Turk J Math
26 (2002) , 339 – 353.

On Some Properties Connecting Infinite Series

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Abstract

We prove several theorems connecting infinite series of real terms in relation to Borel and Baire classification of sets and functions, connectedness of mappings, porosity character of sets etc.

Key words and phrases: Borel sets, connectedness, Baire classification of functions, porosity of sets.

1. Introduction

The motivation for this paper arises from those of ([6], [7]; see also [2], [5]), where in [6], [7] the authors proved several interesting and deep theorems on power series $\sum_{n=0}^{\infty} a_n x^n$, a_n's and x are real, in relation to Borel classification of sets, Baire classification of functions, connectedness of mappings, porosity character and Hausdorff dimension of sets etc. Our approach in this paper is somewhat different. Instead of power series, we consider infinite series of real terms and after defining a mapping suitably, we mainly study the behaviour of the mapping from various aspects. Since our context is different from [6], [7], the technique of proofs of the theorems also differ considerably from [6],[7].

1991 *AMS Subject Classification*: 40A05.

2. Definitions

We consider the set s of all real sequences $a = \{a_k\}$ with the Fréchet metric $d(a,b)$ given by

$$d(a,b) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k - b_k|}{1 + |a_k - b_k|},$$

where $a = \{a_k\} \in s$ and $b = \{b_k\} \in s$. It is known that the metric space (s,d) is complete and has the power of the continuum.

We introduce a mapping σ on the set s in the following way. If $a = \{a_k\} \in s$, then $\sigma(a) = \limsup_{n\to\infty} S_n$ where S_n is the nth partial sum of the series Σa_k i.e. $S_n = a_1 + a_2 + \ldots + a_n$. Clearly σ is a mapping from s to $[-\propto, \propto]$. In particular if $a = \{a_k\} \in s$ be such that Σa_k is convergent or properly divergent then $\sigma(a) = \Sigma a_k$. If x, y etc. are members of s, we shall represent them generally by $x = \{x_k\}$, $y = \{y_k\}$ etc. Also N denotes the set of positive integers and R denotes the set of real numbers.

3. Some set-theoretic properties of the function σ

We first enquire if the mapping σ is injective and surjective. The following theorem gives an answer to the query.

Theorem 1 *The mapping σ is surjective but not injective.*

Proof. It is clear that σ is not injective. We show that σ is surjective. Let $t \in [-\propto, \propto]$. We find a sequence $\{q_n\}$ of real numbers such that $q_n \to t$. Let $a_1 = q_1, a_2 = q_2 - q_1$, $a_3 = q_3 - q_2, \ldots, a_n = q_n - q_{n-1}$ and so on. Then $a = \{a_n\} \in s$ and $\sigma(a) = t$. □

Theorem 1 can be strengthened in one part as follows. For this, let H_t denote the set

$$H_t = \{x \in s; \sigma(x) = t\}$$

where $t \in [0, \propto]$.

Theorem 2 *The set H_t has the cardinality c where c is the power of the continuum.*

Proof. We can find a strictly monotone sequence $\{q_n\}$ of real numbers such that $q_n > 0 \forall\ n$ and $q_n \to t$ as $n \to \propto$. Construct all the sequences of the form $\{\in_n .q_n\}$ where $\{\in_n\}$ runs over sequences of 0's and 1's with infinite number of 1's. From a single sequence $\{\in_n .q_n\}$ we construct a sequence $b = \{b_n\}$ as in Theorem 1. Then $\sigma(b) = t$. One may easily verify that different sequences $\{\in_n q_n\}$ correspond to different sequences $\{b_n\} \in s$. Since the cardinality of all the sequences of the form $\{\in_n q_n\}$ is c, the theorem follows. □

We now investigate some properties of H_t in terms of Borel classification and Baire category of sets. For this, we first prove the following Lemma.

Lemma 1 *For arbitrary $t \in (-\infty, \infty)$, the set*

$$P_t = \{x \in s; \sigma(x) < t\}$$

is a F_σ-set in s.

