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On Derivations of Prime Gamma Rings

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Abstract

We consider some results in a Γ-ring M with derivation which is related to Q, and the quotient Γ-ring of M.

Key words and phrases: Derivation, gamma ring, prime gamma ring, quotient gamma ring.

1. Introduction

Nobusawa [3] introduced the notion of a Γ-ring, an obiect more general than a ring. Barnes [1] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Öztürk et al. [4, 5] studied extended centroid of prime Γ-rings. In this paper, we consider the main results as follows. (1) Let M be a prime Γ-ring of characteristic 2, U a non-zero ideal of M, and d_1 and d_2 two non-zero derivations of M. If $d_1d_2(U) = (0)$, there exists $\lambda \in C_\Gamma$ such that $d_2 = \lambda\alpha d_1$ for all $\alpha \in \Gamma$ where C_Γ is the extended centroid of M. (2) Let M be a prime Γ-ring, U a non-zero right ideal of M and d a non-zero derivation of M. If $d(U)\Gamma a = (0)$ where a is a fixed element of M, then there exists an element q of Q such that $q\gamma a = 0$ and $q\gamma u = 0$ for all $u \in U$ and $\gamma \in \Gamma$. (3) Let M be a prime Γ-ring with $charM \neq 2$, U a non-zero right ideal of M and d_1 and d_2 two non-zero derivations of M. If $d_1d_2(U) = (0)$, then there exists two elements p, q of Q such that $q\Gamma U = (0)$ and $p\Gamma U = (0)$.

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2. Preliminaries

Let M and Γ be (additive) abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ the conditions

$$\begin{array}{lll} (1) & a\alpha b \in M, & \\ (2) & (a+b)\alpha c & = a\alpha b + a\alpha c, \\ & a(\alpha+\beta)b & = a\alpha b + a\beta b, \\ & a\alpha(b+c) & = a\alpha b + a\alpha c, \\ (3) & (a\alpha b)\beta c & = a\alpha(b\beta c). \end{array}$$

are satisfied, then we call M a *Γ-ring.* Let M be a Γ-ring. The subset

$$Z = \{x \in M \mid x\gamma m = m\gamma x \text{ for all } m \in M \text{ and } \gamma \in \Gamma\}$$

is called the *center* of M. By a *right* (resp. *left*) *ideal* of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left ideal, then we say that U is an *ideal* of M. For each a of a Γ-ring M the smallest right ideal containing a is called the *principal right ideal generated* by a and is denoted by $\langle a\rangle_r$. Similarly we define $\langle a\rangle_l$ (resp. $\langle a\rangle$), the *principal left* (resp. *two sided*) *ideal generated* by a. An ideal P of a Γ-ring M is is said to be *prime* if for any ideals U and V of M, $U\Gamma V \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$. A Γ-ring M is said to be *prime* if the zero ideal is prime.

Theorem 2.1 ([2, Theorem 4])**.** *If M is a Γ-ring, the following conditions are equivalent:*

(i) *M is a prime Γ-ring.*

(ii) *If $a, b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$.*

(iii) *If $\langle a\rangle$ and $\langle b\rangle$ are principal ideals of M such that $\langle a\rangle\Gamma\langle b\rangle = (0)$, then $a = 0$ or $b = 0$.*

(iv) *If U and V are right ideals of M such that $U\Gamma V = (0)$, then $U = (0)$ or $V = (0)$.*

(v) *If U and V are left ideals of M such that $U\Gamma V = (0)$, then $U = (0)$ or $V = (0)$.*

Let M be a prime Γ-ring such that $M\Gamma M \neq M$. Denote

$$\begin{aligned} \mathcal{M} := \{(U, f) \mid \quad & U(\neq 0) \text{ is an ideal of } M \text{ and} \\ & f : U \to M \text{ is a right } M\text{-module homomorphism}\}. \end{aligned}$$

