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbf{C} \right\}.$$

The algebra A is not unitary, not right-sided and its Jacobson radical is the unique right ideal of A. Since A is two-dimensional, every proper ideal I of A is one-dimensional. So $I = \mathbf{C} \begin{pmatrix} i & j \\ 0 & 0 \end{pmatrix}$ where i and j are fixed elements of $\mathbf{C}$. But I is a right ideal only if $i = 0$. Indeed, if $i \neq 0$ the equation $\begin{pmatrix} i & j \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} i & j \\ 0 & 0 \end{pmatrix}$, where a and b are not equal to zero and $b \neq \frac{aj}{i}$, is equivalent to $\lambda = a$ and $b = \frac{aj}{i}$: a contradiction.

Remark 1.7 *Let A be an algebra such that $AI \subset IA$ for every right ideal I of A. If $A^2x \subset AxA$ for every $x \in A$, then $A^2x \subset xA^2$ for any $x \in A$. Then A is right-sided when $A^2 = A$. Indeed, for $x \in A$ and $J = xA$, we have by hypothesis $AJ \subset JA$. Hence $A^2x \subset A(xA) \subset xA^2$.*

The right-sidedness is preserved by Cartesian products, inductive limits, tensor products, unitization and quotients by right ideals. So, if A is right-sided, then this is so for the algebra $A/Rad(A)$. The converse is false in general as we can see from the following examples.

Example 1.8 *Let x and y be two symbols such that $x^2 = 0$, $y^2 = 0$ and $xyx = yxy = 0$; and consider $A = [x, y]$ the algebra spanned by the two symbols x and y . It is a radical* *algebra, of dimension 4 and admits* $\{x, y, xy, yx\}$ *as a basis. It is not right-sided because* $Ax = [yx] \not\subset [xy] = xA$.

Example 1.9 *Let* $A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbf{C} \right\}$. *It is a non unitary algebra, but satisfies* $A^2 = A$. *Also* $Rad(A) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbf{C} \right\}$ *and* $A/Rad(A)$ *is right-sided. As* $Rad(A)A = \{0\}$, *the algebra* A *is not right-sided because* $\{0\} \neq Rad(A) = ARad(A) \not\subset Rad(A)A = \{0\}$.

Here is a condition that makes of a right-sided algebra a two-sided one (see page 2, for notations).

Proposition 1.10 *Let* A *be a right-sided algebra. The following propositions are equivalent.*

(i) A *is two-sided.*

(ii) There exists a function g *of right-sidedness such that for every* $x \in A$, *the partial application* $t \mapsto g_x(t) = g(t, x)$, *from* $A \mapsto A$, *is onto* B_x.

Proof. $(i) \Rightarrow (ii)$ Let $x \in A$ be fixed and $y \in B_x$. As A is left-sided, there exists $v \in A$ such that $xy = vx$. Let g be a function of right-sidedness. Then $vx = xg(v, x)$ and $x(y - g(v, x)) = 0$. So $y - g(v, x) \in Ann_d(x) \cap B_x$. But $Ann_d(x) \cap B_x = \{0\}$. Then for every $y \in B_x$, there exists $v \in A$ such that $y = g_x(v)$.

$(ii) \Rightarrow (i)$. Let $x \in A$ be fixed. Every $z \in A$ is written as $z = z_1 + z_2$, with $z_1 \in Ann_d(x)$ and $z_2 \in B_x$. Then there exists $y \in A$ such that $z_2 = g_x(y)$. As $yx = xg(y, x)$, we have $xz = xg_x(y) = yx$. So, there exists $y \in A$ such that $xz = yx$. And so A left-sided. □

Remark 1.11 *(i) In a right-sided algebra that satisfies* $A^2 = A$, *every right maximal ideal is also left maximal.*

(ii) We know ([1]) that, in a unitary two-sided algebra , an element is invertible if, and only if, it does not belong to any maximal ideal. If now A *is unitary, right-sided,*

then by (i), an element is invertible if, and only if, it does not belong to any right maximal ideal.

(iii) In a unitary (resp. not unitary), right-sided algebra, the set $X^*(A)$ of non zero characters of A, can be identified with the set $m(A)$(resp. $m_r(A)$) of right ideals (resp. regular right ideals) of codimension 1.

