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On Summand Sum and Summand Intersection Property of Modules

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Abstract

R will be an associative ring with identity and modules M will be unital left $R-$modules. In this work, extending modules and lifting modules with the SSP (or SIP) are studied. A necessary and sufficient condition for a module M to have the SSP is that for every decomposition $M = A \oplus B$ and $f \in \mathrm{Hom}(A, B)$, $\mathrm{Im}(f)$ is a direct summand of B. Among others it is shown also that a (C_3) module with the SIP has the SSP, and a (D_3) module with SSP has the SIP.

Key Words: SIP modules, SSP modules, extending modules, lifting modules.

Throughout this work all rings will be associative with identity and modules will be unital left modules. Let R be a ring and M a module. $N \leq M$ will mean N is submodule of M. A submodule N of a module M is called *small* in M, denoted by $N << M$, whenever for some submodule L of M, $N + L = M$ implies $L = M$. A module M is said to be *small* if M is small in its injective hull $E(M)$. $0 \neq N \leq M$ is said to be an *essential* submodule of M, denoted by $N \leq_{ess} M$, if for every $0 \neq L \leq M$, $N \cap L \neq 0$. We write $N \leq_d M$ to abbreviate N is a (direct) summand of M.

We recall some definitions and properties as follows

(SSP) A module M has the summand sum property (SSP) if the sum of two direct summands is a direct summand of M;

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(C_1) Every submodule of M is essential in a summand of M;

(C_2) If a submodule A of M is isomorphic to a summand of M, then A is summand of M; and

(C_3) If M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a summand of M.

A submodule N of M is said to be *closed* in M if there is no proper essential extension of N in M and denoted by $N \leq_c M$. Modules with C_1 are called **extending** (or **CS**)-modules. A module M is an extending module if and only if every closed submodule in M is direct summand of M. A module M is called **quasi-continuous** if M has (C_1) and (C_3), and **continuous** if M has (C_1) and (C_2). We then have

$(C_2) \Rightarrow (C_3)$, $SSP \Rightarrow (C_3)$ and continuous $\Rightarrow$ quasi-continuous.

(SIP) An $R-$module M has the summand intersection property (SIP) if the intersection of two summands is again a summand, and M has the strong summand intersection property (SSIP) if the intersection of any number of summands is again a summand.

Now recall the conditions (D_i) dual of the conditions (C_i)
respectively:

(D_1) For every submodule A of a module M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 << M_2$.

(D_2) If $A \leq M$ such that M/A is isomorphic to a summand of M, then A is a summand of M.

(D_3) If A and B are summands of M with $A + B = M$, then $A \cap B$ is summand of M.

Modules with (D_1) are called **lifting** and modules with (D_1) and (D_2) are called **discrete**, and modules with (D_1) and (D_3) are called **quasi-discrete** modules.

We have the implications $(D_2) \Rightarrow (D_3)$, SIP $\Rightarrow (D_3)$, Discrete $\Rightarrow$ Quasi-discrete.

Modules having the SSP and the SIP were motivated by the works of Kaplansky and Fuchs. Kaplansky proves in his book [6] that if M is a free module over a principal ideal domain R, then M has the SIP. And Fuchs suggested the following problem in his book Infinite Abelian Groups.

Problem 9 Characterize the abelian groups in which the intersection of two direct summands is again a summand.

So arose naturally the problem of modules having SSP and their endomorphism rings if they have the SSP or the SIP. Garcia studied this problem in [4] while Wilson studied modules having SIP over Noetherian domains in [11].

In this note we study D_i-modules $(i = 1, 2, 3)$ with SIP and C_i-modules $(i = 1, 2, 3)$ with the SSP. We start with Example 1 below.

There exist modules with D_2 but have neither the SIP nor the SSP.

Example 1 *Let F be a field and let R denote the following ring:*

$$R = \{ \begin{pmatrix} a & 0 & 0 & 0 \\ y & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & x & a \end{pmatrix} : a, b, x, y \in F \}$$

We consider R as a left R-module. Then R satisfies (D_2) since every projective module satisfies (D_2). We show that R does not have neither the SIP nor the SSP. Let $N = \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & x & 0 \end{pmatrix} : b, x \in F \}$ *and* $K = \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & x & 0 \end{pmatrix} : b, x \in F \}$ *be left ideals of R. Then N and K are direct summands of R, and since* $N \cap K = \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix} : x \in F \}$ *is nilpotent the left ideal, $N \cap K$ is not a direct summand of R. It is easy to check that the left ideal* $N + K = \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u & v & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & z & 0 \end{pmatrix} : u, v, z \in F \}$ *is a proper essential left ideal of R and so not a direct summand of R. Then R does not have the SSP.*

We state and prove Lemma 2 for an easy reference.