Define a relation $\sim$ on $\mathcal{M}$ by

$$(U, f) \sim (V, g) \Leftrightarrow \exists W(\neq 0) \subset U \cap V \text{ such that } f = g \text{ on } W.$$

Since M is a prime Γ-ring, it is possible to find a non-zero W and so " $\sim$ " is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We then denote the equivalence class by $Cl(U, f) = \hat{f}$, where

$$\hat{f} := \{g : V \to M \mid (U, f) \sim (V, g)\},$$

and denote by Q the set of all equivalence classes. Then Q is a Γ-ring, which is called the *quotient* Γ*-ring* of M (see [4]). The set

$$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$$

is called the *extended centroid* of M (See [4]).

Lemma 2.2 ([4, p. 476]). *Let M be a prime Γ-ring such that $M\Gamma M \neq M$ and C_Γ the extended centroid of M. If a_i and b_i are non-zero elements of M such that $\sum a_i\gamma_i x\beta_i b_i = 0$ for all $x \in M$ and $\gamma_i, \beta_i \in \Gamma$, then the a_i's (also b_i's) are linearly dependent over C_Γ. Moreover, if $a\gamma x\beta b = b\gamma x\beta a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ where $a(\neq 0), b \in M$ are fixed, then there exists $\lambda \in C_\Gamma$ such that $b = \lambda\alpha a$ for all $\alpha \in \Gamma$.*

Theorem 2.3 ([6, Theorem 3.5]). *The Γ-ring Q satisfies the following properties :*

(i) *For any element $q \in Q$, there exists an ideal $U_q \in F$ such that $q(U_q) \subseteq M$ (or $q\gamma U_q \subseteq M$ for all $\gamma \in \Gamma$).*

(ii) *If $q \in Q$ and $q(U) = (0)$ for some $U \in F$ (or $q\gamma U_q = (0)$ for some $U \in F$ and for all $\gamma \in \Gamma$), then $q = 0$.*

(iii) *If $U \in F$ and $\Psi : U \to M$ is a right M-module homomorphism, then there exists an element $q \in Q$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = q\gamma u$ for all $u \in U$ and $\gamma \in \Gamma$).*

(iv) *Let W be a submodule (an (M, M)-subbimodule) in Q and $\Psi : W \to Q$ a right M-module homomorphism. If W contains the ideal U of the Γ-ring M such that $\Psi(U) \subseteq M$ and $AnnU = Ann_rW$, then there is an element $q \in Q$ such that $\Psi(b) = q(b)$ for any $b \in W$ (or $\Psi(b) = q\gamma b$ for any $b \in W$ and $\gamma \in \Gamma$) and $q(a) = 0$ for any $a \in Ann_rW$ (or $q\gamma a = 0$ for any $a \in Ann_rW$ and $\gamma \in \Gamma$).*

Let M be a Γ-ring. A map $d : M \to M$ is called a *derivation* if

$$d(x + y) = d(x) + d(y) \text{ and } d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$$

for all $x, y \in M$ and $\gamma \in \Gamma$.

Lemma 2.4 ([8, Lemma 3]). *Let M be a prime Γ-ring, U a non-zero ideal of M, and d a derivation of M. If $a\Gamma d(U) = (0)$ $(d(U)\Gamma a = (0))$ for all $a \in M$, then $a = 0$ or $d = 0$.*

Lemma 2.5 ([8, Lemma 1]). *Let M be a prime Γ-ring and Z the center of M.*

(i) *If $a, b, c \in M$ and $\beta, \gamma \in \Gamma$, then*

$$[a\gamma b, c]_\beta = a\gamma[b, c]_\beta + [a, c]_\beta\gamma b + a\gamma(c\beta b) - a\beta(c\gamma b)$$

where $[a, b]_\gamma$ is $a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

(ii) *If $a \in Z$, then $[a\gamma b, c]_\beta = a\gamma[b, c]_\beta$ where $[a, b]_\gamma$ is $a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.*