The following result will be useful.

Proposition 1.12 *Let A be a unitary, finite dimensional algebra such that $A = Rad(A) \oplus \mathbf{C}\varepsilon$, with ε an idempotent element of A. Then A is right-sided if, and only if, ε is the unit of A and $Rad(A)$ is right-sided.*

Proof. The sufficient condition is a particular case of the unitization of a right-sided algebra. For the necessary condition it is easy to see that ε is the unit of A. We show now that $Rad(A)$ is right -sided. Let $r, s \in Rad(A)$ such that $rs \neq 0$. There is $t \in Rad(A)$ and $\lambda \in \mathbf{C}$ such that $rs - st = \lambda s$. Suppose that $\lambda \neq 0$. Putting $u = \frac{r}{\lambda}$ and $v = \frac{t}{\lambda}$, the precedent equation is equivalent to $us - sv = s$. For the resolution of this equation recall that there is $n \in N^*$ such that

$$\{0\} = (Rad(A))^n = \{u_1 u_2 ... u_n : u_1, u_2, ..., u_n \in Rad(A)\}.$$

Multiplying the equation $us - sv = s$ by $u_1 u_2 ... u_m$ successively for $m = n-2, n-3, ..., 1$, and for any $u_1, u_2, ... u_m \in Rad(A)$, we obtain that $su_1 u_2 ... u_m = 0$ for $m = n-3, ..., 1$. So, we have $s = 0$: a contradiction. So $\lambda = 0$. □

Remark 1.13 *If we replace in the previous proposition, the condition "finite dimensional", by "Artinian ", the result is also valid; because the essential in the proof, is that $Rad(A)$ is nilpotent.*

2. Right-sided Banach algebras

First, some examples of right but not two-sided Banach algebras.

Example 2.1 *Let*

$$l^1(A) = \left\{ x = \lambda(x,0)e_1e_2 + \sum_{i=1}^{\infty} \lambda(x,i)e_i \in A : \sum_{i=1}^{\infty} |\lambda(x,i)| < \infty \right\},$$

where A is the algebra of Example I-3. It is clear that $l^1(A)$ is a Banach space for the norm

$$x \mapsto \|x\| = \sum_{i \in N} |\lambda(x,i)| .$$

Furthermore, for every $x, y \in l^1(A)$, we have

$$\|xy\| = \sum_{i \in N} 2^{-i} |\lambda(x,i)| \, |\lambda(y,i+1)| \leq \sum_{i \in N} |\lambda(x,i)| \, |\lambda(y,i+1)| \leq \|x\| \, \|y\| .$$

So $l^1(A)$ is a Banach algebra containing A. The same proof as that of Example I-3, shows that $l^1(A)$ is right but no left-sided.

Example 2.2 *Every product of a right but not left-sided Banach algebra and of a two-sided Banach algebra is right but not two-sided.*

Recall that a sub-algebra and the completion of a normed right-sided algebra are not necessarily of the same type ([1], p. 23). But as in [1], we have the following proposition.

Proposition 2.3 *Let A be a normed right-sided algebra, $\hat{A}$ its completion and g a function of right-sidedness. For every fixed $y \in A$, let g_y be the partial function $x \mapsto g_y(x) = g(x,y)$ from $A \mapsto A$. Then*

(i) For every $y \in A$, the application g_y is linear from A into B_y.

(ii) For every $y \in A$, the application $x \mapsto yg_y(x)$ is linear and continuous.

(iii) $xy = yg(x,y)$, for every $x \in \hat{A}$ and every $y \in A$.

(iv) If for every $x \in \hat{A}$, the application $y \mapsto yg(x,y)$ is continuous or locally bounded, then $\hat{A}$ is right-sided.

Now we give some structure results.