Lemma 2 [7] *Let M_1 be a simple module and M_2 an uniserial module with composition series $0 \subset U \subset M_2$. Then $M = M_1 \oplus M_2$ is a lifting module.*

Proof. Let L be a non-zero submodule of M. We show that there exists a submodule K of M such that $M = K \oplus K'$, $K \leq L$ and $L \cap K'$ is small in K' for some submodule K' of M. If $M_1 \cap (L_1 + M_2) = 0$ then $L \leq M_2$. Hence L is a small submodule or direct summand of M. Suppose that $M_1 \cap (L + M_2) \neq 0$. Then $M_1 \leq L + M_2$ and $M = L + M_2$. If $L \cap M_2 = M_2$ or $L \cap M_2 = 0$ or $L \cap M_2 = U$ and $L \cap M_1 = M_1$ we are done. Assume $L \cap M_2 = U$ and $L \cap M_1 = 0$. Then $U \leq L$. Hence $M = L \oplus M_1$. Thus M has (D_1). □

There are modules having the SSP and (D_1) but not the SIP.

Example 3 *Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ be the ring of upper triangular matrices over F, $N = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $L = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ left ideals of R and $M = R/L$. Let $U = N \oplus M$. Then by [4, Remark on page 81] and Lemma 2, U has the SSP and (D_1) but has not the SIP as left R-module.*

There are modules having the SIP but not the SSP.

Example 4 *Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ be the ring of upper triangular matrices over F, $N = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $L = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ left ideals of R and $M = R/L$. Let $U = N \oplus M$. Then by [4, Remark on page 81] the ring $S = End\, U$ has the SIP on each side but does not have the SSP on the left.*

Example 5 *Let M denote the $\mathbb{Z}-$module $\mathbb{Z} \oplus \mathbb{Z}$. Let N be a submodule of M. It is easy to check that N is a direct summand of M if and only if N has the form $N = \mathbb{Z}(a, b)$ for some integers a, b with the property that the greatest common divisor of a and b is 1. Consider the submodules $\mathbb{Z}(2, 3)$ and $\mathbb{Z}(3, 2)$ of M. Then they are direct summands of M and $[\mathbb{Z}(2, 3)] \cap [\mathbb{Z}(3, 2)] = 0$, and it is clear that $\mathbb{Z}(2, 3) \oplus \mathbb{Z}(3, 2)$ is not a direct summand* *of M. Hence M has not the SSP. Also, for any two distinct direct summands K and N of M their intersection $K \cap N$ is always zero. It follows that M has the SIP.*

As an easy reference we record the following properties of modules with the SIP and the SSP from [4, 11]

Proposition 6 *(i) M has the SIP (resp. the SSP) if and only if for every pair of summand S and T with $\pi : M \to S$ the projection map, the kernel of the restricted map $\pi_{|T}$ (resp. the image of the restricted map $\pi_{|T}$) is summand.*

(ii)If M has the SIP (resp. the SSP) and $S \oplus T$ is summand of M, then the kernel of any homomorphism from S to T (resp. the image of any homomorphism from S to T) is a summand.

Proposition 7 [3] *The R-module M has the summand intersection property if and only if for every decomposition $M = A \oplus B$ and every homomorphism f from A to B, the kernel of f is a direct summand.*

One way of the following Theorem is given as an exercise 39.17 (3) (i) in [12] on page 339 and it is proved in [4]. We prove the other way.

Theorem 8 *The R-module M has the summand sum property if and only if for every decomposition $M = A \oplus B$ and every homomorphism f from A to B, the image of f is a direct summand of B.*

Proof. The necessity is proved in [4]. For the sufficiency assume that for every decomposition $M = A \oplus B$ and every homomorphism f from A to B, the image of f is a direct summand of B. Let N and K be direct summands of M and $M = N \oplus N'$ and $M = K \oplus K'$ for some $N' \leq M$ and $K' \leq M$. We prove $N + K$ is direct summand. Let π_K and $\pi_{N'}$ denote the projections of M onto K and N', respectively. Let A denote $\pi_{N'}(\pi_K(N))$. Then $A = (N + K') \cap (N + K) \cap N'$ and, by assumption, A is a direct summand and $M = A \oplus L$ for some $L \leq M$. Hence $N' = A \oplus (N' \cap L)$. Then $(N + K) \cap [(N + K') \cap (N' \cap L)] = [(N + K) \cap (N + K') \cap N'] \cap (N' \cap L) = A \cap (N' \cap L) = 0$. To show that $N + K$ is direct summand, it is enough to prove that $M = (N + K) + [(N + K') \cap (N' \cap L)]$. Since $A \leq N + K$ and $A \leq N + K'$, the modular law and $M = N \oplus N' = (N \oplus A) \oplus (N' \cap L)$ imply $N + K = (N \oplus A) \oplus [(N + K) \cap (N' \cap L)]$ and, $N + K' = (N \oplus A) \oplus [(N + K') \cap (N' \cap L)]$.