Lemma 2.6 ([8, Lemma 2]). *Let M be a prime Γ-ring, U a non-zero right (resp. left) ideal of M and $a \in M$. If $U\Gamma a = (0)$ (resp. $a\Gamma U = (0)$), then $a = 0$.*

3. Main results

In what follows, let M denote a prime Γ-ring such that $M\Gamma M \neq M$, Z is the center of M, C_Γ is the extended centroid of M and $[a, b]_\gamma = a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Lemma 3.1. *Let M be a prime Γ-ring of characteristic 2. Let d_1 and d_2 two non-zero derivations of M and right M-module homomorphisms. If*

$$d_1 d_2(x) = 0 \text{ for all } x \in M, \tag{3.1}$$

then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $\alpha \in \Gamma$ and $x \in M$

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Replacing x by $x\gamma y$ in (3.1), it follows from $charM = 2$ that for all $x, y \in M$ and $\gamma \in \Gamma$

$$d_1(x)\gamma d_2(y) = d_2(x)\gamma d_1(y). \tag{3.2}$$

Replacing x by $x\beta z$ in (3.2), we get

$$d_1(x)\beta z\gamma d_2(y) = d_2(x)\beta z\gamma d_1(y) \tag{3.3}$$

for all $x, y \in M$ and $\gamma \in \Gamma$. Now, if we replace y by x in (3.3), then we obtain

$$d_1(x)\beta z\gamma d_2(x) = d_2(x)\beta z\gamma d_1(x) \tag{3.4}$$

for all $x \in M$ and $\gamma, \beta \in \Gamma$. If $d_1(x) \neq 0$, then there exists $\lambda(x) \in C_\Gamma$ such that $d_2(x) = \lambda(x)\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$ by Lemma 2.2. Thus, if $d_1(x) \neq 0 \neq d_1(y)$, then (3.3) implies that

$$(\lambda(y) - \lambda(x))\alpha d_1(x)\beta z\gamma d_2(x) = 0. \tag{3.5}$$

Since M is a prime Γ-ring, we conclude by using Lemma 2.4 that $\lambda(y) = \lambda(x)$ for all $x, y \in M$. Hence we proved that there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$ with $d_1(x) \neq 0$. On the other hand, if $d_1(x) = 0$, then $d_2(x) = 0$ as well. Therefore, $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$. This completes the proof. □

Proposition 3.2. *Let M be a prime Γ-ring of characteristic 2 and d a non-zero derivation of M. If*

$$d(x) \in Z \text{ for all } x \in M, \tag{3.6}$$

then there exists $\lambda(m) \in C_\Gamma$ such that $d(m) = \lambda(m)\alpha d(z)$ for all $m, z \in M$ and $\alpha \in \Gamma$ or M is commutative.

Proof. From (3.6), we have

$$[d(x), y]_\beta = 0 \text{ for all } x, y \in M \text{ and } \beta \in \Gamma. \tag{3.7}$$

Replacing x by $x\gamma z$ in (3.7), it follows from Lemma 2.5 that

$$d(x)\gamma[z, y]_\beta + d(z)\gamma[x, y]_\beta = 0 \tag{3.8}$$

for all $x, y, z \in M$ and $\gamma, \beta \in \Gamma$. Replacing z by $d(z)$ in (3.8), we obtain

$$d^2(z)\gamma[x, y]_\beta = 0 \text{ for all } x, y, z \in M \text{ and } \gamma, \beta \in \Gamma. \tag{3.9}$$

Now, substituting $z\alpha m$ for z in (3.9), it follows from $charM = 2$ that

$$d^2(z)\alpha m\gamma[x, y]_\beta = 0 \tag{3.10}$$

for all $x, y, z, m \in M$ and $\gamma, \beta, \alpha \in \Gamma$. Since M is a prime Γ-ring, we obtain

$$d^2(z) = 0 \;\; \forall z \in M \text{ or } [x, y]_\beta = 0 \;\; \forall x, y \in M \text{ and } \forall \beta \in \Gamma. \tag{3.11}$$