Proposition 2.4 *Every Artinian (in particular, of finite dimension) is right-sided but not a radical Banach algebra A, is written as $A = Rad(A) \oplus B$, where B is isomorphic to $\mathbf{C}^n$, for a certain $n \in N^*$; where the sum is taken relatively to vector spaces.*

Proof. By ([4], theorem 27, p. 315), the Artinian algebra $A/Rad(A)$, is isomorphic to a product $\prod_{i=1}^{i=n} A_i$ of Banach algebras, where A_i is simple for every $i = 1, .., n$. The algebra A_i is right-sided and so all of its right-sided ideals are two-sided. Hence it admits no proper right ideals. Consequently, A_i is a field, or a zero-algebra of dimension 1, for every i . As the algebra A_i is not radical, A_i is a field, for every i. By the Gelfand-Mazur theorem, it is isomorphic to $\mathbf{C}$. So $A/Rad(A)$ is of finite dimension. We conclude by theorem 1 of [2]. □

As a consequence, we obtain the following.

Corollary 2.5 *Let A be an Artinian right-sided but not a radical Banach algebra. Then, it is isomorphic to a finite product of algebras as follows:*

(1) $A \simeq \prod_{i=1}^{i=n}(Rad(A_i) \oplus \mathbf{C}e_i)$, if A is unitary.

(2) $A \simeq (\prod_{i=1}^{i=n} Rad(A_i) \oplus \mathbf{C}e_i) \times R_{n+1}$, if A is not unitary; where R_{n+1} is a radical right-sided algebra.

In both cases, e_i is idempotent and $Rad(A_i)$ is right-sided for every $i = 1, ..., n$.

Proof. By proposition II-4, the algebra A is isomorphic to $Rad(A) \oplus \prod_{i=1}^{i=n} \mathbf{C}e_i$, where e_i is idempotent for every i. If A is unitary, then, arguing as in remark I-13, the unit e of A is nothing else than $(e_1, e_2, ..., e_n)$ and we have $e = \sum_{i=1}^{i=n} e_i^*$, with $e_i^* = (0, ..., 0, e_i, 0, ..., 0)$. Let $A_i = Ae_i^*$. Then $Rad(A) = \prod_{i=1}^{i=n} Rad(A_i)$ and A_i is isomorphic to $Rad(A_i) \oplus \mathbf{C}e_i$. So the algebra A is isomorphic to the product $\prod_{i=1}^{i=n} Rad(A_i) \oplus \mathbf{C}e_i$. As A is right-sided, any $Rad(A_i) \oplus \mathbf{C}e_i$ is right-sided. By proposition I-12, any $Rad(A_i)$ is right-sided. If A is not unitary, the unitization $B = A \oplus \mathbf{C}e$ of A, is right-sided. Let $e_{n+1} = e - \sum_{i=1}^{i=n} e_i$. Then $e_{n+1}e_i = e_ie_{n+1} = 0$ and $e_{n+1}^2 = e_{n+1}$. Consequently we have $B = Be = Be_1 \oplus Be_2 \oplus ... \oplus Be_n$, and so any algebra $Be_i = Rad(A)e_i \oplus \mathbf{C}e_{n+1}$ is right-sided for every $i = 1, ..., n$. On the other hand, we have $Be_{n+1} = Rad(A)e_{n+1} \oplus \mathbf{C}e_{n+1}$, because $(\prod_{i=1}^{i=n} \mathbf{C}e_i)e_{n+1} = \{0\}$. By proposition I-12, the algebra $Rad(A)e_{n+1}$ is right-sided. Consequently the algebra B is isomorphic to the product $\prod_{i=1}^{i=n+1}(Rad(A_i) \oplus \mathbf{C}e_i)$, where $Rad(A_i) = Rad(A)e_i$, for every $i = 1, ..., n+1$. But A is isomorphic to Ae. And with the fact that $\mathbf{C}^n e_{n+1} = \{0\}$, we have $eA = \sum_{i=1}^{i=n+1}(Rad(A) \oplus \mathbf{C}^n)e_i = (\prod_{i=1}^{i=n}(Rad(A_i) \oplus \mathbf{C}e_i))(Rad(A_{n+1})$. □

Using corollary II-5 and proposition II-8 of [1], we obtain the following consequence.

Corollary 2.6 *Let A be an Artinian and right-sided Banach algebra. Then A is two-sided if, and only if, $Rad(A)$ is two-sided.*

Right-sidedness is sufficient to imply commutativity modulo the Jacobson radical, and then we have an improvement of proposition II-9 of [1].