Hence $M = N + K' + K = (N \oplus A) + [(N + K) \cap (N' \cap L)] + [(N + K') \cap (N' \cap L)] \subseteq (N + K) + [(N + K') \cap (N' \cap L)]$. Thus $N + K$ is direct summand and so M has the SSP □

We use Theorem 8 to prove the following Theorem 9 and 10 which are Exercises 39.17 (3)(ii) and (iii) in the book [12] on Page 339.

Theorem 9 *Let R be a ring. The following are equivalent for R:*

1. *R is semisimple*
2. *Every R-module has the SSP*
3. *Every projective R-module has the SSP.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is trivial. Assume that (3) holds. We show that R is semisimple. Let K be a submodule of R. Choose a free module F and an epimorphism τ from F onto K. By assumption, the projective module $F \oplus R$ has the SSP. Let ι denote the injection map from K to R and $f = \iota\tau$ the homomorphism from F to R. Then $\mathrm{Im}f = K$ is a direct summand of R by Theorem 8. Hence R is semisimple ring. □

Theorem 10 *A ring R is left hereditary if and only if every injective R-module has the SSP.*

Proof. Suppose that R is a left hereditary ring. The every factor module of every injective R-module is injective. Let M be an injective module which has a decomposition $M = A \oplus B$. Let f be a homomorphism from A to B. Then A is injective. By assumption, $\mathrm{Im}f \cong A/\mathrm{Ker}f$ is injective. Hence $\mathrm{Im}f$ is direct summand of B. Thus it follows from Theorem 8 that M has the SSP. To prove the converse assume that every injective R-module has the SSP. Let M be an injective module and N a submodule of M. By assumption the injective hull $E(M/N)$ of M/N and the injective module $M \oplus E(M/N)$ have the SSP. Let ϕ denote the canonical mapping from M onto M/N and ι the injection of M/N into $E(M/N)$ and f the composition of $\iota\phi$. Then $\mathrm{Im}f = M/N$. By Theorem 8, M/N is direct summand of $E(M/N)$. Hence M/N is injective. Thus R is a left hereditary ring. □

Let $N \leq M$. Whenever $N \leq_{ess} K \leq M$ implies $N = K$, N is called (essentially) closed in M and we denote by $N \leq_c M$. A module M is said to be a *polyform* module if for every $K \leq M$ and $f \in \mathrm{Hom}(K, M)$ $\mathrm{Ker} f \leq_c K$ (see [2, 12]).

Lemma 11 *Let M be an extending polyform module. Then M has the SIP.*

Proof. Let M be an extending polyform module, and let $M = A \oplus B$ be a decomposition of M and $f \in \mathrm{Hom}(A, B)$. Being M polyform, $\mathrm{Ker}(f)$ is closed in K. Then $\mathrm{Ker}(f)$ is direct summand as a closed submodule of an extending module M. Hence M has the SIP. □

A module M is said to be *copolyform* if for $B \leq A \leq M$ and $A/B << M/B$ implies $\mathrm{Hom}(M/B, A/Y) = 0$ for $B \leq Y \leq A$ (see [5]).

Lemma 12 *Let M be a lifting coplyform module. Then M has the SSP.*

Proof. Let M be lifting coplyform module, and let A and B be direct summands of M and π projection from M onto A. Let K denote the image $\pi_{|B}(B)$ of the restriction of π to B. Since A is lifting module as a direct summand of M, there exists a decomposition $A = K_1 \oplus K_2$ such that $K_1 \leq K$ and $K \cap K_2 << K_2$. Then $K \cap K_2$ is also small in A and M and $K = K_1 \oplus (K \cap K_2)$. Hence we have a mapping from M onto $K \cap K_2$. Since M is coplyform, $K \cap K_2 = 0$ and so $K = K_1$ is direct summand of A.□

We consider the following conditions for a module M.

If $M_1 \leq_d M$, $M_2 \leq_d M$ with $M_1 + M_2 \leq_{ess} M$, then $M_1 + M_2 = M$ .... $(*)$

If $M_1 \leq_d M$, $M_2 \leq_d M$ with $M_1 \cap M_2 << M$, then $M_1 \cap M_2 = 0$ .... $(**)$

Lemma 13 *Let M be a module. If M satisfies $(*)$ (or $(**)$) then each direct summand of M satisfies $(*)$ (or $(**)$).*

Proof. Assume that the module M satisfies $(*)$. Let A be a direct summand such that $M = A \oplus B$ for some $B \leq M$ and A_1 and A_2 summands of A with $A_1 + A_2 \leq_{ess} A$. Then $A_2 + B$ and A_1 are direct summands of M and $A_1 + (A_2 + B) \leq_{ess} M$. Hence $A_1 + (A_2 + B) = M$ and so $A_1 + A_2 = A$. The remaining is proved dually. □

Proposition 14 *Let M be an extending module.*

1. *M has the SSP.*

2. *M satisfies $(*)$.*

3. *For any two direct summands M_1 and M_2 of M and for each homomorphism f from M_1 to M_2 with $Imf \leq_{ess} M_2$, $Imf = M_2$.*

Then $(1) \Longleftrightarrow (2)$ *and* $(3) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2)$ Clear.