From (3.11), if $d^2(z) = 0$ for all $z \in M$, then replacing z by $z\gamma m$ in this last relation, it follows from $d(x) \in Z$ that

$$d(z)\gamma d(m) = d(m)\gamma d(z) \text{ for all } z, m \in M \text{ and } \gamma \in \Gamma. \tag{3.12}$$

Replacing z by $z\alpha n$ in (3.12), it follows from (3.6) that for all $z, m, n \in M$ and $\gamma, \alpha \in \Gamma$

$$d(z)\alpha n\gamma d(m) = d(m)\alpha n\gamma d(z). \tag{3.13}$$

If $d(z) \neq 0$, then there exists $\lambda(m) \in C_\Gamma$ such that $d(m) = \lambda(m)\alpha d(z)$ for all $z, m \in M$ and $\alpha \in \Gamma$ by Lemma 2.2. On the other hand, it follows from (3.11) that if $[x, y]_\beta = 0$ for all $x, y \in M$ and $\beta \in \Gamma$, then M is commutative. This completes the proof. □

Theorem 3.3. *Let M be a prime Γ-ring of characteristic 2, d_1 and d_2 two non-zero derivations of M and U a non-zero ideal of M. If*

$$d_1 d_2(u) = 0 \textit{ for all } u \in U \tag{3.14}$$

then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $\alpha \in \Gamma$ and $x \in M$.

Proof. Let $u, v \in U$ and $\gamma \in \Gamma$. Replacing u by $d_2(u)\gamma v$ in (3.14), we get

$$d_2^2(u)\gamma d_1(v) = 0 \text{ for all } u, v \in U \text{ and } \gamma \in \Gamma. \tag{3.15}$$

Since $d_1 \neq 0$, it follows from Lemma 2.4 that $d_2^2(u) = 0$ for all $u \in U$, so from $charM = 2$ that $d_2^2 = 0$. Now, substituting $u\gamma d_2(x)$ for u in (3.14), we get

$$d_2(u)\gamma d_1(d_2(x)) = 0 \text{ for all } u \in U,\, x \in M \text{ and } \gamma \in \Gamma. \tag{3.16}$$

Since $d_2 \neq 0$, we get $d_1(d_2(x)) = 0$ for all $x \in M$ by Lemma 2.4. Hence there exists $\lambda \in C_\Gamma$ such that $d_2 = \lambda\alpha d_1$ for all $\alpha \in \Gamma$ by Lemma 3.1. □

Theorem 3.4. *Let M be a prime Γ-ring, U a non-zero right ideal of M and d a non-zero derivation of M. If*

$$d(u)\gamma a = 0 \text{ for all } u \in U \text{ and } \gamma \in \Gamma \tag{3.17}$$

where a is a fixed element of M, then there exists an element q of Q such that $q\gamma a = 0$ and $q\gamma u = 0$ for all $u \in U$ and $\gamma \in \Gamma$.

Proof. Let $u \in U$, $x \in M$ and $\beta \in \Gamma$. Since U is a right ideal of M, we have $u\beta x \in U$. Replacing u by $u\beta x$ in (3.17), we get

$$d(u)\beta x\gamma a + u\beta d(x)\gamma a = 0 \tag{3.18}$$

for all $u \in U$, $x \in M$ and $\gamma, \beta \in \Gamma$. Hence $d(u)\beta x\gamma a\alpha m + u\beta d(x)\gamma a\alpha m = 0$ for any $m \in M$ and $\alpha \in \Gamma$, and so $d(u)\beta(\sum x\gamma a\alpha m) = -(u\beta(\sum d(x)\gamma a\alpha m))$. Therefore, for any $v \in V = M\Gamma a\Gamma M$ which is a non-zero ideal of M, we have