Proof. We can write

$$P_t = \{x \in \mathbf{s}; \sigma(x) < t\} = \bigcup_{j=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p+1}^{\infty} F(n,j)$$

where

$$F(n,j) = \left\{x \in \mathbf{s}; S_n = x_1 + \ldots + x_n \leq t - \frac{1}{j}\right\} (n, j \in N).$$

We consider the set $F(n,j)$ for fixed n and j. Let $\{x^{(r)}\}_{r=1}^{\propto} \in F(n,j)$ and $x^{(r)} \to x$ as $r \to \infty$ where $x \in \mathbf{s}$. Let $x^{(r)} = \{x_n^{(r)}\}$ and $x = \{x_n\}$. Since $x^{(r)} \in F(n,j)$,

$$x_1^{(r)} + x_2^{(r)} + \ldots + x_n^{(r)} \leq t - \frac{1}{j}$$

for $r = 1, 2, \ldots$ Because the convergence in s is equivalent to coordinate convergence, we see that

$$x_1 + x_2 + \ldots + x_n \leq t - \frac{1}{j},$$

i.e. $x \in F(n,j)$ and $F(n,j)$ becomes closed in s. This proves that P_t is F_σ. □

Corollary 1 *For arbitrary $t \in (-\infty, \infty)$, the set*

$$P'_t = \{x \in s; \sigma(x) \geq t\}$$

is a G_δ-set in s.

Theorem 3 *The set $H_\propto = \{x \in s;\ \sigma(x) = \infty\}$ is a G_δ-set and $H_{-\infty} = \{x \in s; \sigma(x) = -\infty\}$ is a $F_{\sigma\delta}$-set.*

Proof. We can write $H_\propto = \bigcap_{m=1}^{\infty} P'_m$ and Corollary 1 finishes the first part. Again $H_{-\propto} = \bigcap_{m=1}^{\infty} P_{-m}$ where by Lemma 1 each P_{-m} is a F_σ-set. Therefore, $H_{-\propto}$ is a $F_{\sigma\delta}$-set. □

Corollary 2 *The set $P_\propto = \{x \in \mathbf{s}; \sigma(x) < \infty\}$ is a F_σ-set.*

Corollary 3 *The set $G = \{x \in \mathbf{s}; -\infty < \sigma(x) < \infty\}$ is a $G_{\delta\sigma}$ set in s.*

Proof. Corollary 3 follows from Theorem 3 because $G = s - (H_\infty \cup H_{-\infty})$ and in a metric space every G_δ-set is an $F_{\sigma\delta}$-set. □

Theorem 4 *The set $H_\propto = \{x \in \mathbf{s}; \sigma(x) = \propto\}$ is residual in* $\mathbf{s}$.

Proof. By Theorem 3, $H_\propto$ is a G_δ-set in s. Consider all the members of s of the form

$$x = \{x_1, x_2, \ldots, x_p,\ p+1, p+2, \ldots\},$$

where p runs over all positive integers and $x_1, \ldots, x_p$ run over all real numbers. Clearly, for any such $x, \sigma(x) = \propto$ and so all such x's belong to $H_\propto$.

Now, let $x = \{x_t\}$ be an arbitrary element of s. We consider the elements

$$\begin{aligned}
x^{(1)} &= \{x_1, 2, 3, \ldots\} \\
x^{(2)} &= \{x_1, x_2, 3, 4, \ldots\} \\
&\vdots \\
x^{(r)} &= \{x_1, x_2, \ldots, x_r, r+1, r+2, \ldots\}
\end{aligned}$$

etc.
Then each $x^{(r)}$ belongs to $H_\propto$ and $x^{(r)} \to x$ as $r \to\propto$. This shows that H_∞ is dense in s. Hence H_∞ as a dense G_δ-set is residual in s ([3], p. 48). □

Corollary 4 *The set* $\{x \in \mathbf{s}; -\infty \leq \sigma(x) < \infty\}$ *is a set of the first category in* s.

4. Connected properties of the mapping σ

In this section we analyse the connected properties of the mapping $\sigma : s \to [-\propto, \propto]$. We observe that the map of a connected subset of s by σ is not necessarily connected in $[-\propto, \propto]$. In other words, σ is not a connected mapping. While going to prove this result, we came across an interesting allied result which we first prove.

Theorem 5 *Given any connected set* A *in* $(-\propto, \propto)$ *there is a connected set* B *in* s *such that* $\sigma(B) = A$.

Proof. We may clearly assume that A is an interval and we suppose first that A is a bounded closed interval.