$(2) \Longrightarrow (1)$. Assume that M satisfies $(*)$ and let M_1 and M_2 be direct summands of M. We prove that $M_1 + M_2$ is direct summand. Being M extending module there exists a direct summand A of M such that $M_1 + M_2$ is essential in A and $M = A \oplus B$ for some submodule B in M. By Lemma 13 $A = M_1 + M_2$.

$(3) \Longrightarrow (1)$. Assume that $M = A \oplus B$ is a decomposition with a homomorphism f from A to B. We show that $f(A)$ is a direct summand of B. $f(A)$ is either summand of B or contained essentially in a closed submodule C.

If $f(A)$ is a direct summand of B, there is nothing to prove in this case. Assume that $f(A)$ is contained essentially in a closed submodule C of B. By hypothesis C is direct summand of M and so is that of B, and then $B = C \oplus C'$ for some $C' \leq B$. Define the homomorphism $f \oplus 1$ from $A \oplus C'$ to B by $(f \oplus 1)(a + c') = f(a) + c'$ where $a \in A$ and $c' \in C'$. Then $Im(f \oplus 1) = f(A) \oplus C'$ is essential in $C \oplus C'$. By (3) $f \oplus 1$ is epimorphism and so $f(A) = C$. Therefore, (A) is direct summand. □

Note that in Proposition 14 $(1) \Longrightarrow (3)$ is not true in general. In fact let M denote the $\mathbb{Z}-$module $\mathbb{Z}$ and $M_1 = M_2 = M$. It is known that M is an extending module and has the SSP. Consider f as the map defined by $f(n) = 2n$ for $n \in M_1$. Then $\text{Im}f = \mathbb{Z}2 \leq_{ess} M_2$ and $\text{Im}f \neq M_2$.

Proposition 15 *Let M be a lifting module. Then*

1. *M has the SIP.*

2. *M satisfies $(**)$.*

3. *For any two direct summands M_1 and M_2 of M and for each homomorphism f from M_1 to M_2 with Ker$(f) << M_1$, Ker$(f) = 0$.*

Then (1) $\Longleftrightarrow$ (2) *and* (3) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2). It is trivial.

(2) $\Longrightarrow$ (1). Assume that M satisfies $(**)$. Let M_1 and M_2 be direct summands of M. We prove $M_1 \cap M_2$ is also a direct summand. We separate two cases:

If $M_1 \cap M_2$ is small in M then by $(**)$ $M_1 \cap M_2 = 0$.

Suppose that $M_1 \cap M_2$ is not small in M. Being M lifting module there exists a direct summand A of M such that $A \leq M_1 \cap M_2$, $M = A \oplus B$ and $(M_1 \cap M_2) \cap B << B$ for some $B \leq M$. Then $(M_1 \cap M_2) \cap B << M$, $M_1 \cap B \leq_d B$, $M_2 \cap B \leq_d B$ and $(M_1 \cap B) \cap (M_2 \cap B) << B$. By Lemma 13, $(M_1 \cap B) \cap (M_2 \cap B) = 0$. Hence $M_1 \cap M_2 = A$.

(3) $\Longrightarrow$ (1). To prove M has the SIP we use Proposition 7 and assume that M has the decomposition $M = A \oplus B$ and a homomorphism f from A to B. We show that $\mathrm{Ker}(f)$ is a direct summand. Now we have two cases:

(i) If $\mathrm{Ker}(f) << A$, then by hypothesis we have $\mathrm{Ker}(f) = 0$.