$$d(u)\beta v = u\beta f(v) \tag{3.19}$$

for all $u \in U$. $f(v)$ is independent of u but it is dependent on v. Since M is a prime Γ-ring, $f(v)$ is well-defined and unique for all $v \in V$. Note that $v\alpha y \in V$ for any $y \in M$, $v \in V$ and $\alpha \in \Gamma$. Replacing v by $v\alpha y$ in (3.19) we get

$$d(u)\beta(v\alpha y) = u\beta f(v\alpha y) \text{ for all } y \in M, \tag{3.20}$$

and so by using (3.19) and (3.20), we have

$$\begin{aligned}(d(u)\beta v)\alpha y = u\beta f(v\alpha y) &\Rightarrow (u\beta f(v))\alpha y = u\beta f(v\alpha y)\\ &\Rightarrow u\beta f(v)\alpha y = u\beta f(v\alpha y)\\ &\Rightarrow u\beta(f(v)\alpha y - f(v\alpha y)) = 0,\end{aligned}$$

which implies from Lemma 2.6 that

$$f(v\alpha y) = f(v)\alpha y \tag{3.21}$$

for all $y \in M$, $v \in V$ and $\alpha \in \Gamma$. It follows from (3.21) that $f : V \to M$ is a right M-module homomorphism. In this case, $q = Cl(V, f) \in Q$. Moreover, $f(v) = q\beta v$ for all $v \in V$ and $\alpha \in \Gamma$ by Theorem 2.3. Let $x \in M$, $v \in V$, $u \in U$ and $\gamma, \beta \in \Gamma$. Replacing v by $x\gamma v$ in (3.19), we get

$$d(u)\beta(x\gamma v) = u\beta f(x\gamma v) = u\beta(q\beta x\gamma v). \tag{3.22}$$

Also, replacing u by $u\gamma x$ in (3.19), we get

$$d(u)\gamma x\beta v = u\gamma x\beta q\beta v - u\gamma d(x)\beta v. \tag{3.23}$$

Now, replacing β by γ and replacing γ by β in (3.23), we get

$$d(u)\beta x\gamma v = u\beta x\gamma q\gamma v - u\beta d(x)\gamma v. \tag{3.24}$$

Thus, from (3.22) and (3.24) we obtain

$$u\beta(q\beta x - x\gamma q + d(x))\gamma v = 0 \tag{3.25}$$

for all $x \in M$, $v \in V$, $u \in U$ and $\gamma, \beta \in \Gamma$. Hence $d(x) = x\gamma q - q\beta x$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ by Lemma 2.6. Now, we shall prove that q can be chosen in Q such that $q\gamma a = 0$ and $q\gamma u = 0$ for all $u \in U$ and $\gamma \in \Gamma$. Let $u \in U$ and $x \in M, d(u) = q\alpha u - u\beta q$ and $d(x) = q\beta x - x\alpha q$. Then we have $0 = d(u\beta x)\gamma a = (q\alpha(u\beta x) - (u\beta x)\alpha q)\gamma a$. Thus, $q\alpha u\beta x\gamma a = u\beta x\alpha q\gamma a$. If $q\gamma a = 0$, then $q\alpha u\beta x\gamma a = 0$, and so since M is prime Γ-ring, we get $q\Gamma U = (0)$. On the other hand, if $q\gamma a \neq 0$, then $q\gamma u \neq 0$. In fact, if $q\gamma u = 0$, then $q\gamma a = 0$ since $q\alpha u\beta x\gamma a = u\beta x\alpha q\gamma a$. Thus, we may suppose that $q\gamma a \neq 0$ and $q\gamma u \neq 0$ for all $u \in U$ and $\gamma \in \Gamma$. In this case, we get

$$q\alpha u\beta x\gamma a = u\beta x\alpha q\gamma a \tag{3.26}$$

for all $x \in M$, $u \in U$ and $\gamma, \beta, \alpha \in \Gamma$. It follows from Lemma 2.2 that there exists $\lambda \in C_\Gamma$ such that $q\gamma a = \lambda\delta a$ and $q\gamma u = \lambda\delta u$ for all $u \in U$ and $\gamma, \delta, \alpha \in \Gamma$. Hence, if $q' = q - \lambda$, then $q'\Gamma a = 0$ and $q'\Gamma U = (0)$. This completes the proof. □