So let $A = [a, b], -\propto< a < b <\propto$. We find two convergent series Σx_n and Σy_n of real numbers whose sums are a and b, respectively. So $x = \{x_n\}$ and $y = \{y_n\} \in s$ and $\sigma(x) = a$ and $\sigma(y) = b$. For $t \in [0, 1]$, construct $z(t) \in s$ such that

$$z(t) = \{z_n\} \text{ where } z_n = tx_n + (1 - t)y_n \forall\ n.$$

It is clear that $\sigma(z(t)) = ta + (1 - t)b$. Now since the map $f : [0, 1] \to s$ defined by $f(t) = z(t)$ is continuous and every point of $[a, b]$ can be expressed in the form $ta+(1-t)b$ for some $t \in [0, 1]$, we have $f([0, 1]) = B$ (say) is a connected set in s with $\sigma(B) = [a, b]$. After slight modification, the proof can be constructed when A is a bounded open or half-open interval.

If $A = [a, \propto)$, $-\propto< a <\propto$, let m be the least positive integer greater than a. We can write then

$$A = [a, m] \cup [a, m + 1] \cup \ldots\ldots$$

As in the preceding case we can find a non-void connected set B_1 (say) in s (corresponding to convergent series only) such that $\sigma(B_1) = [a, m]$. Again taking a member x from B_1 with $\sigma(x) = m$ and another $y \in s$ (corresponding to a convergent series) with $\sigma(y) = m+1$, we can construct a connected set B_2', as before, such that $\sigma(B_2') = [m, m+1]$ and $x \in B_1 \cap B_2'$. So if $B_2 = B_1 \cup B_2'$, then B_2 is connected and $\sigma(B_2) = [a, m+1]$. Proceeding in this way we obtain a sequence of non-void connected sets $\{B_i\}$ in s with $\sigma(B_i) = [a, m+(i-1)]$ and $B_i \subset B_{i+1} \forall\ i$. Hence $B = \cup B_i$ is a connected set in s and $\sigma(B) = [a, \propto)$. If A is any other type of unbounded interval, the modification in the proof is evident. The theorem is, therefore, proved. □

From the next theorem it follows that σ is not a connected mapping. We need first the following lemma.

Lemma 2 *For $a \in (0, \infty)$, let $B_a = \{x = \{x_n\} \in s$; there exists j (even) such that $x_{j+1} = -(x_1 + \ldots + x_j) + a$, $x_{j+n} = (-1)^{n-1}.a$ for $n \geq 2\}$.*
Then B_a is a connected set in s and $\sigma(B_a) = \{a\}$.

Proof. Clearly B_a is non-void because, for example, the element $\alpha = \{x_1, x_2, -(x_1 + x_2) + a, -a, a, -a, \ldots\}$ is a member of B_a where x_1, x_2 are real numbers. Let $x = \{x_n\} \in B_a$, then $\sigma(x) = a$. By definition there exists j (even) such that $x_{j+n} = (-1)^{n-1}a$ for $n \geq 2$. Let $t \in [0, 1]$. We construct the sequence $b(t) = \{b_n\}$ as follows:

$$\begin{aligned} b_n &= tx_n + (-1)^{n-1}a \text{ for } n \leq j \\ &= -tS_j + a \text{ for } n = j+1, S_j = x_1 + \ldots + x_j \\ &= (-1)^{n-1}a \text{ for } n > j+1. \end{aligned}$$

So $b(t) \in B_a$ for each $t \in [0, 1]$ and we consider the map $f_x : [0, 1] \to s$ defined by $f_x(t) = b(t)$. Then one may easily show that f_x is continuous on $[0, 1]$ and so $f_x[0, 1]$ is a connected set in s. Also $f_x[0, 1] \subset B_a$. This inequality holds for each $x \in B_a$.
So $\underset{x \in B_a}{\cup} f_x\ [0, 1] \subset B_a$.

Further,

$$\begin{aligned} f_x(0) = b(0) &= (-1)^{n-1}a \text{ for } n \leq j \\ &= a \text{ for } n = j+1 \\ &= (-1)^{n-1}a \text{ for } n > j+1. \end{aligned}$$

So for each $x \in B_a$, the above sequence is contained in $f_x[0,1]$. Hence, $\underset{x \in B_a}{\cap} f_x[0,1] \neq \phi$. This shows that $\underset{x \in B_a}{\cup} f_x[0,1]$ is connected and the proof will be complete if we show that $B_a \subset \underset{x \in B_a}{\cup} f_x[0,1]$.