(ii) Assume that $\mathrm{Ker}(f)$ is not small in A. Being M lifting module there exists $C \leq \mathrm{Ker}(f)$ such that $A = C \oplus C'$ and $C' \cap \mathrm{Ker}(f) << A$. Now we define the homomorphism $1 \oplus f : A = C \oplus C' \to C \oplus B$ by $(1 \oplus f)(c + c') = c + f(c')$ where $c \in C$ and $c' \in C'$. Then $\mathrm{Ker}(1 \oplus f) = C' \cap \mathrm{Ker}(f)$. Since $\mathrm{Ker}(1 \oplus f) << A$, we have $\mathrm{Ker}(1 \oplus f) = 0$. On the other hand, $\mathrm{Ker}(f) = C \oplus (C' \cap \mathrm{Ker}(f)) = C$ is a summand of A. This gives that M has the SIP. □

Note that the implication (1) $\Longrightarrow$ (3) in Proposition 15 is not valid in general. Let M denote the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ and $M_1 = M_2 = M$. It is known that M is a lifting module and has the SIP. Let K be a proper submodule of M. Then $M/K \cong M$. Consider π as the canonical map from M onto M/K defined by $\pi(m) = m + K$ for $m \in M_1$. Let g denote the isomorphism $M/K \cong M$ and set $f = g\pi$. Then $\mathrm{Ker}(f)$ is small in M and non-zero submodule of M_1.

Let M be a module. It is well known that for any submodule N of M there exists a closed submodule K such that $N \leq_{ess} K$ and K is called a *closure* of N in M. The module M is called *UC*-module in case every submodule of M has a unique closure (see

[7]). For $B \leq A \leq M$, B is said to be *coessential* submodule of A or A is *coessential extension* of B if $A/B << M/B$. A is said to be *coclosed* in M if A has no coessential submodule in M. Let $B \leq A \leq M$. Then B is called a *coclosure* of A in M if B is coclosed in M and B is coessential in A. Suppose that every submodule A of M has a coessential submodule A^{sc} which is contained in every coessential submodule of A in M. We call M a *unique coclosure module* or *UCC*-module. Recall that a submodule A of M is said to *lie over a direct summand* B if M has a decomposition $M = B \oplus C$, such that $B \leq A$ and $A/B << M/B$. It is known that a module M is a UCC-module if and only if every submodule of M lies over a unique direct summand. In this direction Lemma 16 is proved in [4].

Lemma 16 [4] *Let M be a lifting module. Then M has SSP if and only if M is UCC-module.*

We state Lemma 17 as a dual to Lemma 16 and a generalization of an exercise mentioned in Anderson-Fuller's book (page 214, exercise 7). Note that Lemma 17 is also generalizes Proposition 4 of [11].

Lemma 17 *Let M be an extending module. Then the following are equivalent:*

1. *M is UC-module.*
2. *M has the SIP.*
3. *M has the SSIP.*

Proof. (1) $\Longrightarrow$ (2) Let M be a UC-module. Let N and K be direct summands of M. Then $N \cap K$ is closed in M by Lemma 6 in [10]. By hypothesis $N \cap K \leq_d M$.

(2) $\Longrightarrow$ (1) Assume that M has the SIP. Let $N \leq M$. Suppose that there are $K \leq M$ and $L \leq M$ such that $N \leq_e K \leq_c M$ and $N \leq_e L \leq_c M$. We prove $K = L$. By hypothesis $K \leq_d M$ and $L \leq_d M$ and by (2) $(K \cap L) \oplus T = M$ for some $T \leq M$. Hence $K = (K \cap L) \oplus (K \cap T)$. Since $N \leq_e K$ and $N \cap (K \cap T) = 0$, $K \cap T = 0$. Hence $K = K \cap L$. Similarly, it is shown that $L = K \cap L$. Therefore $K = K \cap L = L$.

(3) $\Longrightarrow$ (2). Clear.

(1) $\Longrightarrow$ (3). Assume that M is UC-module and let $K_i (i \in I)$ be direct summands of M. Then every K_i for $i \in I$ is closed, and so by assumption and Lemma 8 (9) in [10]

$\bigcap_{i\in I} K_i$ is closed in M. By hypothesis $\bigcap_{i\in I} K_i$ is a direct summand. It completes the proof. $\square$

Proposition 18 *Let M be a quasi-continuous module. The following are equivalent:*

1. *M has the SSIP.*
2. *M has the SIP.*
3. *$E(M)$ has the SIP.*
4. *$E(M)$ has the SSIP.*

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) clear from Lemma 17.

(3) $\Rightarrow$ (2) Suppose $E(M)$ has the SIP. Let A and B be direct summands of M. Then there exist A' and B' such that $M = A \oplus A'$ and $M = B \oplus B'$. Then we have that $E(M) = E(A) \oplus L$ and $E(M) = E(B) \oplus L'$ for some submodules L and L' of $E(M)$. Since $E(M)$ has the SIP, $E(M) = [E(A) \cap E(B)] \oplus K$ for some $K \leq E(M)$. Therefore, $M = [(E(A)\cap E(B))\cap M]\oplus(K\cap M)$ by [8, Theorem 2.8]. Now $A \leq_e E(A)$ and $B \leq_e E(B)$ imply $A \leq_e E(A)\cap M$ and $B \leq_e E(B)\cap M$, and since $E(A)\cap M = A\oplus(E(A)\cap M)\cap A'$ and $E(B)\cap M = B\oplus(E(B)\cap M)\cap B'$ it follows that $A = E(A)\cap M$ and $B = E(B)\cap M$. Hence $A\cap B = E(A)\cap E(B)\cap M$ is a direct summand of M.