Proposition 2.7 *Let A be a right-sided Banach algebra.*

(1) When M is a regular right maximal ideal of A. We have just two possibilities.

(i) M is a kernel of a continuous character of A.

(ii) M is a hyperplane, of A, of codimension 1 and contains A^2. In particular, this is the case when M is closed but not regular.

(2) $A/Rad(A)$ is commutative.

Proof. (1) As A is right-sided, this is also so for $B = A/M$. Furthermore B admits no proper right ideal. We have $B^2 = \{0\}$ with $\dim(B) = 1$ or B is a field. The first case is nothing else than (ii). If now M is regular, then B is unitary. So $B^2 \neq \{0\}$. Consequently, B is a field. By the Gelfand-Mazur theorem, it is isomorphic to **C**. So, by (iii) of remark I-11, there exists a character χ of A such that $M = Ker(\chi)$. So, we have (i).

(2) If $A = Rad(A)$, the conclusion is trivial. If $A \neq Rad(A)$, then A admits regular ideals. Let M a right maximal ideal of A. By (i) of (1), we have $xy - yx \in M$ for every $x, y \in A$. So we have $xy - yx \in Rad(A)$ for every $x, y \in A$. □

Remark 2.8 *Example I-9 shows that the converse of (2) of the precedent proposition is false. Indeed, in this case $A/Rad(A)$ is isomorphic to* **C**. *In the following we are going to give conditions that make it valid.*

Definition 2.9 *([2]). A $B(\infty)$ direct sum of a sequence of algebras $\{B_i : i \in N\}$, is the* *completion of the algebra*

$$B = \left\{ b = (b_i)_{i \in N} \in \prod_{i=0}^{i=\infty} B_i : ((\exists i_b \geq 0) : b_i = 0, i \geq i_b) \right\}$$

for a specific algebra norm.

Lemma 2.10 *Let A be a unitary right-sided Banach algebra such that $Rad(A)$ is of finite dimension and $A/Rad(A)$ is a $B(\infty)$ direct sum of total matrix and finite dimensional B_i's. Then A is isomorphic to the Cartesian product $B \times C$ of two right-sided algebras B and C, with B of finite dimension.*

Proof. Denote by 1 the unit element of A. By theorem 2 of [2], there exists an idempotent e of A and three algebras B, C and D such that $A = B \oplus C \oplus D$, with $BC = CB = \{0\}$, $D = (1-e)Ae \oplus eA(1-e) \subset Rad(A)$, $B = eAe$ is of finite dimension and $C = (1-e)A(1-e)$. By right-sidedness of A, we have $D = \{0\}$. So A is isomorphic to the cartesien product $B \times C$. Consequently B and C are right-sided and B is of finite dimension. □

Lemma 2.11 *Let A be a unitary right-sided Banach algebra such that $Rad(A)$ is of finite dimension and $A/Rad(A)$ is a completely continuous C^*-algebra. Then A is isomorphic to the Cartesian product $B \times (S \oplus R)$ of a right-sided algebra of finite dimension $B = Rad(B) \oplus \mathbf{C}^n$ and a right-sided (vector sum) $S \oplus R$ with S commutative and R radical.*

Proof. It is proved in [5], that a completely continuous C^* -algebra is a $B(\infty)$ direct sum of finite dimensional and total matrix algebras. By lemma II-10, the algebra A is isomorphic to the a Cartesian product $B \times C$ of a finite dimensional right-sided algebra B and a right-sided algebra . By proposition II-4, the algebra B is isomorphic to $Rad(B) \oplus \mathbf{C}^n$, where $Rad(B)$ is right-sided algebra. On the other hand, by theorem 3 of [2], the algebra C is isomorphic to the vector sum $S \oplus Rad(C)$. But as $BC = CB = \{0\}$, we have $\mathbf{C}^n S = S\mathbf{C}^n = \{0\}$. By theorem 3 of [2], we know that the Cartesian product $\mathbf{C}^n \times S$ is isomorphic to $A/Rad(A)$. Finally, as by proposition II-7, the algebra $A/Rad(A)$ is commutative, it is also so of S. □