Let $\alpha \in B_a$. Then α can be written in the form

$$\alpha = \{x_1, x_2, \ldots, x_j (j \text{ even}), -(x_1 + \ldots + x_j) + a, -a, a, \ldots\}.$$

We show that $\alpha \in f_x[0,1]$ for some $x \in B_a$. For this, let

$$x = \{x_1 - a, x_2 + a, \ldots, x_j + a, -(x_1 + \ldots + x_j) + a, -a, a, \ldots\}.$$

Then $x \in B_a$. Now if

$$b_1 = t(x_1 - a) + a, b_2 = t(x_2 + a) - a, \ldots, b_j = t(x_j + a) - a,$$

$$b_{j+1} = -t(x_1 + \ldots + x_j) + a,\ b_{j+n} = (-1)^{n+1}a \text{ for } n \geq 2,$$

then $b(t) = \{b_n\} \in B_a$ and also $f_x(t) = b(t)$ and therefore

$$f_x(1) = b(1) = \{x_1, x_2, \ldots, x_j, -(x_1 + \ldots + x_j) + a, -a, a \ldots\} = \alpha.$$

So $\alpha \in f_x[0,1]$ and thus $B_a \subset \underset{x \in B_a}{\cup} f_x[0,1]$. This, in conjunction with the reverse inequality, gives

$$B_a = \underset{x \in B_a}{\cup} f_x[0,1].$$

So B_a is connected and $\sigma(B_a) = \{a\}$. This proves the lemma. □

Theorem 6 *Let A be an arbitrary non-void subset of $(0, \propto)$. Then there exists a connected set $B \subset s$ such that $\sigma(B) = A$.*

Proof. Let $B_a, a \in A$ have the same meaning as in Lemma 2. Put $B = \underset{a\in A}{\cup} B_a$. Then by Lemma 2, $\sigma(B) = A$. We show that B is connected and for this we prove that no two of the sets $\{B_a; a \in A\}$ are separated {see [1], p. 54}. Let $a_1, a_2 \in A, a_1 \neq a_2$. Let $x = \{x_n\} \in B_{a_1}$ and let $\in > 0$ be given. From the definition of B_{a_1}, there is a j (even) such that

$$x_{j+1} = -(x_1 + \ldots + x_j) + a_1, x_{j+n} = (-1)^{n-1}a_1 \forall\ n \geq 2.$$

Choose an even $i \in N$ such that $\sum\limits_{k>i} \frac{1}{2^k} < \in$. Now construct $y = \{y_k\} \in s$ as follows:

$$\begin{aligned} y_k &= x_k \text{ for } k \leq i \\ &= -(x_1 + \ldots + x_i) + a_2 \text{ for } k = i+1 \\ &= (-1)^{k-i-1}a_2 \text{ for } k > i+1. \end{aligned}$$

Then $y \in B_{a_2}$ and we have also $d(x, y) < \in$. This shows that every $\in$-ball of x contains a member of B_{a_2} which implies that $x \in \overline{B}_{a_2}$. Thus $B_{a_1} \subset \overline{B}_{a_2}$. Similarly $B_{a_2} \subset \overline{B}_{a_1}$. This completes the proof. □

5. Baire Classification of σ

The following lemma is needed.

Lemma 3 *Let $a = \{a_n\} \in s$ and $\delta > 0$. Then $\sigma(B(a, \delta)) = [-\propto, \propto]$ where $B(a, \delta) = \{x \in s; d(a, x) < \delta\}$.*

Proof. Choose $m \in N$ such that $\sum\limits_{k=m+1}^{\infty} \frac{1}{2^k} < \delta$. Let $t \in [-\propto, \propto]$.

Let $\{q_n\}$ be a sequence in R such that $\lim q_n = t$. We construct a sequence $y = \{y_n\} \in s$ as follows:

$$\begin{aligned} y_n &= a_n \text{ for } n \leq m \\ &= -a_{n-m} \text{ for } n = m+1 \text{ to } 2m \\ &= b_n \text{ for } n > 2m, \end{aligned}$$

where $b_{2m+1} = q_1, b_{2m+2} = q_2 - q_1, b_{2m+3} = q_3 - q_2$ and so on. Then clearly $d(a, y) < \delta$ i.e. $y \in B(a, \delta)$ and $\sigma(y) = t$. This proves the lemma. □

Before going to prove the next theorem we state, as a convention, that any ray like $x > a$, where a is a real number is to be treated as a neighbourhood of $+\infty$. Similarly for $-\infty$.