(2) $\Rightarrow$ (3) Assume M has the SIP and let A and B be direct summands of $E(M)$ and $E(M) = A \oplus A'$ and $E(M) = B \oplus B'$ for some $A' \leq E(M)$ and $B' \leq E(M)$ and $A = E(A)$ and $B = E(B)$. By [8, Theorem 2.8] $A\cap M$ and $B\cap M$ are direct summands of M. By assumption $A\cap B\cap M$ is direct summand of M, and so $(A\cap B\cap M)\oplus L = M$ for some $L \leq M$. Since $A\cap M \leq_e A$ and $B\cap M \leq_e B$ $A\cap B\cap M \leq_e A\cap B$. Hence $E(M) = E(A\cap B\cap M)\oplus E(L) = E(A\cap B)\oplus E(L)$. Therefore, $A = E(A\cap B)\oplus(E(L)\cap A)$ and $B = E(A\cap B)\oplus(E(L)\cap B)$. Then $E(A\cap B) \leq A\cap B \leq E(A\cap B)$ implies $A\cap B = E(A\cap B)$ is a direct summand of $E(M)$. $\square$

It is proved in [4] that a quasi-injective (resp. quasi-projective) module with the SIP (resp. the SSP) has the SSP (resp. the SIP). In this direction, we prove the following Lemma.

Lemma 19 *Let M be a module.*

1. *Let M be a (C_3) module. If M has the SIP then M has the SSP.*

2. *Let M be a (D_3) module. If M has the SSP then M has the SIP.*

Proof. (1). Let M be a (C_3) module. Assume M has the SIP. Let N and T be a direct summands of M. We show that $N + T$ is direct summand of M. Since M has the SIP then there exists $L \leq M$ such that $(N \cap T) \oplus L = M$. By modularity law, we get that $N = (N \cap T) \oplus (L \cap N)$ and $T = (N \cap T) \oplus (L \cap T)$. Then we have $N+T = (N\cap T)+[(L\cap N)\oplus(L\cap T)]$. Next we prove that $(N\cap T)\cap[(L\cap N)\oplus(L\cap T)] = 0$. For if, $x \in (N \cap T) \cap [(L \cap N \oplus (L \cap T)]$, then $x = n_1 + n_2$ where $n_1 \in L \cap N$ and $n_2 \in L \cap T$. We have $n_2 = x - n_1 \in [(N \cap T) + (L \cap N)] \cap (L \cap T) \leq N \cap (L \cap T) = 0$. Hence $n_2 = 0$ and $x = n_1$. Now $x = n_1 \in (N \cap T) \cap (L \cap N) = N \cap T \cap L = 0$. Thus $N + T = (N \cap T) \oplus (L \cap N) \oplus (L \cap T) = T \oplus (L \cap N)$. Since M has the SIP and L, N are direct summands then $L \cap N$ is a direct summand and so by (C_3) it follows that $N + T = T \oplus (L \cap N)$ is a direct summand of M. Thus M has the SSP.

(2). Let M be a (D_3) module. Assume M has the SSP. Let X and Y be direct summands of M. We prove that $X \cap Y$ is a direct summand of M. Since M has the SSP then $X + Y$ is a direct summand, and so there exists $Z \leq M$ such that $M = (X + Y) \oplus Z$. Since X, Y and Z are direct summands and M has the SSP then $X + Z$ and $Y + Z$ are direct summands, and since M is (D_3) and $M = (X + Z) + (Y + Z)$ then $(X + Z) \cap (Y + Z)$ is direct summand, and so there exists $U \leq M$ such that $M = [(X + Z) \cap (Y + Z)] \oplus U$. Now $(X + Z) \cap (Y + Z) = [X \cap (Y + Z)] + Z$ and $X \cap (Y + Z) \leq X \cap Y$ and $M = [(X + Z) \cap (Y + Z)] \oplus U$ imply $M = (X \cap Y) \oplus Z \oplus U$. □

Corollary 20 *Let M be a module having the SIP. Then M is (C_3) module if and only if M has the SSP.*

Proof. Let M be a module having the SIP. Assume that M is (C_3) module. Then by Lemma 19 M has the SSP. The converse is clear since every module having the SSP is a (C_3) module. □

Note that the converse statements (1) and (2) in Lemma 19 need not be true in general. There are (C_3) modules with the SSP but not the SIP. Namely the module in Example 3 is a module having the SSP and therefore (C_3) but does not have the SIP.

There are (D_3) modules having the SIP but not the SSP.