Theorem 3.5. *Let M be a prime Γ-ring with char$M \neq 2$, U a non-zero right ideal of M and d a non-zero derivation of M. Then the subring of M generated by $d(U)$ contains no non-zero right ideals of M if and only if $d(U)\Gamma U = (0)$.*

Proof. Let A be the subring generated by $d(U)$. Let $S = A \cap U$, $u \in U$, $s \in S$ and $\gamma \in \Gamma$. Then $d(s\gamma u) = d(s)\gamma u + s\gamma d(u) \in A$, and so we have $d(s)\gamma u \in S$. Thus $d(S)\Gamma U$ is a right ideal of M. In this case, $d(S)\Gamma U = (0)$ by hypothesis. $d(u\gamma a) = d(u)\gamma a + u\gamma d(a) \in S$ and $d(u)\gamma a \in S$ where $u \in U, a \in A$. Thus, we have $u\gamma d(a) \in S$. Therefore, $0 = d(u\gamma d(a))\beta u = (u\gamma d^2(a) + d(u)\gamma d(a))\beta u$. Since M is a prime Γ -ring, it follows from Lemma 2.6 that

$$u\gamma d^2(a) + d(u)\gamma d(a) = 0 \tag{3.27}$$

for all $u \in U, \gamma \in \Gamma$ and $a \in A$. Replacing u by $u\beta v$ where $v \in U, \beta \in \Gamma$ in (3.27), we get, for all $u, v \in U, \beta, \gamma \in \Gamma$ and $a \in A$

$$d(u)\beta v\gamma d(a) = 0. \tag{3.28}$$

Since M is a prime Γ-ring, we get $d(U)\Gamma U = (0)$ or $d(A)\Gamma U = (0)$. If $d(A)\Gamma U = (0)$, then $d^2(U)\Gamma U = (0)$. Let $u, v \in U$ and $\beta \in \Gamma$. Then $0 = d(d(u\beta v)) = u\beta d^2(v) + d(u)\beta d(v) + d(v)\beta d(u) + d^2(u)\beta v$, and so we have $d(u)\beta d(v) = 0$ for all $u, v \in U$ and $\beta \in \Gamma$ by $char M \neq 2$. Replacing u by $u\gamma w$ where $w \in U, \gamma \in \Gamma$ in last relation, we have $d(u)\gamma w\beta d(v) = 0$ which yields $d(u)\gamma v = 0$ for all $u, v \in U$ and $\gamma \in \Gamma$.

Conversely assume that $d(U)\Gamma U = (0)$. Then $A\Gamma U = (0)$. Since M is a prime Γ-ring, A contains no non-zero right ideals. □

Theorem 3.6. *Let M be a prime Γ-ring with $char M \neq 2$, U a non-zero right ideal of M and d_1 and d_2 two non-zero derivations of M. If $d_1 d_2(U) = (0)$, then there exists two elements p, q of Q such that $q\Gamma U = (0)$ and $p\Gamma U = (0)$.*

Proof. If $d_1 d_2(U) = (0)$, then $d_1(A) = (0)$ where A is a subring generated by $d_2(U)$. Since $d \neq 0, A$ contains no non-zero right ideals of M. Thus, from Theorem 3.5, we have $d_2(u)\gamma v = 0$ for all $u, v \in U$ and $\gamma \in \Gamma$. Also, there exists $q \in Q$ such that $q\Gamma U = (0)$ by Theorem 3.4. Therefore $d_2(u\gamma v) = u\gamma d_2(v)$ for all $u, v \in U$ and $\gamma \in \Gamma$. In this case, $0 = d_1 d_2(u\gamma v) = d_1(u\gamma d_2(v)) = d_1(u)\gamma d_2(v)$, and since M is a prime Γ-ring, we get $d_2(u)\gamma v = 0$ for all $u, v \in U$ and $\gamma \in \Gamma$. Again, by Theorem 3.4, there exists $p \in Q$ such that $p\Gamma U = (0)$. This completes the proof. □