Theorem 7 *The function σ is discontinuous everywhere in s.*

Proof. Let $a \in s$ and suppose first that $-\infty < \sigma(a) < +\infty$. Then by Lemma 3, given $\in > 0$ there is no $\delta > 0$ such that $\sigma\ (B\ (a,\ \delta)) \subset (\sigma(a) - \in, \sigma(a) + \in)$. So σ is discontinuous at a. If $\sigma(a) = +\infty$ (or $-\infty$), similar argument shows that σ is discontinuous at a. This proves the theorem. □

The next theorem is related to the idea of Baire classification of functions. As it appears, these classes may not be widely known, we state in brief, the ideas pertaining to these classes for ready reference.

All real-valued continuous functions defined on $[a, b]$ or in a metric space are said to belong to Baire class 0. Functions $f(x)$ which do not belong to Baire class 0 but are representable in the form

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{1}$$

where all $f_n(x)$ belong to Baire class 0, are said to belong to Baire class 1. In general, the functions of Baire class m are those which do not belong to any of the preceding classes but are representable in the form (1) where all $f_n(x)$ belong to Baire class $m - 1$ and so on. We denote this sequence of classes by

$$B_0, B_1, \dots, B_m, \dots \tag{2}$$

There may exist a function $f(x)$ which does not belong to any of the classes (2) but is representable in the form (1) where the functions $f_n(x)$ belong to any of the classes in (2). Then such a function is said to belong to the class H_ω where ω is the order type of the set of all positive integers written in their natural order. To have an idea about further classification one may see ([4], p. 128; cf. [8], p. 185).

Theorem 8 *The function σ belongs exactly to the second Baire class.*

Proof. Since σ is discontinuous everywhere in s (Theorem 7), it cannot belong to zero or first Baire class (cf. [8], p. 185). We show that σ belongs to second Baire class.

Let $a = \{a_n\} \in s$. We put

$$\sigma_m(a) = \sup\{S_m, S_{m+1}, \dots\}, m = 1, 2, 3, \dots$$

where $S_n = a_1 + a_2 + \dots + a_n$. Then $\sigma(a) = \lim_{m\to\infty} \sigma_m(a)$.

We put further

$$\sigma_{m,k}(a) = \max\{S_m, \dots, S_{m+k}\}, m, k = 1, 2, 3, \dots$$

Then

$$\sigma(a) = \lim_{m\to\propto} \lim_{k\to\propto} \sigma_{m,k}(a). \tag{3}$$

We shall show that for fixed m, k, the function $\sigma_{m,k} : s \to R$ is continuous and the assertion then follows from (3).

Let $\in > 0$ be given. Choose $\delta > 0$ such that $\frac{\delta}{1-\delta} = \frac{\in}{2(m+k)}$ and let $b = \{b_n\} \in B(a, \frac{\delta}{2^{m+k}})$. Then $d(a, b) < \frac{\delta}{2^{m+k}}$,

$$i.e. \qquad \frac{|a_j - b_j|}{1 + |a_j - b_j|} < \delta \text{ for } j = 1, 2, \dots, m+k,$$

which implies that

$$|a_j - b_j| < \frac{\delta}{1-\delta} = \frac{\in}{2(m+k)} \text{ for } j = 1, 2, \dots, m+k.$$

If

$$S'_n = b_1 + b_2 + \dots + b_n,$$

then

$$|S_l - S'_l| \le |a_1 - b_1| + |a_2 - b_2| + \dots + |a_l - b_l|$$

$$< l.\frac{\in}{2(m+k)} < \in \text{ for } l = m, m+1 \dots, m+k.$$

If $\sigma_{m,k}(a) = S_i$ and $\sigma_{m,k}(b) = S'_p, m \leq i, p \leq m + k$, then because S_i and S'_p are respectively maximum values, we obtain

$$\sigma_{m,k}(b) = S'_p \geq S'_i \geq S_i - \in = \sigma_{m,k}(a) - \in$$

and

$$\sigma_{m,k}(a) = S_i \geq S_p \geq S'_p - \in = \sigma_{m,k}(b) - \in .$$

Hence we have $|\sigma_{m,k}(a) - \sigma_{m,k}(b)| < \in$. This proves the theorem. □

6. Porosity character of the set $P_\propto$

We have seen in Corollary 2 that the set $P_\infty = \{x \in s; \sigma(x) < \infty\}$ is F_σ. In this section we study more closely the behaviour of the set $P_\propto$ from the view point of porosity character of sets.