Example 21 *Let K be a field and M denote the left R-module* $R = \begin{pmatrix} K & 0 & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$. *Let e_{ij} denote the matrix units in R. Then it is easy to check that $A = R(e_{11} + e_{13})$, $B = Re_{22}$, $A \oplus B$, $C = R(e_{11} + e_{22})$, $D = R(e_{13} + e_{22} + e_{33})$, $E = R(e_{13} + e_{33})$, $F = Re_{11}$ and $G = R(e_{11} + e_{33})$ are only direct summands of M and their intersections are also direct summands and $A \oplus B \oplus F$ is an essential submodule of M. Then M has the SIP. Also M has (D_3) as a projective module over R. Now $A \cap C = 0$ and $A \oplus C = A \oplus B \oplus F$ is not a direct summand. Hence M does not have the SSP.*

It is proved in [4] that for any ring R and any module M, M has the SSP and the SIP if and only if $S = \mathrm{End}M$ has the SSP. Now we prove Theorem 22 that also generalizes Corollary 2.4 in [4].

Theorem 22 *Let M be a module. Then*

1. *If M has (D_3) then M has the SSP if and only if $S = EndM$ has the SSP.*
2. *If M has (C_3) then M has the SIP if and only if $S = EndM$ has the SSP.*

Proof. (i) Assume S has the SSP. Then M has the SSP and SIP.

Assume M has the SSP. Since M has (D_3) then by lemma 19, M has the SIP. Then S has the SSP.

(ii)Assume M has the SIP. Then by lemma 19, M has the SSP and so M has the SIP and SSP implies S has the SSP.

Assume S has the SSP. Then M has the SSP and SIP by [4,Theorem 2.3]. □

Let M be a module. The submodule $Z(M) = \{m \in M : l(m) \leq_{ess} M\}$ is called singular submodule of M. In case $Z(M) = 0$, M is called nonsingular module.

Corollary 23 *Assume M is nonsingular quasi-continuous module with $S = End(M)$. Then S has the SSP as a right S-module.*

Proof. Let M be a nonsingular quasi-continuous module with a decomposition $M = A \oplus B$ and $f \in \mathrm{Hom}(A, B)$. Since $Z(M) = 0$ it is easy to prove that $\mathrm{Ker}(f)$ is closed in M. Hence $\mathrm{Ker}(f)$ is direct summand of M since M is an extending module. By Proposition 7 M has the SIP, and by Lemma 19, M has the SSP. Then from [4, Theorem 2.3], S has the SSP as a right S-module. □

Let M be a module. Let $N << M$. Then N is a small module, that is N is small submodule of $E(N)$ and also $E(M)$. In the subsequent $Z^*(M)$ will denote the submodule $\{m \in M : Rm << E(M)\}$ of M(see [9]).

Corollary 24 *Let M be a quasi-discrete module with $Z^*(M) = 0$ and $S = End(M)$. Then S has the SIP as a right S-module.*

Proof. Let M be a quasi-discrete module and assume $Z^*(M) = 0$ and A a submodule of M. Then there exists a direct summand B such that $M = B \oplus B'$ with $B \leq A$ and $A \cap B'$ is small in M, and hence $A \cap B' \leq Z^*(M) = 0$. It follows that $A = B$ and A is direct summand. Thus M is semisimple module and so M has the SIP and the SSP. By [12, 37.7] S is regular ring in the sense of von Neumann. Let $I = eS$ and $I' = fS$ be right ideals of S that are direct summands of S for some idempotents e and f of S. Then $eM \cap fM$ is direct summand of M as M has the SIP. If α is the orthogonal projection of M on $eM \cap fM$ then it is easy to check that $\alpha S = eS \cap fS$. Thus $eS \cap fS$ is a direct summand of S. □

Lemma 25 *Let R be a commutative Noetherian ring and $M = M_1 \oplus M_2$ with indecomposable submodules M_1 and M_2. Assume that M has the (C_3) and the SIP, then*

1. *$Hom(M_1, M_2) = 0$ or*
2. *M_1 is isomorphic to M_2 and there is some prime ideal $A \leq R$ with $ann(x) = A$ for every nonzero $x \in M_1$.*

Proof. Take $0 \neq f \in \text{Hom}(M_1, M_2)$. Since $\text{Ker}(f)$ is a direct summand of M_1 we have $\text{Ker}(f) = 0$. Similarly $\text{Im}f$ is direct summand of M_2 since $M_1 \oplus M_2$ has the SSP. Hence f is onto and so M_1 is isomorphic to M_2.
It remains to show the conditions on annihilators. Let $x, y \in M_1$ be nonzero and assume that there is a in $\text{ann}(x)$ but a is not in $\text{ann}(y)$. Define $g : M_1 \to M_2$ by $g(m) = f(am)$ for $m \in M_1$. Then $x \in \text{Ker}(g)$ and y is not in $\text{Ker}(g)$. Hence $\text{Ker}(g) \neq 0$ and $g \neq 0$. This is a contradiction. Hence $a \in \text{ann}(x)$ implies $a \in \text{ann}(y)$ or $\text{ann}(x) = \text{ann}(y)$. Then $\text{ann}(x)$ is prime follows from [6, Theorem 6]. □