Remark 3.7. (a) Consider the following example. Let R be a ring. A derivation $d : R \to R$ is called an *inner derivation* if there exists $a \in R$ such that $d(x) = [a, x] = ax - xa$ for all $x \in R$. Let S be the 2×2 matrix ring over Galois field $\{0, 1, w, w^2\}$, with inner derivations d_1 and d_2 defined by

$$d_1(x) := \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x \right], \; d_2(x) := \left[\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, x \right]$$

for all $x \in S$. Then the characteristic of S is 2 and we have $d_1 \neq 0,\ d_2 \neq 0,\ d_1 d_2 = 0$ and $d_2^2 = 0$. Also, if we take

$$M := M_{1\times 2}(S) = \{(a, b) \mid a, b \in S\} \text{ and } \Gamma := \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix} \mid n \text{ is an integer} \right\},$$

then M is a prime Γ-ring of characteristic 2. Define an additive map $D_1 : M \to M$ by $D_1(x, y) = (d_1(x), d_1(y))$. Since $(x, y) \begin{pmatrix} n \\ 0 \end{pmatrix} (a, b) = (nxa, nxb)$, therefore D_1 is a derivation on M. Similarly $D_2 : M \to M$ given by $D_2(x, y) = (d_2(x), d_2(y))$ is a derivation. In this case, we have $D_1 \neq 0,\ D_2 \neq 0,\ D_1 D_2 = 0$ and $D_2^2 = 0$ (see [7]). Thus we know that there exist two derivations $D_1,\ D_2$ of M such that $D_1 D_2(M) = (0)$ but $D_1(M)\Gamma M \neq (0)$ and $D_2(M)\Gamma M \neq (0)$. Therefore the condition of $charM \neq 2$ in Theorems 3.5 and 3.6 is necessary.

(b) In Theorems 3.4 and 3.6, if $a\gamma(c\beta b) = a\beta(c\gamma b)$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$, then $d(x) = [q, x]_\gamma = q\gamma x - x\gamma q$ for all $x \in M,\ \gamma \in \Gamma$ and for some $q \in Q$ is inner derivation and also $d_1(x) = [q, x]_\gamma$ and $d_2(x) = [q, x]_\beta$ for all $x \in M,\ \gamma, \beta \in \Gamma$ and for some elements $q, p \in Q$ are inner derivations by Lemma 2.5(i).

References

[1] Barnes, W. E.: *On the* Γ*-rings of Nobusawa*, Pacific J. Math. 18, 411-422, (1966).

[2] Kyuno, S.: *On prime gamma rings*, Pacific J. Math. 75(1), 185-190, (1978).

[3] Nobusawa, N.: *On a generalization of the ring theory*, Osaka J. Math. 1, 81-89, (1964).

[4] Öztürk, M. A. and Jun, Y. B.: *On the centroid of the prime gamma rings*, Comm. Korean Math. Soc. 15(3), 469-479, (2000).

[5] Öztürk, M. A. and Jun, Y. B.: *On the centroid of the prime gamma rings II*, Tr. J. of Math. (to appear).

[6] Öztürk, M. A.: *On the quotient gamma ring of the semi-prime gamma rings* (to appear).

[7] Sapanci M. and Öztürk, M. A.: *A note on gamma rings* , Atatürk University 40th foundation year Math. Symposium, Special Press (20-22 May 1998), Erzurum, 48-51.

[8] Soytürk, M.: *The commutativity in prime gamma rings with derivation*, Tr. J. of Math. 18(4), 149-155, (1994).

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