For convenience we quote several definitions and notations from the theory of porosity of sets ([7], [9]-[11]). Let (Y, ρ) be a metric space. If $y \in Y$ and $r > 0$, we denote as before, by $B(y, r)$ the ball with centre y and radius r. Let $M \subset Y$. Let

$\nu(y, r, M) = \sup\{t > 0;$ there is a $z \in B(y, r)$ such that $B(z, t) \subset B(y, r)$ and $M \cap B(z, t) = \phi\}$.

Further let

$$\overline{p}(y, M) = \limsup_{r \to 0^+} \frac{\nu(y, r, M)}{r},$$

$$\underline{p}(y, M) = \liminf_{r \to 0^+} \frac{\nu(y, r, M)}{r},$$

and if $\overline{p}(y, M) = \underline{p}(y, M)$, then we set

$$p(y, M) = \overline{p}(y, M) = \underline{p}(y, M) = \lim_{r \to 0^+} \frac{\nu(y, r, M)}{r}.$$

The set $M \subset Y$ is said to be *porous* at $y \in Y$ if $\overline{p}(y, M) > 0$ and σ-porous at $y \in Y$ provided $M = \bigcup_{n=1}^{\infty} M_n$ and each of the sets M_n is porous at y. M is said to be porous or σ-porous in $Y_0 \subset Y$ if it is so at each point $y \in Y_0$.

The set $M \subset Y$ is said to be *very porous* at $y \in Y$ if $\underline{p}(y, M) > 0$ and *very strongly porous* at $y \in Y$ if $p(y, M) = 1$. M is very (strongly) porous in $Y_0 \subset Y$ if it is so at each $y \in Y_0$.

Further, $M \subset Y$ is said to be *uniformly very porous* [11] in $Y_0 \subset Y$ if there is a $c > 0$ such that for each $y \in Y_0$ we have $\underline{p}(y, M) \geq c$.

Definition [7] (a) $M \subset Y$ is said to be *uniformly σ-very porous* in $Y_0 \subset Y$ provided that $M = \bigcup_{n=1}^{\infty} M_n$ and there is a $c > 0$ such that for each $y \in Y_0$, and for each $n = 1, 2, \ldots$ we have $\underline{p}(y, M_n) \geq c$.

(b) $M \subset Y$ is said to be *uniformly σ-very strongly porous* in $Y_0 \subset Y$ if $M = \bigcup_{n=1}^{\infty} M_n$ and for each $y \in Y_0$ and for each $n = 1, 2, \ldots$ we have $p(y, M_n) = 1$.

Theorem 9 *The set $P_\propto$ is uniformly σ-very strongly porous in $s - P_\propto$.*

Proof. By Corollary 2, $P_\propto$ is an F_σ-set. The proof now follows from the following lemma. □

Lemma A[7] Let (Y, ρ) be a metric space. Let $M \subset Y, M$ be an F_σ-set in Y. Then M is uniformly σ-very strongly porous in $Y - M$.

The following theorem describes the porosity character of the set $P_\propto$ in the whole space s.

Theorem 10 *The set $P_\propto$ is uniformly σ-very porous in the space s.*

Proof. Evidently $P_\propto = \{x \in s; \sigma(x) <\propto\}$

$$= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{n,k},$$

where $B_{n,k} = \{x = \{x_i\} \in s; S_j = x_1 + \ldots + x_j \leq n \text{ for all } j \geq k\}$. We shall prove that at each $y \in s, \underline{p}(y, B_{n,k}) \geq \frac{1}{8}$ $(n, k = 1, 2, \ldots)$. Let $y = \{y_i\} \in s$ and $r > 0$. We consider the ball $B(y, r)$. For fixed n, k, we may assume without any loss of generality that $r < 2^{-k}$. So there is a $m \in N$ such that $2^{-m-1} \leq r \leq 2^{-m}$ and clearly $m \geq k$. We now construct the sequence $z = \{z_i\}$ as follows:

$$z_i = y_i \text{ for } i \neq m+2$$

and

$$z_{m+2} = n + 1 + (m+2) - (y_1 + y_2 + \ldots + y_{m+1}).$$

Then $z \in s$ and $d(z, y) < 2^{-m-2}$ so that $z \in B(y, r)$. We shall show that there is a $\delta > 0$ such that

a) $B(z, \delta) \subset B(y, r)$

and

b) $B(z, \delta) \cap B_{n,k} = \phi$.

If we take $\delta = 2^{-m-3}$, then if $x \in B(z, \delta)$, we have $d(x, z) < \delta < 2^{-m-2}$ and so

$$d(x, y) \leq d(x, z) + d(z, y) < 2^{-m-2} + 2^{-m-2} = 2^{-m-1} < r.$$

Hence $x \in B(y, r)$ and so $B(z, \delta) \subset B(y, r)$, i.e. (a) holds.

For (b) let $t = \{t_i\} \in B(z, \delta)$. Then from the definition of the metric d we see that $d(t, z) < \delta$

$$\text{i.e.} \qquad \frac{1}{2^i} \frac{|t_i - z_i|}{1 + |t_i - z_i|} < \delta = 2^{-m-3} \ \forall \ i,$$

i.e.

$$\frac{|t_i - z_i|}{1 + |t_i - z_i|} < \frac{1}{2} \text{ for } i = 1, 2, \ldots, m+2.$$

Since the function $\frac{x}{1+x} (x > 0)$ is monotonically increasing, we have from the above

$$|t_i - z_i| < 1 \text{ for } i = 1, 2, \ldots, m+2.$$

Now if we denote by $S_p = z_1 + z_2 + \ldots + z_p$ and $S'_p = t_1 + t_2 \ldots + t_p$, then from above

$$|S'_{m+2} - S_{m+2}| < (m+2).$$

Hence $S'_{m+2} > S_{m+2} - (m+2) > n + 1 + (m+2) - (m+2) = n+1$.

Since $m + 2 > k,\ t \notin B_{n,k}$. This proves (b). So we get

$$\frac{\nu(y, r, B_{n,k})}{r} \geq \frac{\delta}{r} \geq 2^{-m-3}.2^m = \frac{1}{8}.$$

Hence $\underline{p}(y, B_{n,k}) \geq \frac{1}{8}$ and this is true for all $n, k = 1, 2, \ldots$ and for all $y \in s$. This proves the theorem. $\square$

Acknowledgement

The authors are thankful to the referee for his kind suggestions which considerably improved the presentation of the paper.

References

[1] Kelley, J.L.: *General Topology*, New York -London, D. Van Nostrand Comp. INC 1955.

[2] Kostyrko, P. and Śalát, T.: On the exponent of convergence, *Rend. Circ. Matem. Palermo.*, 31, 187-194 (1982).

[3] Kuratowski, K.: *Topologie I*, Warszawa, PWN 1958.

[4] Natanson, I.P.: *Theory of functions of a real variable II*, New York, Frederick Ungar Publishing Co., 1960.

[5] Śalát, T.: On exponents of convergence of subsequences, *Czechosl. Math. J.*, 34 (109), 362-370 (1984).

[6] Śalát, T.: and Tóth, J.T.: On radii of convergence of power series, *Bull. Math. Soc. Sci. Math. Roum.*, 38, 183-198 (1994-95).

[7] Śalát, T., Taylor, S.J. and Tóth, J.T.: Radii of convergence of power series, *Real Anal. Exchange*, 24(1), 263-274 (1998-99).

[8] Sikorski, R.: *Funkeje rzeczywiste I*, Warszawa, PWN 1958.

[9] Tkadlec, J.: Constructions of some non-σ-porous sets of real line, *Real Anal. Exchange*, 9, 473-482 (1983-84).

[10] Zajićek, L.: Sets of σ-porosity and sets of σ-porosity (q), *Cas. Pest. Mat.*, 101, 350-359 (1976).

[11] Zajićek, L.: Porosity and σ-porosity, *Real Anal. Exchange*, 13, 314-350 (1987-88).

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Received 14.02.2002