Theorem 26 *Let M have a decomposition $M = M_1 \oplus M_2$ with M_1 local module and M_2 simple module.*

1. *Assume Hom$(M_1, M_2) \neq 0$. Then M has not the SIP.*
2. *Assume Hom$(M_2, M_1) \neq 0$. Then M has not the SSP.*

Proof. (1). Assume that $M = M_1 \oplus M_2$ has the SIP. Let $f \in \text{Hom}(M_1, M_2)$ be a nonzero homomorphism. Then $\text{Ker}(f) \neq 0$. Since M has the SIP, by Proposition 7 $\text{Ker}(f)$ is a direct summand of M_1. This gives a contradiction. Therefore, M have not the SIP.

(2). Suppose that $M = M_1 \oplus M_2$ has the SSP. Let $f \in \text{Hom}(M_2, M_1)$ be a nonzero homomorphism. Then $\text{Im}f \neq M_1$. Since M has the SSP, by Theorem 8 $\text{Im}f$ is a direct summand of M_1. This is not possible. It follows that M has not the SSP. □

Corollary 27 *Let M have a decomposition $M = M_1 \oplus M_2$ with M_1 uniserial module and M_2 simple module.*

1. *Assume Hom$(M_1, M_2) \neq 0$. Then M has not the SIP.*
2. *Assume Hom$(M_2, M_1) \neq 0$. Then M has not the SSP.*

Proof. Clear. □

The following example is known. We study here as an illustration of Theorem 26.

Example 28 *Let p be a prime integer. Let $M_1 = \mathbb{Z}/\mathbb{Z}p^2$ and $M_2 = \mathbb{Z}/\mathbb{Z}p$ be $\mathbb{Z}-$modules and $M = M_1 \oplus M_2$. Then M has neither the SIP nor the SSP.*

Proof. Let $f : M_1 \to M_2$ be defined by $f(x + \mathbb{Z}p^2) = y + \mathbb{Z}p$ where $x + \mathbb{Z}p^2 \in M_1$ and $y + \mathbb{Z}p \in M_2$ and y is the remainder when x is divided by p. Then $\mathrm{Ker}(f) = M_1 p$ which is not a direct summand of M_1. Hence M has not the SIP. Let $f : M_2 \to M_1$ be defined by $f(x + \mathbb{Z}p) = px + \mathbb{Z}p^2$ where $x + \mathbb{Z}p \in M_2$. Then $\mathrm{Im}(f) = M_1 p$ which is not a direct summand. Hence M has not the SSP. □

Theorem 29 *Let M be a module with $S = End(M)$.*

1. *If M is (C_2)-module then $M \oplus M$ has the SIP if and only if S is regular ring.*
2. *If M is (D_2)-module then $M \oplus M$ has the SSP if and only if S is regular ring.*

Proof. (1). Let M be (C_2)-module. Necessity: Assume that the module $M \oplus M$ has the SIP. Let $f \in S$. Then f is a homomorphism from a direct summand of $M \oplus M$ to a direct summand of $M \oplus M$. By assumption and Proposition 7, $\mathrm{Ker}(f)$ is direct summand of M. Then $\mathrm{Im}(f)$ is isomorphic to a direct summand of M. By (C_2), $\mathrm{Im}(f)$ is direct summand of M. Thus S is a regular ring from [12, 37.7]. Sufficiency: Suppose that $S = \mathrm{End}(M)$ is a regular ring. By [12, 37.9 (c)], $\mathrm{End}(M \oplus M)$ is also regular ring as a 2×2 matrix ring over the regular ring S, and so $\mathrm{Ker}(f)$ of every $f \in \mathrm{End}(M \oplus M)$ is a direct summand of $M \oplus M$. Hence $M \oplus M$ has the SIP by Proposition 7. Thus M has the SIP as a direct summand of $M \oplus M$.

(2). Let the module M has (D_2). Necessity: Assume now that $M \oplus M$ has the SSP. Let $f \in S$. By assumption and by Propsition 8, $\mathrm{Im}(f)$ is a direct summand of M. Since $\mathrm{Im}(f) \cong M/\mathrm{Ker}(f)$ and M has the (D_2), $\mathrm{Ker}(f)$ is a direct summand of M. By [12, 37.7] S is a regular ring. The proof of sufficiency of (2) is proved in the same way as the sufficiency of (1). This completes the proof. □

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