Recall that two sub-algebras E and F of the same algebra G are said to be transcommutative if

$$(\forall a \in E)(\forall b \in F) : ab = ba.$$

Proposition 2.12 *Let A be a unitary Banach algebra such that $Rad(A)$ is right-sided and of finite dimension. If in addition $A/Rad(A)$ is a commutative completely continuous C^* -algebra such that $A/Rad(A)$ is transcommutative with $Rad(A)$, then A is right-sided.*

Proof. By theorem 2 of [2], we have $A = B \oplus C \oplus D$, with $BC = CB = \{0\}$. By the right-sidedness of $Rad(A)$, the definition of D and the fact that $D \subset R$, we have $D = \{0\}$. Consequently, the algebra A is isomorphic to $A = B \times C$. By theorem 3 of [2], there exists two algebras T and S such that the product $B \times C$ is isomorphic to $(T + Rad(B)) \times (S + Rad(C))$. As $Rad(A) = Rad(B) \times Rad(C)$ is right-sided, $Rad(B)$ and $Rad(C)$ are right-sided. On the other hand, $T \times S$ is isomorphic to $A/Rad(A)$. So, T and S are commutative. As $A/Rad(A)$ is transcommutative with $Rad(A)$, it is also so for S and $Rad(C)$. Consequently $S \oplus Rad(C)$ is right-sided. The algebra T is commutative, semi-simple and finite dimensional. So it is isomorphic to $\mathbf{C}^n$. And then B is isomorphic to $Rad(B) \oplus \mathbf{C}^n$. A decomposition as that in the proof of (1) of corollary II-5 shows that B is isomorphic to a product $\prod_{i=1}^{i=n}(Rad(A_i) \oplus \mathbf{C}e_i)$. As $Rad(A_i) = Rad(A)e_i$, it is easy to see that $Rad(A_i)$ is right-sided; and it is true also for $Rad(A_i) \oplus \mathbf{C}e_i$. Consequently B is right-sided. But A is isomorphic to the Cartesian product $B \times (S \oplus Rad(C))$; so it is right-sided. □

Proposition 2.13 *([2], p. 776). An l_1-algebra is the commutative Banach algebra of all sums $\sum_{i=0}^{i=\infty} \alpha_i e_i$, with the $\alpha_i \in \mathbf{C}$,*

$$\left\| \sum_{i=0}^{i=\infty} \alpha_i e_i \right\| = \sum_{i=0}^{i=\infty} |\alpha_i| < \infty,$$

where $(e_i)_i$ is a family of orthogonal, primitive and idempotent elements.

Proposition 2.14 *Let A be a Banach algebra such that $A/Rad(A)$ is an l_1-algebra and $Rad(A)$ is right-sided and of finite dimension. Then A is right-sided.*

Proof. If A is unitary, the unit e of A is $e = \sum_{i \in N} e_i$. By theorem 4 of [2], $A = S \oplus Rad(A)$ with S isomorphic to $A/Rad(A)$. But, for every $i \in N$, the algebra $Rad(A_i) \oplus \mathbf{C}e_i$ is right-sided as unitization of a right-sided algebra. Consequently A is isomorphic to the product $\prod_{i=0}^{i=\infty}(Rad(A_i) \oplus \mathbf{C}e_i)$. It is then right-sided. For the non unitary case, we use the same arguments and the proof of (2) of corollary II-5. □

References

[1] El Kinani, A., Najmi, A., Oudadess, M.: Algèbres de Banach Bilatérales. *Bull. Greek. Math. Soc.* 45, 17-29 (2001).

[2] Feldman, C.: The Wedderburn principal theorem in Banach algebras. *Proc. Amer. Math. Soc.* 2, 771-772 (1951).

[3] Hille, E., Philips, R. S.: Functional analysis and semi-groups. In: Colloquium Publications. *Amer. Math. Soc.* 31 (1957).

[4] Jacobson, N.: The radical and semi-simplicity for arbitrary rings. *Amer. J. Math.* 67, 320-333 (1945).

[5] Kaplansky, I.: Normed algebras. *Duke Math. J.* 16, 399-418 (1949).

A. El KINANI, A. NAJMI, M. OUDADESS Received 19.12.2001

Ecole Normale Supérieure
B. P. 5118, Takaddoum.
10105 Rabat (Maroc).
e-mail: anajmi@ens-rabat.ac.ma